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Local Representations of Quartic Splines *

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Abstract

Quartic polynomial splines on the real knotset can be used with B-spline basis or with various local representations using some local parameters according to the user's choice or to the problem solved (the most frequently used are function values, mean values or derivative values). Local parameters of a quartic spline are usually computed from continuity conditions, some interpolation and boundary conditions. The aim of this contribution is to give an overview on possible choices of local parameters and corresponding local representations of quartic splines. The general form of continuity conditions as recurrences for local parameters used in such representations is described to form a kernel of computational algorithms.

Key words: Quartic splines, interpolation of function values, mean values, values of the derivative or another linear functional.

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1 Introduction

Splines were introduced to the mathematical community in the middle of our century and they proved to be a very useful tool of function and data approximation (interpolation, least-squares approximation, smoothing, curves and surfaces

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design in computer graphics). We can find at least two different approaches to polynomial splines. The first one uses local parameters of a polynomial spline $s(x)$ (the values $s^{(k)}(t_j), k = 0, 1, \ldots$ on the local interval, mean value over such local interval or its part) to formulate all conditions which determine the spline we search for—and only this approach will be followed in this contribution devoted to quartic interpolatory splines. We will not mention the second possible approach which uses some basis of the spline space (e.g. B-splines, see [3]) to formulate conditions determining spline under search and which eventually turns the computed results from global parameters (B-spline coordinates) to local parameters (e.g. for graphing purposes—see [3], [10]).

Definition 1 Let us have a simple and monotone knotset on the real axis $(\Delta x) = \{x_i, i = 0(1)n + 1\}$ and denote $I_k = [x_k, x_{k+1}]$, $h_k = x_{k+1} - x_k$, $k = 0(1)n$. The quartic spline on this knotset is a function $s_4(x) = s(x)$ with properties
1^0 $s(x)$ is a quartic polynomial on each interval I_k, $k = 0(1)n$;
2^0 $s(x) \in C^3[x_0, x_{n+1}]$ (the continuity condition—CC).

The linear space $S_4(\Delta x)$ of all such quartic splines on the given knotset (Δx) has dimension (see [4]) $\dim S_4(\Delta x) = n + 5$. We can use such splines to interpolate values of different functionals of our interest over intervals I_k:

- function values interpolation (FVI) at the points of interpolation t_i, ($t_i \neq x_i$ in general) with given values $g_i = s(t_i)$, (for the case $t_i = x_i$ see [9]);
- mean values interpolation (MVI) over intervals I_k (or some part of them) with given mean values $g_k = \frac{1}{h_k} \int_{x_k}^{x_{k+1}} s(x)dx$, $k = 0(1)n$;
- derivative values interpolation (DVI) at the points t_i with prescribed values $g_i = s'(t_i)$ (or values of s'', s''');
- interpolation of the values of some more general linear functionals f_k over interval $[x_0, x_{n+1}]$.

The main idea of our approach how to use and compute the local representation of a quartic interpolatory spline can be described in the following way. We use the value of interpolated functional (FV, MV, DV, ...) in the local representation of the spline under search together with some four another local parameters. From the simplicity and computational efficiency purposes we choose these local parameters as two pairs of some Taylor's polynomial parameters at the end points of local intervals I_k—to be common for these intervals. Then we have to express the continuity conditions (CC) between neighbouring intervals, to add the conditions of interpolation (CI) and eventually to complete such a system of equations by some another conditions (usually boundary conditions—(BC)) to make the number of conditions equal to the dimension of the space of splines used. The structure of a matrix of such completed system of linear equations and the computational complexity of its solution depend heavily on the local parameters used. The approach just mentioned leads to systems with some four blocks of tridiagonal matrices—see e.g. [5], [13].

2 Local parameters and representations

2.1 Choice of local parameters

We will use the following notation for the values of the derivatives of the quartic spline $s(x)$ at the knots x_i

$$s_i = s(x_i), \quad m_i = s'(x_i), \quad M_i = s''(x_i), \quad T_i = s'''(x_i), \quad Q_i = s^{(4)}(x_i + 0). \quad (1)$$

Let us first mention two possible extremal cases how to express the (CC) for a quartic spline. Using all five local parameters given in (1), the (CC) at each spline knot $x = x_i$, $i = 1(1)n$ can be written as

$$\begin{aligned} &s_i + h_i m_i + \tfrac{1}{2} h_i^2 M_i + \tfrac{1}{6} h_i^3 T_i + \tfrac{1}{24} h_i^4 Q_i = s_{i+1}, \\ &m_i + h_i M_i + \tfrac{1}{2} h_i^2 T_i + \tfrac{1}{6} h_i^3 Q_i = m_{i+1}, \\ &M_i + h_i T_i + \tfrac{1}{2} h_i^2 Q_i = M_{i+1}, \\ &T_i + h_i Q_i = T_{i+1}. \end{aligned} \quad (2)$$

Completing this system with the conditions of interpolation (in case that we have to interpolate some functional not mentioned in (2)) and eventually with some boundary conditions—both expressed as some linear relations between local parameters used—we obtain as a result some big block-structured system of about $5n$ linear equations. We could solve it and to obtain five local (Taylor) parameters for each knot to store.

The opposite extremal approach was used at first stages of spline theory. Its main idea was to express all given conditions (CC, CI, BC) as recurrences between two kinds of the local parameters used: the given function values at knots and the unknown first or second derivatives of cubic or quadratic spline, the fourth derivatives at quintic splines—see [1], [14], [4]. The techniques used to obtain such relations are quite simple for quadratic and cubic splines—they use the hermitian local representation to express all conditions. The more sophisticated technique of divided differences was used in case of quintic splines on simple grid (see [1]) and for quartic splines on simple grid ($t_i = x_i$, see examples in [6], [9]) and composed grid (with $t_i \neq x_i$ in [6], [8]). The advantage of such approach is to be seen in the simple band structure of the resulting matrix (of the size about n—see e.g. [7], [1]) of the system for computing the values of the parameter chosen. Some disadvantages may be here

1. the lengthy computations which we need to formulate all conditions (CC, CI) in terms of one parameter only;
2. the necessity to compute remaining local parameters from some another recurrences to have full information about the spline (which makes the nice computational complexity slightly worse).

The first of them can be seen as elimination of all another parameters from the whole system (1) and the second one as the back step process. The contemporary abilities of symbolic computing could be of some help in realization of these steps.

Some compromise between two extremal cases mentioned we can see in the technique following the cubic and quadratic case, but adapted to the quartic case. The number of local parameters of the quartic spline on the interval I_k is five—let us put one to denote the subject we have to interpolate and choose the remaining four as the values of two kinds of Taylor's parameters (1) at the endpoints of the interval. The first problem we have to solve now is to have some overview of all possible local representations with respect to functional interpolated. Some contraexamples are known demonstrating the impossibility to prescribe the first derivative in the middle of the interval (see e.g. [6]). In the approximation theory some partial answer for the problem of Birkhoff's interpolation is given in Atkinson-Sharma's Theorem (see e.g. [2]). We could not find similar result giving the answer for the problem of mean value interpolation or the problem with general linear functional, as formulated in the following Lemma.

Lemma 1 *Let us use the local parameters (1) and the values g_k of some linear functionals $f_k : I = [x_0, x_{n+1}] \mapsto \mathrm{R}$ for a local representation of a quartic spline. Then the following conditions are necessary and sufficient for unique determination of a quartic polynomial using the values g_k of f_k and some of the following couples of parameters (1) on the interval I_k, $k = 0(1)n$*

a) $\frac{1}{2}[f_k(u^2)+f_k(u^4)] \neq f_k(u^3)$ *for local parameters* $(s, m) = \{s_i, m_i, s_{i+1}, m_{i+1}\}$

b) $\frac{1}{2}[f_k(u) + f_k(u^4)] \neq f_k(u^3)$ *for local parameters* (s, M)

c) $f_k(u) \neq f_k(u^2)$ *for local parameters* (s, T)

d) $f_k(1) \neq 0$ *for local parameters* $(m, M), (m, T)$ *with the local variable* $u = (x - x_i)/h_i \in (0, 1)$.

Proof Let us write a quartic spline on I_k with local variable u and unknown coefficients a_i as

$$s(x) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4.$$

When we use e.g. the representation (s, m) and write down the conditions of interpolation for $s_i, m_i, s_{i+1}, m_{i+1}$, we obtain four equations for the coefficients a_i. Denoting g_i the prescribed value of the linear functional f_i, the fifth equation can be written as

$$g_i = a_0 f_i(1) + a_1 f_i(u) + a_2 f_i(u^2) + a_3 f_i(u^3) + a_4 f_i(u^4).$$

The completed system of this five equations has a unique solution for all values of chosen parameters under condition that its determinant is not equal to zero; it is rewritten in the statement a).

Similarly we can prove the remaining parts of our Lemma.

Remark 1

1. The local parameters (M,T) combined with any functional do not determine a quartic polynomial (Polya's condition violated—see [2]).
2. Taking f_i as a function value at some point $t_i \in (x_i, x_{i+1})$, we can use any couple of local parameters different from (M,T). As we could see from the more detailed proof of the Lemma, we can extend such result also to function values prescribed slightly outside the intervals I_i (which corresponds with the Schoenberg—Whitney Theorem in B-spline theory—see [3]).
3. Similar conclusion we can obtain in case of MVI–the mean values can be taken over some narrowed or extended intervals (see the examples in [11]).
4. In all mentioned couples of local parameters the conditions of the Lemma are violated in case of prescribing the value of the first derivative in the midpoint of the interval. Please take attention that it does not prove nonexistence of such a spline—only the fact, that it have to be described in some another way.

2.2 List of local representations

We will give a short overview of some basic local representations for problems of (FVI), (MVI) or (DVI) stated in the Introduction (for (FVI) and (DVI) only the case $t_i \neq x_i$ will be discussed; for the case $t_i = x_i$ see [9]). In all this cases we can write the local representation of a quartic spline in the lagrangian form—as linear combination of corresponding basis functions multiplied by the value of the local parameter used. In the following we shall denote
ψ *- the basis function corresponding to the functional* f_k *(FV,MV,DV),*
φ_0^j, φ_1^j, $j \in \{1,2,3\}$ *- the basis functions corresponding to the j-th derivative on the boundaries of* I_k*, which is transformed to the reference interval* $[0,1]$ *of the local variable* u*.*

2.2.1 Local parameters (s,m)

The local representation of $s(x) = s(x_i + h_i u)$ with a local variable $u = (x - x_i)/h_i$ we can write as

$$s(x) = \psi(u)g_i + \varphi_0(u)s_i + \varphi_1(u)s_{i+1} + h_i[\varphi_0^1(u)m_i + \varphi_1^1(u)m_{i+1}] \qquad (3)$$

with the corresponding basis functions $\psi, \varphi_0, \varphi_1, \varphi_0^1, \varphi_1^1$ to be found.

For the (FVI) problem with $d_i = (t_i - x_i)/h_i$ these basis functions are

$$\begin{aligned}
\psi(u) &= [u(u-1)/d_i(1-d_i)]^2 \\
\varphi_0(u) &= [d_i^2 - (1+2d_i+3d_i^2)u^2 + 2(1+d_i)^2u^3 - (1+2d_i)u^4]/d_i^2 \\
\varphi_1(u) &= u^2[d_i(-4+3d_i) + 2(2-d_i^2)u + (-3+2d_i)u^2]/(1-d_i)^2 \qquad (4)\\
\varphi_0^1(u) &= u[d_i - (1+2d_i)u + (2+d_i)u^2 - u^3]/d_i \\
\varphi_1^1(u) &= u^2[d_i - (1+d_i)u + u^2]/(1-d_i).
\end{aligned}$$

In the (MVI) problem we have the following basis functions

$$\begin{aligned}\psi(u) &= 30u^2(1-u)^2,\\ \varphi_0(u) = (1-u)^2(5u+1)(1-3u), &\quad \varphi_1(u) = u^2(6-5u)(3u-2)\\ \varphi_0^1(u) = u(u-1)^2(2-5u)/2, &\quad \varphi_1^1(u) = u^2(u-1)(5u-3).\end{aligned} \tag{5}$$

In the (DVI) problem with $g_i = s'(t_i)$, $d_i \neq 0, \frac{1}{2}, 1$ (see Lemma 1) the corresponding basis functions are

$$\begin{aligned}\psi(u) &= u^2(u-1)^2/[2d_i(d_i-1)(2d_i-1)]\\ \varphi_0(u) &= (u-1)^2[1+2u+(3u^2)/(2d_i-1)]\\ \varphi_1(u) &= u^2[-6d_i+4(1+d_i)u-3u^2]/(1-2d_i)\\ \varphi_0^1(u) &= u(u-1)^2\{1+u(1-3d_i)/[2d_i(2d_i-1)]\}\\ \varphi_1^1(u) &= u^2[d_i(4d_i-3)+2(1-2d_i^2)u+(3d_i-2)u^2].\end{aligned} \tag{6}$$

Remark 2 It would be possible to use the fifth parameter g_i as the value of s'' (with $d_i \neq \frac{1}{2}+\frac{\sqrt{3}}{6}, \frac{1}{2}-\frac{\sqrt{3}}{6}$), or s''' (with $d_i \neq \frac{1}{2}$) or even $s^{(4)}$.

2.2.2 Local parameters (s,M)

The local representation of the quartic spline can be written now as

$$s(x) = \psi(u)g_i + \varphi_0(u)s_i + \varphi_1(u)s_{i+1} + h_i^2[\varphi_0^2(u)M_i + \varphi_1^2(u)M_{i+1}] \tag{7}$$

where in case of (FVI) on the general knotset ($t_i \neq x_i$)
with $d_i = (t_i - x_i)/h_i$, $D_i = d_i^2 + d_i - 1$ we have the basis functions

$$\begin{aligned}\psi(u) &= u(u-1)(1+u-u^2)/[d_i(1-d_i)D_i]\\ \varphi_0(u) &= [d_iD_i + (1+d_i+d_i^2-d_i^3)u - 2u^3 + u^4)]/(d_iD_i)\\ \varphi_1(u) &= u[(d_i-2)/(d_i-1) + 2u^2/(d_i-1) + u^3/d_i]/D_i\\ \varphi_0^2(u) &= (h_i^2/6)u[d_i(3-2d_i) + 3D_iu + (5-d_i-d_i^2)u^2 + (d_i-2)u^3]/D_i\\ \varphi_1^2(u) &= (h_i^2/6)u[-d_i^2 + (d_i^2+d_i+1)u^2 - (1+d_i)u^3]/D_i.\end{aligned} \tag{8}$$

In the frequently used case of (FVI) with $d_i = \frac{1}{2}$ the basis functions are

$$\begin{aligned}\psi(u) &= 16u(u-1)(-1+u+u^2)/5\\ \varphi_0(u) &= (5+13u+16u^3-8u^4)/5\\ \varphi_1(u) &= u(1-2u)-3-6u+4u^2)/5\\ \varphi_0^2(u) &= u(1-u)(1-2u)(3u-4)/30\\ \varphi_1^2(u) &= u(1-u)(1-2u)(1+3u)/30.\end{aligned} \tag{9}$$

For the problem of (MVI) the corresponding basis functions are

$$\begin{aligned}
\psi(u) &= 5u(u-1)(-1-u+u^2)\\
\varphi_0(u) &= (u-1)(-2+5u+5u^2-5u^3)/2\\
\varphi_1(u) &= u(-3+10u^2-5u^3)/2\\
\varphi_0^2(u) &= u(u-1)(3-9u+5u^2)/24\\
\varphi_1^2(u) &= -u(u-1)(1+u-5u^2)/24.
\end{aligned}\tag{10}$$

In the (DVI) problem with $d_i \neq (1-\sqrt{3})/2, 1/2, (1+\sqrt{3})/2$ (see Lemma), $D_i = 6(2d_i-1)(-1-2d_i+2d_i^2)$ we can write the basis functions as

$$\begin{aligned}
\psi(u) &= 6h_iu(u-1)(u^2-u-1)/D_i\\
\varphi_0(u) &= 1+6u[2d_i^2(3-2d_i)-2u^2+u^3]/D_i\\
\varphi_1(u) &= 6u[2d_i^2(-3+2d_i)+2u^2-u^3]/D_i\\
\varphi_0^2(u) &= u[d_i(-6+15d_i-8d_i^2)+(D_i/2)u-(5-12d_i+4d_i^3)u^2\\
&\quad +(2-6d_i+3d_i^2)u^3]/D_i\\
\varphi_1^2(u) &= u[d_i^2(3-4d_i)-(1-4d_i^3)u^2+(1-3d_i^2)u^3]/D_i\,.
\end{aligned}\tag{11}$$

It would be possible to choose $g_i = g(t_i) = s''(t_i)$ as the fifth local parameter. In this case we could use the representation

$$s(x) = (1-u)s_i+us_{i+1}+\tfrac{1}{12}h_i^2u(u-1)[2(1+u-u^2)g_i+(1-u)^2M_i+u^2M_{i+1}]. \tag{12}$$

Remark 3 We could use as the fifth parameter the values of s''' (with $d_i \neq \frac{1}{2}$) or $s^{(4)}$.

2.2.3 Local parameters (s,T)

The corresponding local representation we can write now as

$$s(x) = \psi(u)g_i+\varphi_0(u)s_i+\varphi_1(u)s_{i+1}+h_i^3[\varphi_0^3(u)T_i+\varphi_1^3(u)T_{i+1}] \tag{13}$$

where the basis functions for (FVI) problem with $d_i = (t_i - x_i)/h_i$ are now

$$\begin{aligned}
\psi(u) &= u(u-1)/[d_i(d_i-1)]\\
\varphi_0(u) &= (1-u)(1-u/d_i)\\
\varphi_1(u) &= u(u-d_i)/(1-d_i)\\
\varphi_0^3(u) &= u[d_i(3-d_i)+(-3+3d_i+d_i^2)u+4u^2-u^3]/24\\
\varphi_1^3(u) &= u[d_i(1+d_i)+(1+d_i+d_i^2)u+u^3]/24.
\end{aligned}\tag{14}$$

For the (MVI) problem we have now the following basis functions

$$\begin{aligned} \psi(u) &= 6u(1-u) \\ \varphi_0(u) &= (u-1)(3u-1) \\ \varphi_1(u) &= u(3u-2) \\ \varphi_0^3(u) &= u(u-1)(-6+15u-5u^2)/120 \\ \varphi_1^3(u) &= u(u-1)(-4+5u+5u^2)/120. \end{aligned} \tag{15}$$

The problem of (DVI) with $d_i \neq \frac{1}{2}$ leads to the basis functions

$$\begin{aligned} \psi(u) &= h_i u(1-u)/(1-2d_i) \\ \varphi_0(u) &= (1-2d_i+2d_i u-u^2)/(1-2d_i) \\ \varphi_1(u) &= u(u-2d_i)/(1-2d_i) \\ \varphi_0^3(u) &= u[2d_i(3-6d_i+2d_i^2)+(-3+12d_i^2-4d_i^3)u \\ &\quad +4(1-2d_i)u^2-(1-2d_i)u^3]/[24(1-2d_i)] \\ \varphi_1^3(u) &= u[2d_i(1-2d_i^2)+(4d_i^3-1)u+(1-2d_i)u^3]/[24(1-2d_i)]. \end{aligned} \tag{16}$$

Remark 4 The values of s'' could be used (again with exception) as the fifth local parameter too; the Polya's condition is violated in the case of s'''.

2.2.4 Local parameters (m,M)

Local representation can be written similarly as

$$s(x) = \psi(u)g_i + h_i[\varphi_0^1(u)m_i + \varphi_1^1(u)m_{i+1}] + h_i^2[\varphi_0^2(u)M_i + \varphi_1^2(u)M_{i+1}] \tag{17}$$

with the basis functions for the (FVI) problem given as

$$\begin{aligned} \psi(u) &= 1 \\ \varphi_0^1(u) &= u-d_i-(u^3-d_i^3)+(u^4-d_i^4)/2 \\ \varphi_1^1(u) &= u^3-d_i^3-(u^4-d_i^4)/2 \\ \varphi_0^2(u) &= (u^2-d_i^2)/2-2(u^3-d_i^3)/3+(u^4-d_i^4)/4 \\ \varphi_1^2(u) &= -(u^3-d_i^3)/3+(u^4-d_i^4)/4 \end{aligned} \tag{18}$$

and for the problem of (MVI) with the basis functions

$$\begin{aligned} \psi(u) &= 1 \\ \varphi_0^1(u) &= -\tfrac{7}{20}+u-u^3+\tfrac{1}{2}u^4 \\ \varphi_1^1(u) &= -\tfrac{3}{20}+u^3-\tfrac{1}{2}u^4 \\ \varphi_0^2(u) &= -\tfrac{1}{20}+\tfrac{1}{2}u^2-\tfrac{2}{3}u^3+\tfrac{1}{4}u^4 \\ \varphi_1^2(u) &= \tfrac{1}{30}-\tfrac{1}{3}u^3+\tfrac{1}{4}u^4. \end{aligned} \tag{19}$$

The Atkinson–Sharma's Theorem or our Lemma say, that there is not a local representation of $s(x)$ without at least one of parameters chosen as function value.

2.2.5 Local parameters (m,T)

The corresponding local representation of a spline is now

$$s(x) = \psi(u)g_i + h_i[\varphi_0^1(u)m_i + \varphi_1^1(u)m_{i+1}] + h_i^3[\varphi_0^3(u)T_i + \varphi_1^3(u)T_{i+1}] \quad (20)$$

with basis functions for (FVI) problem

$$\begin{aligned} \psi(u) &= 1 \\ \varphi_0^1(u) &= [d_i(d_i-2) + 2u - u^2]/2 \\ \varphi_1^1(u) &= (u^2 - d_i^2)/2 \\ \varphi_0^3(u) &= [d_i^2(d_i-2)^2 - 4u^2 + 4u^3 - u^4]/24 \\ \varphi_1^3(u) &= [d_i^2(2-d_i^2) - 2u^2 + u^4]/24 \end{aligned} \quad (21)$$

and for the problem (MVI) with basis functions

$$\begin{aligned} \psi(u) &= 1 \\ \varphi_0^1(u) &= -\tfrac{1}{3} + u - \tfrac{1}{2}u^2 \\ \varphi_1^1(u) &= -\tfrac{1}{6} + \tfrac{1}{2}u^2 \\ \varphi_0^3(u) &= \tfrac{1}{45} - \tfrac{1}{6}u^2 + \tfrac{1}{6}u^3 - \tfrac{1}{24}u^4 \\ \varphi_1^3(u) &= \tfrac{7}{360} - \tfrac{1}{12}u^2 + \tfrac{1}{24}u^4. \end{aligned} \quad (22)$$

Once again we can not use as local parameter here the value of s'.

Some additional local representations using as the fifth parameter the value of some derivative at the knots can be found in [9].

3 Continuity conditions

We mentioned in Section 1 roughly the possible ways how to express the conditions of continuity for neighbouring segments of a quartic spline—with their advantages and disadvantages. With the local representations from the last section in our disposal we can now quite simple form the (CC) in terms of chosen local parameters (appearing in or related to the remaining conditions determining the spline under search or having some another meaning for the user).

The idea how to obtain such form of continuity conditions is quite simple: we compare the local representations with the same local parameters used for two neighbouring intervals. We suppose implicitly the equality of the two parameters used in the common point; comparing then the values of the remaining two of them we obtain two relations per inner spline knot. They can be seen as recurrence relations between corresponding local parameters used in the representation (see [6]), related to the problem under consideration (FVI, MVI, DVI). Let us mention the existence of another recurrence relations following from the fact, that the derivatives of the quartic spline are splines of corresponding lower order and have to obey the recurrences known for these splines.

In the following we give an overview of continuity conditions which follow from local representations given in Section 1.

3.1 Local parameters (s,m)

For the (FVI) problem on the composed grid with $d_i = (t_i - x_i)/h_i$ the continuity of s, s' is implicitly supposed in our notation. Equating the expressions for s'', s''' obtained from (s,m) representation in the common point x_i of neighbouring intervals, we obtain the recursions of the general form

$$\begin{aligned} a^i_{i-1}s_{i-1} + a^i_i s_i + a^i_{i+1}s_{i+1} + b^i_{i-1}m_{i-1} + b^i_i m_i + b^i_{i+1}m_{i+1} &= c^i_i g_i - c^i_{i-1}g_{i-1} \\ A^i_{i-1}s_{i-1} + A^i_i s_i + A^i_{i+1}s_{i+1} + B^i_{i-1}m_{i-1} + B^i_i m_i + B^i_{i+1}m_{i+1} &= \\ = C^i_i g_i + C^i_{i-1}g_{i-1} \end{aligned} \tag{23}$$

where the coefficients depend on the geometry of the set described by parameters $h_i,\ d_i = (t_i - x_i)/h_i,\ p_i = h_{i-1}/h_i$ and are given by

$$\begin{aligned} a^i_{i-1} &= (-1 + d_{i-1})(3d_{i-1} + 1)/d^2_{i-1} \\ a^i_{i+1} &= -p^2_i d_i(4 - 3d_i)/(1 - d_i)^2 \\ a^i_i &= (-6 + 8d_{i-1} - 3d^2_{i-1})/(1 - d_{i-1})^2 + p^2_i(1 + 2d_i + 3d^2_i)/d^2_i \\ b^i_{i-1} &= -h_{i-1}(1 - d_{i-1})/d_{i-1}, \quad b^i_{i+1} = -h_i d_i p^2_i/(1 - d_i) \\ b^i_i &= h_{i-1}(3 - 2d_{i-1})/(1 - d_{i-1}) + p^2_i h_i(1 + 2d_i)/d_i \\ c^i_i &= p^2_i[d_i(1 - d_i)]^{-2}, \quad c^i_{i-1} = -[d_{i-1}(1 - d_{i-1})]^{-2} \\ A^i_{i-1} &= 2(1 + 2d_{i-1} - d^2_{i-1})/d^2_{i-1}, \quad A^i_{i+1} = 2p^3_i(2 - d^2_i)/(1 - d_i)^2 \\ A^i_i &= 2[(d_{i-1} - 2)/(1 - d_{i-1})]^2 + p^3_i[(1 + d_i)/d_i]^2 \\ B^i_{i-1} &= h_{i-1}(2 - d_{i-1})/d_{i-1}, \quad B^i_{i+1} = -p^3_i h_i(1 + d_i)/(1 - d_i) \\ B^i_i &= -h_{i-1}(3 - d_{i-1})/(1 - d_{i-1}) + p^3_i h_i(2 + d_i)/d_i \\ C^i_{i-1} &= 2[d_{i-1}(1 - d_{i-1})]^{-2}, \quad C^i_i = 2p^3_i[d_i(1 - d_i)]^{-2}. \end{aligned} \tag{24}$$

On the equidistant knotset (Δx) with $h_i = h,\ d_i = \frac{1}{2},\ i = 0(1)n$ the continuity conditions have simple form

$$\begin{aligned} \tfrac{10}{6}\tfrac{1}{2h}(s_{i+1} - s_{i-1}) + \tfrac{1}{6}(-m_{i-1} + 8m_i - m_{i+1}) &= \tfrac{16}{6}\tfrac{1}{h}(g_i - g_{i-1}) \\ \tfrac{1}{32}(7s_{i-1} + 18s_i + 7s_{i+1}) + \tfrac{3h}{64}(m_{i-1} - m_{i+1}) &= \tfrac{1}{2}(g_{i-1} + g_i). \end{aligned} \tag{25}$$

For the (MVI) problem we obtain the conditions of continuity as the pair of recurrences

$$\begin{aligned} -\tfrac{2}{h^2_{i-1}}s_{i-1} + 3\left(\tfrac{1}{h^2_i} - \tfrac{1}{h^2_{i-1}}\right)s_i + \tfrac{2}{h^2_i}s_{i+1} - \\ -\tfrac{1}{4h_{i-1}}m_{i-1} + \tfrac{3}{4}\left(\tfrac{1}{h_{i-1}} + \tfrac{1}{h_i}\right)m_i - \tfrac{1}{4h_i}m_{i+1} &= 5\left(\tfrac{g_i}{h^2_i} - \tfrac{g_{i-1}}{h^2_{i-1}}\right) \\ \tfrac{14}{h^3_{i-1}}s_{i-1} + 16\left(\tfrac{1}{h^3_{i-1}} + \tfrac{1}{h^3_i}\right)s_i + \tfrac{14}{h^3_i}s_{i+1} - \\ -\tfrac{2}{h^2_{i-1}}m_{i-1} + 3\left(\tfrac{1}{h^2_i} - \tfrac{1}{h^2_{i-1}}\right)m_i - \tfrac{2}{h^2_i}m_{i+1} &= 30\left(\tfrac{g_{i-1}}{h^3_{i-1}} + \tfrac{g_i}{h^3_i}\right). \end{aligned} \tag{26}$$

In the (DVI) problem with $d_j \neq \frac{1}{2}$ the continuity of s'', s''' leads to recurrences (23) with coefficients

$$\begin{aligned}
a^i_{i-1} &= d_{i-1}/[h^2_{i-1}(2d_{i-1}-1)], \quad a^i_{i+1} = -d_i/[h^2_i(2d_i-1)] \\
a^i_i &= [(d_i-1)/[h^2_i(2d_i-1)] - (d_{i-1}-1)/[h^2_{i-1}(2d_{i-1}-1)] \\
b^i_{i-1} &= (d_{i-1}-1)(4d_{i-1}-1)/[12d_{i-1}h_{i-1}(2d_{i-1}-1)] \\
b^i_{i+1} &= -d_i(3-4d_i)/[12h_i(d_i-1)(2d_i-1)], \\
b^i_i &= (6-15d_{i-1}+8d^2_{i-1})/[12h_{i-1}(d_{i-1}-1)(2d_{i-1}-1)]- \\
&\quad -(1+d_i-8d^2_i)/[12h_id_i(2d_i-1)] \\
c^i_i &= -1/[12h_id_i(d_i-1)(2d_i-1)] \\
c^i_{i-1} &= 1/[12h_{i-1}d_{i-1}(d_{i-1}-1)(2d_{i-1}-1)] \\
A^i_{i-1} &= 4(d_{i-1}-1)/[h^3_{i-1}(2d_{i-1}-1)] \\
A^i_{i+1} &= 4(1+d_i)/[h^3_i(2d_i-1)] \\
A^i_i &= 4[(2-d_i)/[h^3_i(2d_i-1)] + (2-d_{i-1})/[h^3_{i-1}(2d_{i-1}-1)] \\
B^i_{i-1} &= (5-6d_{i-1}+2d^2_{i-1})/[h^2_{i-1}(d_{i-1}-1)(2d_{i-1}-1)] \\
B^i_{i+1} &= -(3-6d_i+2d^2_i)/[h^2_i(d_i-1)(2d_i-1)] \\
B^i_i &= (3-6d_{i-1}+2d^2_{i-1})/[h^2_{i-1}(d_{i-1}-1)(2d_{i-1}-1)]- \\
&\quad -(1-4d_i+2d^2_i)/[h^2_id_i(2d_i-1)] \\
C^i_i &= 1/[h^2_id_i(d_i-1)(2d_i-1)] \\
C^i_{i-1} &= 1/[h^2_{i-1}d_{i-1}(1-d_{i-1})(2d_{i-1}-1)].
\end{aligned} \tag{27}$$

3.2 Local parameters (s,M)

For the (FVI) problem on the general composed grid the conditions of continuity of s', s''' give the recurrences similar to (23) with the parameters M_j instead of m_j. For their coefficients (using the same notation) with $d_i = (t_i - x_i)/h_i$, $w_i = (h_i/h_{i-1})^3 v_i$, $v_i = d_i(d_i-1)(d^2_i+d_i-1)/[d_{i-1}(d_{i-1}-1)(d^2_{i-1}+d_{i-1}-1)]$ we obtain

$$\begin{aligned}
a^i_{i-1} &= 6v_i(1-2d_{i-1}+2d^2_{i-1}-d^4_{i-1}), \quad a^i_{i+1} = 6(2-d_i)d^3_i \\
a^i_i &= 6v_id_{i-1}(2-2d^2_{i-1}+d^3_{i-1}) + 6(1-2d^3_i+d^4_i) \\
c^i_{i-1} &= c^i_i = 6, \quad C^i_{i-1} = 12w_i, \quad C^i_i = 12 \\
b^i_{i-1} &= v_id_{i-1}h^2_{i-1}(-1+3d_{i-1}-3d^2_{i-1}+d^3_{i-1}) \\
b^i_{i+1} &= h^2_id^3_i(d_i-1) \\
b^i_i &= v_id_{i-1}h^2_{i-1}(1-3d^2_{i-1}+2d^3_{i-1}) + h^2_ih^2_i(3-5d_i+2d^2_i) \\
A^i_{i-1} &= 12w_i(1-d_{i-1}), \quad A^i_{i+1} = 12d_i, \quad A^i_i = 12(d_{i-1}w_i+1-d_i) \\
B^i_{i-1} &= d_{i-1}h^2_{i-1}w_i(-3+2d_{i-1}-4d^2_{i-1}+d^3_{i-1}), \quad B^i_{i+1} = d_ih^2_i(d^3_i-1) \\
B^i_i &= d_{i-1}h^2_{i-1}w_i(3+4d^2_{i-1}-d^3_{i-1}) + d_ih^2_i(-5+6d_i-d^3_i).
\end{aligned} \tag{28}$$

On equidistant knotset with $h_i = h,\ d_i = \frac{1}{2}$ these recurrences have a simple form (see [6])

$$\begin{aligned} \tfrac{1}{32}(3s_{i-1} + 26s_i + 3s_{i+1}) + \tfrac{h^2}{192}(-M_{i-1} + 8M_i - M_{i+1}) &= \tfrac{1}{2}(g_{i-1} + g_i) \\ \tfrac{1}{4}(s_{i-1} + 2s_i + s_{i+1}) - \tfrac{h^2}{384}(7M_{i-1} + 34M_i + 7M_{i+1}) &= \tfrac{1}{2}(g_{i-1} + g_i). \end{aligned} \tag{29}$$

For the (MVI) problem we obtain the continuity conditions ($p_i = h_{i-1}/h_i$)

$$\begin{aligned} s_{i-1} + \tfrac{7}{3}(1+p_i)s_i + p_i s_{i+1} + \tfrac{1}{36}h_{i-1}^2[-M_{i-1} + 3(1+\tfrac{1}{p_i})M_i + \tfrac{1}{p_i}M_{i+1}] &= \\ = \tfrac{10}{3}(g_{i-1} + p_i g_i) & \\ s_{i-1} + (1+p_i^3)s_i + p_i^3 s_{i+1} - \tfrac{1}{20}h_{i-1}^3[M_{i-1} + \tfrac{7}{3}(1+p_i)M_i + p_i M_{i+1}] &= \\ = 2(g_{i-1} + p_i^3 g_i). & \end{aligned} \tag{30}$$

When we use the local parameter $g_i = s'(t_i),\ t_i \neq \frac{1}{2}$ in the representation (11), then the continuity conditions take the form of recurrences (23) with the coefficients (abbreviation $D = 6(2d-1)(2d^2 - 2d - 1)$ used)

$$\begin{aligned} a_{i-1}^i &= 12(d_{i-1} - 1)(1 + d_{i-1} - 2d_{i-1}^2)/(h_{i-1}D_{i-1}) \\ a_{i+1}^i &= 12d_i^2(3 - 2d_i)/(h_i D_i), \quad a_i^i = -a_{i-1}^i - a_{i+1}^i \\ b_{i-1}^i &= h_{i-1}(d_{i-1} - 1)^2(4d_{i-1} - 1)/D_{i-1} \\ b_{i+1}^i &= h_i d_i^2(4d_i - 3)/D_i \\ b_i^i &= h_{i-1}(d_{i-1} - 1)(8d_{i-1}^2 + 8d_{i-1} - 1)/D_{i-1} + h_i d_i(6 - 15d_i + 8d_i^2)/D_i \\ c_{i-1}^i &= 6/D_{i-1}, \quad c_i^i = 6/D_i \\ A_{i-1}^i &= -12/(h_{i-1}^3 D_{i-1}), \quad A_{i+1}^i = 12/(h_i^3 D_i), \quad A_i^i = -A_{i-1}^i - A_{i+1}^i \\ B_{i-1}^i &= -(3 - 12d_{i-1} + 12d_{i-1}^2 - 4d_{i-1}^3)/(h_{i-1}D_{i-1}) \\ B_{i+1}^i &= -(1 - 4d_i^3)/(h_i D_i) \\ B_i^i &= -(3 - 12d_{i-1}^2 + 4d_{i-1}^3)/(h_{i-1}D_{i-1}) - (5 - 12d_i - 4d_i^3)/(h_i D_i) \\ C_{i-1}^i &= 12/(h_{i-1}D_{i-1}), \quad C_i^i = 12/(h_i D_i). \end{aligned} \tag{31}$$

In the case of local parameters $g_i = s''(t_i)$ the continuity conditions for s', s''' derived from the local representation(12) have the coefficients of recurrences (23) equal to

$$\begin{aligned} a_{i-1}^i &= -1/h_{i-1}, \quad a_i^i = 1/h_{i-1} + 1/h_i, \quad a_{i+1}^i = -1/h_i \\ b_{i-1}^i &= h_{i-1}/6, \quad b_{i+1}^i = (h_i/12)(1 - 2d_i)/(1 - d_i) \\ b_i^i &= [4h_i + h_{i-1}(4d_{i-1} - 3)/(d_{i-1} - 1)]/12 \\ c_{i-1}^i &= h_{i-1}/[12d_{i-1}(d_{i-1} - 1)], \quad c_i^i = h_i/[12d_i(d_i - 1)] \\ A_{i-1}^i &= A_i^i = A_{i+1}^i = 0 \\ B_{i-1}^i &= (1 - d_{i-1})/(h_{i-1}d_{i-1}), \quad B_{i+1}^i = d_i/[h_i(1 - d_i)] \\ B_i^i &= (2 - d_{i-1})/[h_{i-1}(1 - d_{i-1})] + (1 + d_i)/(h_i d_i) \\ C_i^i &= 1/[h_i d_i(1 - d_i)], \quad C_{i-1}^i = 1/[h_{i-1}d_{i-1}(1 - d_{i-1})]. \end{aligned} \tag{32}$$

3.3 Local parameters (s,T)

The conditions of continuity of s', s'' for (FVI) problem can be written in similar three-term recurrences form (23) with parameters T_j instead of m_j and coefficients

$$\begin{aligned}
a^i_{i-1} &= [-1+(1/d_{i-1})]/h_{i-1}, \quad a^i_{i+1} = d_i/[(1-d_i)h_i] \\
a^i_i &= (d_{i-1}-2)/[h_{i-1}(d_{i-1}-1)] + [1+(1/d_i)]/h_i \\
b^i_{i-1} &= (h^2_{i-1}/24)(2-3d_{i-1}+d^2_{i-1}) \\
b^i_{i+1} &= -(h^2_i/24)d_i(1+d_i) \\
b^i_i &= (1/24)[h^2_{i-1}(2-d_{i-1}-d^2_{i-1}) + h^2_i d_i(d_i-3)] \\
c^i_{i-1} &= [h_{i-1}d_{i-1}(1-d_{i-1})]^{-1}, \quad c^i_i = [h_i d_i(1-d_i)]^{-1}; \\
A^i_{i-1} &= 1/(d_{i-1}h^2_{i-1}), \quad A^i_{i+1} = -1/[h^2_i(1-d_i)] \\
A^i_i &= 1/[h^2_{i-1}(1-d_{i-1})] - 1/[h^2_i d_i] \\
B^i_{i-1} &= h_{i-1}(3-3d_{i-1}+d^2_{i-1})/24 \\
B^i_{i+1} &= h_i(1+d_i+d^2_i)/24 \\
B^i_i &= [h_{i-1}(5-d_{i-1}-d^2_{i-1}) + h_i(3+3d_i-d^2_i)]/24 \\
C^i_{i-1} &= [h^2_{i-1}d_{i-1}(1-d_{i-1})]^{-1}, \quad C^i_i = [h^2_i d_i(d_i-1)]^{-1}.
\end{aligned} \tag{33}$$

On the equidistant knotset ($h_i = h, d_i = \frac{1}{2}$) we obtain specially (see [6])

$$\begin{aligned}
\tfrac{1}{8}(s_{i-1}+6s_i+s_{i+1}) + \tfrac{h^3}{256}(T_{i-1}+T_{i+1}) &= \tfrac{1}{2}(g_{i-1}+g_i) \\
\tfrac{1}{2h}(s_{i+1}-s_{i-1}) + \tfrac{h^2}{384}(7T_{i-1}+34T_i+7T_{i+1}) &= \tfrac{1}{h}(g_i-g_{i-1}).
\end{aligned} \tag{34}$$

For the (MVI) problem on the general knotset the recurrences are

$$\begin{aligned}
2s_{i-1} + 4(1+p_i)s_i + 2p_i s_{i+1} + \tfrac{1}{30}h^3 T_{i-1} + \tfrac{1}{20}(h^3_{i-1}+p_i h^3_i)T_i + \tfrac{1}{30}p_i T_{i+1} \\
= 6(p_i g_i + g_{i-1}) \qquad (35) \\
s_{i-1} + (1-p^2_i)s_i + s_{i+1} + \tfrac{1}{40}h^3_{i-1}T_{i-1} + \tfrac{7}{120}(h^3_{i-1}+p^3_i h^3_i)T_i + \tfrac{1}{40}p^2_i h^3_i T_{i+1} \\
= 2(g_{i-1} - p^2_i g_i), \quad p_i = h_{i-1}/h_i\,.
\end{aligned}$$

For the (DVI) problem we obtained the following expressions for coefficients in recurrences (23)

$$\begin{aligned}
a^i_{i-1} &= 2(1-d_{i-1})/[h_{i-1}(2d_{i-1}-1)], \quad a^i_{i+1} = -2d_i/[h_i(2d_i-1)] \\
a^i_i &= 2(d_{i-1}-1)/[h_{i-1}(2d_{i-1}-1)] + 2d_i/[h_i(2d_i-1)] \\
b^i_{i-1} &= h^2_{i-1}(1-5d_{i-1}-6d^2_{i-1}+2d^3_{i-1}))/[12(2d_{i-1}-1)], \\
b^i_{i+1} &= h^2_i d_i(1-2d^2_i)/[12(2d_i-1)] \\
b^i_i &= h^2_{i-1}(-1+3d_{i-1}-2d^3_{i-1})/[12(2d_{i-1}-1)] + \\
&\quad + h^2_i d_i(3-6d_i+2d^2_i)/[12(2d_i-1)]
\end{aligned}$$

$$
\begin{aligned}
c_i^i &= 1/(1-2d_i), \quad c_{i-1}^i = 1/(1-2d_{i-1}) \\
A_{i-1}^i &= 2/[h_{i-1}^2(2d_{i-1}-1)], \quad A_{i+1}^i = 2/[h_i^2(2d_i-1)] \\
A_i^i &= 2/[h_{i-1}^2(1-2d_{i-1})] + 2/[(h_i^2(1-2d_i)] \\
B_{i-1}^i &= h_{i-1}(-3+12d_{i-1}-12d_{i-1}^2+4d_{i-1}^3)/[12(2d_{i-1}-1)] \\
B_{i+1}^i &= h_i(4d_i^3-1)/[12(d_i-1)] \\
B_i^i &= [h_{i-1}(-5+12d_{i-1}-4d_{i-1}^3)/(2d_{i-1}-1) - \\
&\quad - h_i(3-12d_i^2+4d_i^3)/(2d_i-1)]/12 \\
C_i^i &= 2/[h_i(2d_i-1)] \\
C_{i-1}^i &= -2/[h_{i-1}(2d_{i-1}-1)].
\end{aligned} \tag{36}
$$

3.4 Local parameters (m,M)

The continuity conditions for s, s''' in the (FVI) problem can be written similarly to the three term recurrences (23) with the coefficients

$$
\begin{aligned}
a_{i-1}^i &= 2h_{i-1}(1-2d_{i-1}+2d_{i-1}^3-d_{i-1}^4), \quad a_{i+1}^i = 2h_i d_i^3(2-d_i) \\
a_i^i &= 2h_{i-1}(1-2d_{i-1}^3+d_{i-1}^4) + 2h_i d_i(2-2d_i^2+d_i^3) \\
b_{i-1}^i &= (h_{i-1}^2/12)(-1+6d_{i-1}^3+8d_{i-1}^3+6d_{i-1}^4) \\
b_{i+1}^i &= (h_i^2/12)d_i^3(-4+3d_i) \\
c_i^i &= -c_{i-1}^i = 1; \quad C_i^i = C_{i-1}^i = 0 \\
A_{i-1}^i &= 1, \quad A_i^i = p_i^2 - 1, \quad A_{i+1}^i = -p_i^2, \quad p_i = h_{i-1}/h_i \\
B_{i-1}^i &= h_{i-1}/3, \quad B_i^i = 2(h_{i-1}+p_i h_i)/3, \quad B_{i+1}^i = h_i p_i/3\,.
\end{aligned} \tag{37}
$$

In case of equidistant knotset we obtain simple recurrences (see [6])

$$
\begin{aligned}
\tfrac{1}{32}(3m_{i-1}+26m_i+3m_{i+1}) + \tfrac{5}{192}h(M_{i-1}-M_{i+1}) &= \tfrac{1}{h}(g_i-g_{i-1}) \\
\tfrac{1}{2h}(m_{i+1}-m_{i-1}) &= \tfrac{1}{6}(M_{i-1}+4M_i+M_{i+1}).
\end{aligned} \tag{38}
$$

For the (MVI) problem we have the following continuity conditions

$$
\begin{aligned}
&\tfrac{3}{20}h_{i-1}m_{i-1} + \tfrac{7}{20}(h_{i-1}+h_i)m_i + \tfrac{3}{20}h_i m_{i+1} + \\
&+\tfrac{1}{30}h_{i-1}^2 M_{i-1} + \tfrac{1}{20}(h_i^2-h_{i-1}^2)M_i - \tfrac{1}{30}h_i^2 M_{i+1} = g_i - g_{i-1} \qquad (39) \\
&m_{i-1} + (p_i^2-1)m_i - p_i^2 m_{i+1} + \tfrac{1}{3}h_{i-1}[M_{i-1}+2(1+p_i)M_i+p_i M_{i+1}] = 0.
\end{aligned}
$$

3.5 Local parameters (m,T)

The continuity of s, s'' for (FVI) problem leads to recurrences with the coefficients

$$
\begin{aligned}
a_{i-1}^i &= h_{i-1}(1-d_{i-1})^2, \quad a_{i+1}^i = -h_i d_i^2 \\
a_i^i &= h_{i-1}(1-d_{i-1}^2) + h_i d_i(2-d_i)
\end{aligned}
$$

$$
\begin{aligned}
b_{i-1}^i &= \tfrac{1}{12}h_{i-1}^2(-1+4d_{i-1}^2-4d_{i-1}^3+d_{i-1}^4), \quad b_{i+1}^i = \tfrac{1}{12}d_i^2h_i^3(2+d_i^2) \\
b_i^i &= \tfrac{1}{12}[h_{i-1}^3(1+2d_{i-1}^2-d_{i-1}^4)-h_i^3d_i^2(4-4d_i+d_i^2)] \\
c_i^i &= -c_{i-1}^i = 2, \quad C_i^i = C_{i-1}^i = 0 \\
A_{i-1}^i &= -\tfrac{1}{h_{i-1}}, \quad A_{i+1}^i = -\tfrac{1}{h_i}, \quad A_i^i = \tfrac{1}{h_{i-1}}+\tfrac{1}{h_i} \\
B_{i-1}^i &= -\tfrac{1}{6}h_{i-1}, \quad B_{i+1}^i = \tfrac{1}{6}h_i, \quad B_i^i = \tfrac{1}{3}(h_{i-1}+h_i).
\end{aligned}
\tag{40}
$$

In the equidistant case we have specially (see [6])

$$
\begin{aligned}
\tfrac{1}{8}(m_{i-1}+6m_i+m_{i+1}) - \tfrac{1}{384}h^2(7T_{i-1}+18T_i+7T_{i+1}) = \tfrac{1}{h}(g_i-g_{i-1}) \\
\tfrac{1}{h^2}(m_{i-1}-2m_i+m_{i+1}) - \tfrac{1}{6}(T_{i-1}+4T_i+T_{i+1}) = 0.
\end{aligned}
\tag{41}
$$

In the (MVI) problem we obtain the following continuity conditions

$$
\begin{aligned}
h_{i-1}m_{i-1}+2(h_{i-1}+h_i)m_i+h_im_{i+1}-\tfrac{1}{60}[7h_{i-1}^3T_{i-1}+8(h_{i-1}^3+h_i^3)T_i+7h_i^3T_{i+1}] \\
= g_i - g_{i-1}, \\
-h_im_{i-1}+(h_{i-1}+h_i)m_i-h_{i-1}m_{i+1}+ \\
+\tfrac{1}{6}h_{i-1}^2T_{i-1}+\tfrac{1}{3}h_{i-1}(h_{i-1}+h_i)T_i+\tfrac{1}{6}h_{i-1}h_iT_{i+1} = 0
\end{aligned}
\tag{42}
$$

Remark 5 The symbolic computing system MATHEMATICA was used by the author to obtain and control the results presented. The description of algorithms for computing local parameters using the continuity conditions presented here was partially mentioned in [5], [10] and will be continued in some another contribution.

References

[1] Ahlberg, J. H., Nilson, E. N., Walsh, J. L.: *The Theory of Splines and Their Applications.* Academic Press, New York, 1967.

[2] Bojanov, B. P., Hakopian, H. A., Sahakian, A. A.: *Spline Functions and Multivariate Interpolation.* Kluwer Acad. Publ., Dordrecht, 1993.

[3] De Boor, C.: *A Practical Guide to Splines.* Springer Verlag, New York, 1978.

[4] Kobza, J.: *Spline Functions.* VUP Olomouc, 1993, 224 pp. (Textbook in Czech).

[5] Kobza, J.: *Some algorithm for computing local parameters of quartic interpolatory splines.* Acta Univ. Palacki. Olomuc., Fac. rer. nat.**33**, Math. 114 (1994), 63–73.

[6] Kobza, J.: *Spline recurrences for quartic splines.* Acta Univ. Palacki. Olomuc., Fac. rer. nat.**34**, Math. (1995), 75–89.

[7] Kobza, J.: *Computing local parameters of biquartic interpolatory splines.* J. Comp. Appl. Math **63** (1995), 229–236.

[8] Kobza, J.: *Quartic interpolatory splines.* Studia Univ. Babes–Bolyai, Math. (1996), to appear.

[9] Kobza, J.: *Quartic and biquartic interpolatory splines on simple grid.* Acta Univ. Palacki. Olomuc., Fac. rer. nat.**35**, Math. (1996), 61–72.

[10] Kobza, J.: *Quadratic and quartic splines in MATLAB.* Folia Fac. sci. Nat. Univ. Masaryk Brunensis, Mathematica 5 (1997), 47–65.

[11] Laurent, P.-J.: *Approximation et optimisation.* Paris, Hermann, 1972.

[12] Schumaker, L. L.: *Spline Functions. Basic Theory.* Wiley, 1981.

[13] Spaeth, H.: *Eindimensionale Spline-Interpolations-Algorithmen.* Oldenbourgh Verlag, 1990.

[14] Stěčkin, S. B., Subbotin, J. N.: *Splines in Numerical Mathematics.* Nauka, Moscow, 1976 (in Russian).

[15] Zavjalov, J. S., Kvasov, B. I., Mirošničenko, V. L.: *Methods of Spline Functions.* Nauka, Moscow, 1980 (in Russian).