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Desargues Theorem for Klingenberg Projective Plane over Certain Local Ring

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Abstract

In this paper the desarguesian configuration condition of Klingenberg projective planes over certain local ring is founded.

Key words: Local ring, free module, incidence structure, Klingenberg plane.

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The notion *desarguesian* Klingenberg plane was introduced in [4] by the algebraic way. The geometric interpretation (configuration condition) was founded in [6]. We present the configuration condition for planes over certain local ring which is a natural generalization of the desarguesian configuration condition for projective plane over a field.

1 Introduction

According to [5] we define:

Definition 1 Let $V = (P, L, I)$ be a incidence structure and let $\bar{V} = (\bar{P}, \bar{L}, \bar{I})$ be a projective plane. If $\bar{\mu} : V \to \bar{V}$ is a homomorphism of this incidence structures such that:

(a) $\forall \mathrm{P}, \mathrm{Q} \in P,\ \bar{\mu}(\mathrm{P}) \neq \bar{\mu}(\mathrm{Q}),\ \exists! \mathrm{p} \in L : \mathrm{P} I \mathrm{p} \wedge \mathrm{Q} I \mathrm{p}$

(b) $\forall \mathrm{p}, \mathrm{q} \in L,\ \bar{\mu}(\mathrm{p}) \neq \bar{\mu}(\mathrm{q}),\ \exists! \mathrm{P} \in P : \mathrm{P} I \mathrm{p} \wedge \mathrm{P} I \mathrm{q}$

then the triple $(V, \bar{V}, \bar{\mu})$ is called the *Klingenberg projective plane*.

The points $\mathrm{P}, \mathrm{Q} \in P$, resp. lines $\mathrm{p}, \mathrm{q} \in L$, such that $\bar{\mu}(\mathrm{P}) = \bar{\mu}(\mathrm{Q})$, resp. $\bar{\mu}(\mathrm{p}) = \bar{\mu}(\mathrm{q})$, are called *neighbour points*, resp. *neighbour lines*. In the opposite case they are called *non-neighbour*.

The following important theorem is assumed from [5]:

Theorem 2 *Let* $\mathbf{A}$ *be a local ring*[1], *with the maximal ideal* a. *Let us denote* $\mathbf{M} = \mathbf{A}^3$, $\bar{\mathbf{M}} = \mathbf{M}/a\mathbf{M}$, $\bar{\mathbf{A}} = \mathbf{A}/a$, *and let* μ *be a natural homomorphism* $\mathbf{M} \to \bar{\mathbf{M}}$.

Then the triple $(V_A, \bar{V}_{\bar{A}}, \bar{\mu})$, *where*

- V_A *is a incidence structure such that:*
  - *the points are just all free one dimensional submodules of* $\mathbf{M}$,
  - *the lines are just all submodules* $[\mathbf{x}, \mathbf{y}]$ *of* $\mathbf{M}$ *for which* $\mu(\mathbf{x}), \mu(\mathbf{y})$ *forms a linearly independent subset of* $\bar{\mathbf{M}}$,
  - *the incidence relation is the inclusion,*
- $\bar{V}_{\bar{A}}$ *is a projective plane over the vector space* $\bar{\mathbf{M}}$,
- $\bar{\mu} : V_A \to \bar{V}_{\bar{A}}$*is a homomorphism of this incidence structures, which is naturally induced by* μ[2],

is a Klingenberg projective plane and will be called coordinate projective Klingenberg plane over the ring $\mathbf{A}$.

Remark 3 The points $\mathrm{P} = [\mathbf{p}]$, $\mathrm{Q} = [\mathbf{q}]$ are neighbour if and only if their arithmetical representatives $\mathbf{p}$, $\mathbf{q}$ forms linearly dependent subset of $\mathbf{M}$.

In the following text we denote the Klingenberg plane $(V, \bar{V}, \bar{\mu})$ only by V.

Agreement 4 In this paper we have deal with the local ring $\mathbf{A}$ the maximal ideal a of which has the following properties:

(a) $a = \eta \mathbf{A}$,

(b) $\exists m \in \boldsymbol{N} : \eta^m = 0 \wedge \eta^{m-1} \neq 0$,

We may prove easily:

Lemma 5 *For every* $\alpha \in \mathbf{A}$ *there exists a unit* α' *such that*

$$\alpha = \eta^k \alpha'.$$

[1] The ring $\mathbf{A}$ may be noncommutative, generally (see [5]).

[2] I.e. $\bar{\mu}([\mathbf{x}]) = [\mu(\mathbf{x})]$ and $\bar{\mu}([\mathbf{x}, \mathbf{y}]) = [\mu(\mathbf{x}), \mu(\mathbf{y})]$.

Agreement 6 *In the following we denote by* $\mathbf{A}$ *the local ring according to 4 with the maximal ideal* $\eta\mathbf{A}$*. By the capital* $\mathbf{M}$ *will be denote the free* n*-dimensional* $\mathbf{A}$*-modul (so called* $\mathbf{A}$*-space in the sence of [7]). By* V_A *will be denote the coordinate Klingenberg projective plane over the ring* $\mathbf{A}$*. The coset* $\mu(\alpha) \in \bar{\mathbf{A}}$*, resp.* $\mu(\mathbf{x}) \in \bar{\mathbf{M}}$ *will be denote by* $\bar{\alpha}$*, resp.* $\bar{\mathbf{x}}$*.*

Following qualities (7., 8.) of this ring and of free modules over it are assumed from [3]. Free submodules of $\mathbf{M}$ will be called $\mathbf{A}$-*subspaces.*

Proposition 7

(a) If the $\mathbf{A}$*-space* $\mathbf{M}$ *has one basis consisting of* n *elements then any its basis consists of the same number* n *elements. The number* n *is called the dimension of* $\mathbf{M}$*. (It is true for every free module over a commutative ring*[3]*.)*

(b) From every system of generators of $\mathbf{M}$ *we may select a basis of* $\mathbf{M}$*. (It is valid over every local ring (according to Nakayama lemma*[4]*.)*

Moreover in our case:

(c) Any linearly independent system can be completed to a basis of $\mathbf{M}$*.*

(d) Every maximal linearly independent system in $\mathbf{M}$ *forms a basis of* $\mathbf{M}$*.*

Theorem 8 *Let* K, L *be* $\mathbf{A}$*-subspaces of* $\mathbf{A}$*-space* $\mathbf{M}$*. Then* $K + L$ *is an* $\mathbf{A}$*-subspace if and only if the* $K \cap L$ *is an* $\mathbf{A}$*-subspace and in this case the dimensions of* $\mathbf{A}$*-subspaces* $K, L, K \cap L, K + L$ *fulfil the following relation:*
$\dim(K + L) + \dim(K \cap L) = \dim K + \dim L$.

Lemma 9 *Let* $\mathbf{x}$ *as well as* $\mathbf{y}$ *be a linearly independent element of* $\mathbf{M}$*. If*

$$\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{o}, \tag{1}$$

then either $\alpha = \beta = 0$ *or there exists* k*,* $0 \le k < m - 1$*, such that* $\{\alpha, \beta\} \subseteq \eta^k\mathbf{A} - \eta^{k+1}\mathbf{A}$*.*

Proof If $\alpha = 0$ then the linear independence of $\mathbf{y}$ implies $\beta = 0 \cdot \beta = 0$ implies $\alpha = 0$, analogously.

Let $\alpha, \beta \neq 0$. Then they may be written by $\alpha = \eta^k\alpha'$, $\beta = \eta^h\beta'$ where α', β' are units and $0 \le k, h \le m - 1$ (due to lemma 5).

If f.e. $k < h$ consequently $h = k + r$, $r \in N$ we obtain from (1)

$$\eta^k\alpha'\mathbf{x} + \eta^{k+r}\beta'\mathbf{y} = \mathbf{o}.$$

Multiplying the last identity by η^{m-k-r} we get $\eta^{m-r}\mathbf{x} = \mathbf{o}$—a contradiction to the linear independence of $\mathbf{x}$.

[3] See [1].
[4] See [8].

2 Desargues theorem in the Klingenberg plane

Proposition 10 *The lines of Kligenberg plane V_A are just all 2-dimensional* **A***-subspaces of* **M**.

Proof We must prove that the linear independence of the couple $\mathbf{x}, \mathbf{y} \in \mathbf{M}$ is equivalent to the linear independence of the cosets $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ in $\bar{\mathbf{M}}$.

First, let us prove if $\mathbf{x}$, $\mathbf{y}$ determines a line (therefore $\bar{\mathbf{x}} \neq \mathbf{o} \neq \bar{\mathbf{y}}$) then both $\mathbf{x}$ and $\mathbf{y}$ are linear independent elements:

If $\bar{\mathbf{z}} \neq \bar{\mathbf{o}}$, i.e. $\mathbf{z} \in \mathbf{M} \backslash \eta \mathbf{M}$, then at least one of their coordinates $\zeta_1, \zeta_2, \zeta_3$ over an arbitrary basis of the **A**-space **M** is a unit. It implies the linear independence of $\mathbf{z}$, clearly.

(a) Let the couple $\mathbf{x}$, $\mathbf{y}$ be linearly independent and let the couple $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$ be dependent.

Then there exist $\alpha, \beta \in \mathbf{M}$, at least one of them is a unit, such that $\bar{\alpha}\bar{\mathbf{x}} + \bar{\beta}\bar{\mathbf{y}} = \bar{\mathbf{o}}$ which means $\alpha\mathbf{x} + \beta\mathbf{y} \in \eta\mathbf{M}$. It follows from this $(\eta^{m-1}\alpha)\mathbf{x} + (\eta^{m-1}\beta)\mathbf{y} = \mathbf{o}$—it is a contradiction.

(b) Conversely, let the couple $\mathbf{x}$, $\mathbf{y}$ be linearly dependent. Then there exist $\alpha, \beta \in \mathbf{M}$, $\{\alpha, \beta\} \neq \{0\}$, such that

$$\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{o}.$$

It (due to lemma 8) may be written by $\eta^k(\alpha'\mathbf{x} + \beta'\mathbf{y}) = \mathbf{o}$, which means $\alpha'\mathbf{x} + \beta'\mathbf{y} \in \eta\mathbf{M}$. We get $\bar{\alpha}'\bar{\mathbf{x}} + \bar{\beta}'\bar{\mathbf{y}} = \bar{\mathbf{o}}$ where α', β' are units—the couple $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ is linearly dependent as well.

Proposition 11 *Two points of Kligenberg plane V_A are neighbour if and only if their arithmetical representatives forms linear dependent subset of* **M**.

Proof The neighbouring of the points $[\mathbf{x}], [\mathbf{y}]$ is equivalent to the linear dependence of the cosets $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$ and it is (according to the proof above) the necessary and sufficients condition of the linear dependence of the $\mathbf{x}$, $\mathbf{y}$.

Theorem 12 (Desargues) *Let* $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}', \mathrm{B}', \mathrm{C}'$ *and S be points of the plane V_A such that:*

(a) $\mathrm{A}, \mathrm{B}, \mathrm{C}$ *and* $\mathrm{A}', \mathrm{B}', \mathrm{C}'$ *are triples of linearly independent points*[5],

(b) A, A', B, B', C, C' *are couples of non-neighbour points,*

(c) the point S is not neighbour with any of lines AB, BC, AC, A', B', B', C', A', C'.

(d) the point S is the intersection point of lines AA', BB', CC'.

Then the points $\mathrm{X} \in \mathrm{AB} \cap \mathrm{A'B'}$, $\mathrm{Y} \in \mathrm{AC} \cap \mathrm{A'C'}$, $\mathrm{Z} \in \mathrm{BC} \cap \mathrm{B'C'}$ *are determined uniquely and belong to unique line.*

[5]It means that their arithmetical representatives form the linearly independent subset.

Proof Let us denote: A=[a], B=[b], C=[c], analogously for points A', B', C', X=[x], Y=[y], Z=[z], S=[s].

First, let us prove the points X,Y,Z are determined uniquely: Since **a**, **b**, **c** form a basis of **M** we may **s** write in the form

$$\mathbf{s} = \sigma_1 \mathbf{a} + \sigma_2 \mathbf{b} + \sigma_3 \mathbf{c}.$$

Moreover, the all σ_i are units. In the opposite case multiplying the expression of **s** by η^{m-1} we obtain a contradiction with the linear independence of s or with the supposed non-neighbourness of S and any of the lines AB, BC, AC (by 11).

It follows from this that every of **a**, **b**, **c**, **s** may be written by a linear combination of the others of them. Thus (according to 7) $\{\mathbf{a}, \mathbf{b}, \mathbf{s}\}$, $\{\mathbf{a}, \mathbf{s}, \mathbf{c}\}$, $\{\mathbf{s}, \mathbf{b}, \mathbf{c}\}$ are other basis of **M**.

Now, let us investigate the intersection $[\mathbf{a}, \mathbf{b}] \cap [\mathbf{a}', \mathbf{b}']$. Supposing (d) we may write:

$$\mathbf{s} = \alpha \mathbf{a} + \alpha' \mathbf{a}'.$$

Then $\mathbf{M} = [\mathbf{a}, \mathbf{b}, \mathbf{s}] \subseteq [\mathbf{a}, \mathbf{b}, \mathbf{a}'] \subseteq [\mathbf{a}, \mathbf{b}] + [\mathbf{a}', \mathbf{b}']$, i.e. $\mathbf{M} = [\mathbf{a}, \mathbf{b}] + [\mathbf{a}', \mathbf{b}']$, which is an **A**-space. Due to the theorem 8 we get $[\mathbf{a}, \mathbf{b}] \cap [\mathbf{a}', \mathbf{b}']$ is also an **A**-space and the dimension of it is 1. On the other words lines AB and A'B' have exactly one intersection point—X.

The unicity of the points Y,Z may be proved analogously.

Let us prove the unicity of the line XY—i.e. the points X,Y are non-neighbour (see (a) in the definition 1):

Considering $[\mathbf{x}] = [\mathbf{a}, \mathbf{b}] \cap [\mathbf{a}', \mathbf{b}']$ and $[\mathbf{y}] = [\mathbf{a}, \mathbf{c}] \cap [\mathbf{a}', \mathbf{c}']$ we write:

$$\mathbf{x} = \alpha_1 \mathbf{a} + \beta \mathbf{b} \qquad (2)$$

$$\mathbf{x} = \alpha_1' \mathbf{a} + \beta' \mathbf{b}' \qquad (3)$$

$$\mathbf{y} = \alpha_2 \mathbf{a} + \gamma \mathbf{c} \qquad (4)$$

$$\mathbf{y} = \alpha_2' \mathbf{a}' + \gamma' \mathbf{c}' \qquad (5)$$

Let us suppose neighbouring of the points X,Y. Then (by 11) the couple of **x**, **y** is linearly dependent.

Choose representatives **x**, **y** such that (using lemma 1):

$$\eta^k \mathbf{x} + \eta^k \mathbf{y} = \mathbf{o}, \quad 0 \leq k \leq m-1.$$

Multiplying the equalities (2), (4) by η^k we obtain after summing:

$$\mathbf{o} = \eta^k (\alpha_1 + \alpha_2) \mathbf{a} + \eta^k \beta \mathbf{b} + \eta^k \gamma \mathbf{c}$$

and due to (3), (5):

$$\mathbf{o} = \eta^k (\alpha_1' + \alpha_2') \mathbf{a}' + \eta^k \beta' \mathbf{b} + \eta^k \gamma' \mathbf{c}'.$$

Since A,B,C and A', B', C', are linearly independent triples it implies:

$$\eta^k(\alpha_1 + \alpha_2) = 0, \quad \eta^k \beta = 0, \quad \eta^k \gamma = 0, \quad \eta^k(\alpha_1' + \alpha_2') = 0, \quad \eta^k \beta' = 0, \quad \eta^k \gamma' = 0$$

which yields $\beta, \gamma, \beta', \gamma'$ are not units and α_1, α_1' must satisfied just one of the following conditions:

$$\alpha_1 \in \eta\mathbf{A} \wedge \alpha_1' \notin \eta\mathbf{A} \tag{i}$$

$$\alpha_1 \in \eta\mathbf{A} \wedge \alpha_1' \in \eta\mathbf{A} \tag{ii}$$

$$\alpha_1 \notin \eta\mathbf{A} \wedge \alpha_1' \in \eta\mathbf{A} \tag{iii}$$

$$\alpha_1 \notin \eta\mathbf{A} \wedge \alpha_1' \notin \eta\mathbf{A} \tag{iv}$$

Multiplying (2) and (3) by η^{m-1} and using $\beta, \beta' \in \eta\mathbf{A}$ we obtain:

$$\eta^{m-1}\mathbf{x} = (\eta^{m-1}\alpha_1)\mathbf{a} = (\eta^{m-1}\alpha_1')\mathbf{a}'.$$

In the cases $(i), (ii), (iii)$ we have $\eta^{m-1}\mathbf{x} = \mathbf{o}$, which contradicts to the linear independence of $\mathbf{x}$.

In the case (iv) we have $(\eta^{m-1}\alpha_1)\mathbf{a} = (\eta^{m-1}\alpha_1')\mathbf{a}'$, $\eta^{m-1}\alpha_1 \neq 0 \neq \eta^{m-1}\alpha_1'$, which contradicts to the non-neighbouring of points A, A'.

The unicity of the line XY is proved.

Now, let us show that the point Z belongs to this line. The supposition (d) implies:

$$\mathbf{s} = \delta\mathbf{a} + \delta'\mathbf{a}' = \varepsilon\mathbf{b} + \varepsilon'\mathbf{b}' = \varphi\mathbf{c} + \varphi'\mathbf{c}'.$$

Let us prove that all coeficients of this linear combination are units: Multiply it by η^{m-1} and consider following cases:

a) $\delta, \delta' \in \eta\mathbf{A}$, then $\eta^{m-1}\mathbf{s} = \mathbf{o}$—a contradiction with the linear independence of $\mathbf{s}$.

b) $\delta \notin \eta\mathbf{A}$, $\delta' \in \eta\mathbf{A}$ (for example). Then $\eta^{m-1}\mathbf{s} = (\eta^{m-1}\delta)\mathbf{a}$—a contradiction with the non-neighbouring of the points S, A.

By the analogical way we derive that others coeficients are units. The considered expression of $\mathbf{s}$ implies:

$$\begin{aligned} \delta\mathbf{a} - \varepsilon\mathbf{b} &= \varepsilon'\mathbf{b}' - \delta'\mathbf{a}', \\ \delta\mathbf{a} - \varphi\mathbf{c} &= \varphi'\mathbf{c}' - \delta'\mathbf{a}', \\ \varepsilon\mathbf{b} - \varphi\mathbf{c} &= \varphi'\mathbf{c}' - \varepsilon'\mathbf{b}'. \end{aligned}$$

Let us denote this elements in order $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$.

Clearly, $\mathbf{x}' \in [\mathbf{x}]$. Because δ, ε are units and the couple $\mathbf{a}$, $\mathbf{b}$ is linearly independent the element $\mathbf{x}'$ is linearly independent. Thus $[\mathbf{x}'] = [\mathbf{x}]$, which means $\mathbf{x}'$ is a representative of the point X. For elements $\mathbf{y}'$, $\mathbf{z}'$ we obtain the same situation.

Evidently, $\mathbf{z}' = \mathbf{y}' - \mathbf{x}'$. It follows from this $[\mathbf{z}] \subseteq [\mathbf{x}, \mathbf{y}]$, which means $\mathrm{Z} \in \mathrm{XY}$.

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