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Incidence Structures and Closure Spaces

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Abstract

A relationship between closure spaces and incidence structures with corresponding concept lattices is studied in this paper.

Key words: Closure spaces, incidence structures, corresponding complete lattices.

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Definition 1 Let G be a set and $\mathcal{G}$ be a family of its subsets. Then the pair $(G, \mathcal{G})$ is called a *closure space* if $\mathcal{G}$ is intersection closed and $G \in \mathcal{G}$. If $A \subseteq G$, then the intersection $\langle A \rangle$ of all sets of $\mathcal{G}$ containing A is called a *closure* of A.

Obviously, $\mathcal{G}$ forms a complete lattice under the set inclusion, which will be denoted by $L(G, \mathcal{G})$ in what follows. If $\mathcal{A} \subseteq \mathcal{G}$, then the infimum $\bigwedge \mathcal{A}$ is the intersection of all sets of $\mathcal{A}$ and the supremum $\bigvee \mathcal{A}$ is the intersection of all sets of $\mathcal{G}$ containing the union of sets of $\mathcal{A}$.

The closures of subsets A, B of G have the following properties:

$A \subseteq \langle A \rangle$,

$A \subseteq B \Rightarrow \langle A \rangle \subseteq \langle B \rangle$,

$\langle A \rangle = \langle \langle A \rangle \rangle$,

$\langle A \rangle = \bigvee\limits_{g \in A} \langle \{g\} \rangle$

Definition 2 Let $L = (G, \leq)$ be a complete lattice. If $A \subseteq G$, then the set $U(A) = \{x \in G \mid a \leq x \ \forall a \in A\}$ is called the *upper cone* of A. If $P \subseteq G$, then $U^P(A) := U(A) \cap P$ for an arbitrary set $A \subseteq G$. Particularly, if $A = \{x\}$, then $U^P(x) := U^P(\{x\})$.

Remark 1 In what follows, the lattice operations in L are denoted, as usual, by symbols $\bigwedge, \bigvee$, the greatest or the least elements of L are denoted by 1_L or 0_L, respectively. We suppose that $\bigwedge \emptyset = 1_L$ and $\bigvee \emptyset = 0_L$.

Theorem 1 *Let $L = (G, \leq)$ be a complete lattice and $P \subseteq G$. If $\mathcal{U}^P = \{U^P(x) \mid x \in G\}$, then $(P, \mathcal{U}^P)$ is a closure space, in which $\langle A \rangle = U^P(\bigwedge A)$ for an arbitrary subset $A \subseteq P$.*

Proof Let $A \subseteq G$. Obviously, $z \in U^P(A) \Leftrightarrow z \in P$, $a \leq z$ for all $a \in A \Leftrightarrow z \in P$, $z \in U(a)$ for all $a \in A \Leftrightarrow z \in U^P(a)$ for all $a \in A \Leftrightarrow z \in \bigcap_{a \in A} U^P(a)$.

Hence

$$U^P(A) = \bigcap_{a \in A} U^P(a) \tag{1}$$

Similarly, $x \in U^P(A) \Leftrightarrow x \in P$, $a \leq x$ for all $a \in A \Leftrightarrow x \in P$, $\bigvee A \leq x \Leftrightarrow x \in U^P(\bigvee A)$.

Hence

$$U^P(A) = U^P\left(\bigvee A\right) \tag{2}$$

Because of $U(0_L) = G$, we obtain $U^P(0_L) = G \cap P = P$ and $P \in \mathcal{U}^P$. Consider $\mathcal{U}_1 \subseteq \mathcal{U}^P$. Then $\mathcal{U}_1 = \{U^P(x) \mid x \in A\}$ for a certain set $A \subseteq G$. According to (1), (2), $\bigcap_{x \in A} U^P(x) = U^P(A) = U^P(\bigvee A) \in \mathcal{U}^P$.

Let $A \subseteq P$. Then $A \subseteq U^P(x)$ iff $x \leq a$ for all $a \in A$ iff $x \leq \bigwedge A$. We obtain that

$$z \in \langle A \rangle \Leftrightarrow z \in \bigcap_{A \subseteq U^P(x)} U^P(x) \Leftrightarrow z \in \bigcap_{x \leq \wedge A} U^P(x) \Leftrightarrow z \in U^P(x)$$

for all $x \leq \bigwedge A \Leftrightarrow z \in P$, $x \leq z$ for all $x \leq \bigwedge A \Leftrightarrow z \in P$, $\bigwedge A \leq z \Leftrightarrow z \in U^P(\bigwedge A)$.

Therefore $\langle A \rangle = U^P(\bigwedge A)$.

Definition 3 Let $L = (G, \leq)$ be a complete lattice. A set $P \subseteq G$ is called *infimally dense* in L if for each $x \in G$ there exists a subset $X \subseteq P$ such that $x = \bigwedge X$.

Theorem 2 *Let $L = (G, \leq)$ be a complete lattice and let $P \subseteq G$. Then the map $\varphi : G \to \mathcal{U}^P$ in which $U^P(x)$ corresponds to every $x \in G$ is a bijective map of the set G onto the set $\mathcal{U}^P$ if and only if P is an infimally dense set in L. In this case, φ is even an antiisomorphism of the lattices L and $L(P, \mathcal{U}^P)$.*

Proof 1. Let us assume that P is an infimally dense set in L. Obviously, φ is a surjective map onto $\mathcal{U}^P$. Let $U^P(x) \subseteq U^P(y)$ for certain $x, y \in G$. Since P is infimally dense in L, there exists a set $X \subseteq P$ such that $x = \bigwedge X$. Thus $x \leq z$ for all $z \in X$ and $X \subseteq U(x)$, therefore $X \subseteq U^P(x)$.

We obtained $X \subseteq U^P(y)$ and $y \leq z$ for all $z \in X$, from which $y \leq \bigwedge X$ and $y \leq x$ immediately follows. Thus $U^P(x) = U^P(y)$ implies $x = y$ and φ is injective.

2. Let us assume that $\varphi : x \mapsto U^P(x)$ is a bijective map of the set G onto $\mathcal{U}^P$, and that there exists an element $x \in L$ such that $x \neq \bigwedge X$ for every subset $X \subseteq P$. There exists the infimum $q = \bigwedge U^P(x)$ in L, and since $U^P(x) \subseteq P$, we get $q \neq x$. Because of $x \leq z$ for all $z \in U^P(x)$, it follows that $x \leq \bigwedge U^P(x)$, and therefore $x \leq q$.

Hence $U^P(q) \subseteq U^P(x)$. If $z \in U^P(x)$, then $q \leq z$ and $z \in U^P(q)$, and hence $U^P(x) \subseteq U^P(q)$.

We obtained $U^P(x) = U^P(q)$ and $\varphi(x) = \varphi(q)$ for $x \neq q$, which is a contradiction to the injectivity of φ.

So, to each element $x \in L$ there exists a set $X \subseteq P$ such that $x = \bigwedge X$.

3. Let P be an infimally dense set in L. According to Theorem 1, $(P, \mathcal{U}^P)$ is a closure space and $L(P, \mathcal{U}^P)$ is a complete lattice which we can join to that space by Definition 1. We will denote the lattice operations in L and in $L(P, \mathcal{U}^P)$ by the symbols $\bigwedge, \bigvee$ and $\bigwedge', \bigvee'$, respectively.

We will prove that the map $\varphi : x \mapsto U^P(x)$ is an antiisomorphism of the complete lattices L and $L(P, \mathcal{U}^P)$: Consider a set $A \subseteq G$. Then

$$\bigvee_{a \in A}{}' U^P(a) = \bigcap U^P(z),$$

where $U^P(a) \subseteq U^P(z)$ for all $a \in A$.

According to the first part of this proof, we obtain that $U^P(a) \subseteq U^P(z)$ for all $a \in A$ if and only if $z \leq a$ for all $a \in A$ if and only if $z \leq \bigwedge A$. Thus

$$\bigvee_{a \in A}{}' U^P(a) = \bigcap_{z \leq \wedge A} U^P(z).$$

Now we obtain that $x \in U^P(\bigwedge A) \Leftrightarrow x \in P$, $\bigwedge A \leq x \Leftrightarrow x \in P$, $z \leq x$ for all $z \leq \bigwedge A \Leftrightarrow x \in U^P(z)$ for all $z \leq \bigwedge A \Leftrightarrow x \in \bigcap_{z \leq \wedge A} U^P(z) \Leftrightarrow x \in \bigvee'_{a \in A} U^P(a)$. Hence

$$\varphi\Big(\bigwedge A\Big) = U^P\Big(\bigwedge A\Big) = \bigvee_{a \in A}{}' U^P(a)$$
$$= \bigvee{}' \{U^P(a) \mid a \in A\} = \bigvee{}' \{\varphi(a) \mid a \in A\} = \bigvee{}' \varphi(A).$$

Similarly we can prove that $\varphi(\bigvee A) = \bigwedge' \varphi(A)$. Hence, φ is an antiisomorphism of L and $L(P, \mathcal{U}^P)$.

Remark 2 If P_1, P_2 are infimally dense sets in a complete lattice L, then the complete lattices $L(P_1, \mathcal{U}^{P_1})$, $L(P_2, \mathcal{U}^{P_2})$ are isomorphic.

Remark 3 We can define the *lower cone* $L(A)$ for $A \subseteq G$ in a complete lattice $L = (G, \leq)$ as $L(A) = \{x \in G \mid x \leq a \ \forall a \in A\}$ and also the sets $L^P(A) = L(A) \cap P$, $\mathcal{L}^P = \{L^P(x) \mid x \in G\}$ for an arbitrary set $P \subseteq G$. The pair $(P, \mathcal{L}^{\mathcal{P}})$ is a closure space and $L(P, \mathcal{L}^{\mathcal{P}})$ is a complete lattice.

In a similar way we can define a supremally dense set Q in L. The complete lattices L and $L(P, \mathcal{L}^{\mathcal{P}})$ are isomorphic if and only if P is supremally dense in L. If P and Q are infimally and supremally dense in L, respectively, then the complete lattices $L(P, \mathcal{U}^{\mathcal{P}})$, $L(Q, \mathcal{L}^{\mathcal{Q}})$ are antiisomorphic.

Definition 4 Let G and M be sets and $I \subseteq G \times M$. Then the triple $\mathfrak{I} = (G, M, I)$ is called an *incidence structure*. If $A \subseteq G, B \subseteq M$ are non-empty sets, then we denote $A^{\uparrow} = \{m \in M \mid gIm \ \forall g \in A\}$, $B^{\downarrow} = \{g \in G \mid gIm \ \forall m \in B\}$. And moreover, we also denote $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$, $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$ for $A \subseteq G$, $B \subseteq M$, $g^{\uparrow} := \{g\}^{\uparrow}$, $m^{\downarrow} := \{m\}^{\downarrow}$ for $g \in G$, $m \in M$.

Let $\mathfrak{I} = (G, M, I)$ be an incidence structure. Then it is obvious that:

$A_1 \subseteq A_2 \Rightarrow A_2^{\uparrow} \subseteq A_1^{\uparrow}$ for $A_1, A_2 \subseteq G$,

$B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}$ for $B_1, B_2 \subseteq M$,

$A \subseteq A^{\uparrow\downarrow}$, $B \subseteq B^{\downarrow\uparrow}$ for $A \subseteq G$, $B \subseteq M$,

$A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$, $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$ for $A \subseteq G$, $B \subseteq M$,

$\left(\bigcup_{i \in J} A_i\right)^{\uparrow} = \bigcap_{i \in J} A_i{}^{\uparrow}$, $\left(\bigcup_{i \in J} B_i\right)^{\downarrow} = \bigcap_{i \in J} B_i^{\downarrow}$ for $A_i \subseteq G$, $B_i \subseteq M$,

$A^{\uparrow\downarrow} = \bigcap_{m \in A^{\uparrow}} m^{\downarrow}$, $B^{\downarrow\uparrow} = \bigcap_{g \in B^{\downarrow}} g^{\uparrow}$ for $A \subseteq G$, $B \subseteq M$.

Theorem 3 *Let $\mathfrak{I} = (G, M, I)$ be an incidence structure. If we put*

$$\mathcal{G}_{\mathfrak{I}} = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\}, \qquad \mathcal{M}_{\mathfrak{I}} = \{B \subseteq M \mid B = B^{\downarrow\uparrow}\}$$

then the pairs $(G, \mathcal{G}_{\mathfrak{I}})$, $(M, \mathcal{M}_{\mathfrak{I}})$ are closure spaces and the complete lattices $L(G, \mathcal{G}_{\mathfrak{I}})$, $L(M, \mathcal{M}_{\mathfrak{I}})$ are antiisomorphic.

Proof First we will prove that $(G, \mathcal{G}_{\mathfrak{I}})$ is a closure space. Because of $A \subseteq A^{\uparrow\downarrow}$ for an arbitrary set $A \subseteq G$, we get $G \subseteq G^{\uparrow\downarrow}$ and $G = G^{\uparrow\downarrow}$, and thus $G \in \mathcal{G}_{\mathfrak{I}}$.

Consider sets $A_i \in \mathcal{G}_{\mathfrak{I}}$, $i \in J$. Then

$$\Big(\bigcap_{i \in J} A_i\Big)^{\uparrow\downarrow} = \Big(\bigcap_{i \in J} A_i{}^{\uparrow\downarrow}\Big)^{\uparrow\downarrow} = \Big(\Big(\bigcup_{i \in J} A_i{}^{\uparrow}\Big)^{\downarrow}\Big)^{\uparrow\downarrow} = \Big(\bigcup_{i \in J} A_i{}^{\uparrow}\Big)^{\downarrow} = \bigcap_{i \in J} A_i{}^{\uparrow\downarrow} = \bigcap_{i \in J} A_i$$

and hence $\bigcap_{i \in J} A_i \in \mathcal{G}_{\mathfrak{I}}$.

Similarly, we can prove that $(M, \mathcal{M}_{\mathfrak{I}})$ is also a closure space. If we denote the lattice operations in the complete lattice $L(G, \mathcal{G}_{\mathfrak{I}})$ by $\bigwedge, \bigvee$, then

$$\bigwedge_{i \in J} A_i = \bigcap_{i \in J} A_i \quad \text{and} \quad \bigvee_{i \in J} A_i = \Big(\bigcup_{i \in J} A_i\Big)^{\uparrow\downarrow},$$

where $A_i \in \mathcal{G}_{\mathfrak{I}}$ for all $i \in J$.

Similarly, for the lattice operations $\bigwedge', \bigvee'$ in $L(M, \mathcal{M}_\mathfrak{I})$ we obtain

$$\bigwedge_{i\in J}{}' B_i = \bigcap_{i\in J} B_i, \qquad \bigvee_{i\in J}{}' B_i = \Big(\bigcup_{i\in J} B_i\Big)^{\downarrow\uparrow},$$

where $B_i \in \mathcal{M}_\mathfrak{I}$ for all $i \in J$.

Consider the map $\varphi : A \to A^\uparrow,\ A \in \mathcal{G}_\mathfrak{I}$.

Because of $(A^\uparrow)^{\downarrow\uparrow} = A^\uparrow$, we get $A^\uparrow \in \mathcal{M}_\mathfrak{I}$ and φ is a map into $\mathcal{M}_\mathfrak{I}$. Take $B \in \mathcal{M}_\mathfrak{I}$. Then $B^\downarrow \in \mathcal{G}_\mathfrak{I}$ and $\varphi(B^\downarrow) = B^{\downarrow\uparrow} = B$.

Thus φ is a surjective map onto $\mathcal{M}_\mathfrak{I}$. If $A_1{}^\uparrow = A_2{}^\uparrow$ for $A_1, A_2 \in \mathcal{G}_\mathfrak{I}$ then $A_1{}^{\uparrow\downarrow} = A_2{}^{\uparrow\downarrow} = A_1 = A_2$ and φ is injective. Take sets $A_i \in \mathcal{G}_\mathfrak{I}$, $i \in J$.

Then

$$\varphi\Big(\bigwedge_{i\in J} A_i\Big) = \varphi\Big(\bigcap_{i\in J} A_i\Big) = \Big(\bigcap_{i\in J} A_i\Big)^\uparrow = \Big(\bigcap_{i\in J} A_i{}^{\uparrow\downarrow}\Big)^\uparrow$$
$$= \Big(\bigcup_{i\in J} A_i{}^\uparrow\Big)^{\downarrow\uparrow} = \bigvee_{i\in J} A_i{}^\uparrow = \bigvee_{i\in J}{}' \varphi(A_i).$$

And similarly,

$$\varphi\Big(\bigvee_{i\in J} A_i\Big) = \varphi\bigg(\Big(\bigcup_{i\in J} A_i\Big)^{\uparrow\downarrow}\bigg) = \Big(\bigcup_{i\in J} A_i\Big)^{\uparrow\downarrow\uparrow}$$
$$= \Big(\bigcup_{i\in J} A_i\Big)^\uparrow = \bigcap_{i\in J} A_i{}^\uparrow = \bigwedge_{i\in J}{}' \varphi(A_i)$$

Remark 4 If $A \subseteq G$, $B \subseteq M$, then $\langle A\rangle = A^{\uparrow\downarrow}$ and $\langle B\rangle = B^{\downarrow\uparrow}$ in $(G, \mathcal{G}_\mathfrak{I})$ and $(M, \mathcal{M}_\mathfrak{I})$, respectively.

Theorem 4 *Let $\mathfrak{I} = (G, M, I)$ be an incidence structure and let*

$$\mathcal{A}_\mathfrak{I} = \{g^{\uparrow\downarrow} \mid g \in G\}, \qquad \mathcal{B}_\mathfrak{I} = \{m^\downarrow \mid m \in M\}.$$

Then the set $\mathcal{B}_\mathfrak{I}$ is infimally dense and the set $\mathcal{A}_\mathfrak{I}$ is supremally dense in $L(G, \mathcal{G}_\mathfrak{I})$. And moreover, gIm if and only if $g^{\uparrow\downarrow} \subseteq m^\downarrow$. (See literature [1], [2].)

Theorem 5 *Let $(G, \mathcal{G})$, $(M, \mathcal{M})$ be closure spaces and the sets $2^G, 2^M, \mathcal{G}, \mathcal{M}$ be ordered by the set inclusion. The following conditions are equivalent:*

(1) There exist antitone (i.e. order converting) maps $\varphi_1 : 2^G \to \mathcal{M}$, $\varphi_2 : 2^M \to \mathcal{G}$ such that $\langle A\rangle = \varphi_2\varphi_1(A)$ and $\langle B\rangle = \varphi_1\varphi_2(B)$ for all $A \in 2^G, B \in 2^M$.

(2) The complete lattices $L(G, \mathcal{G}), L(M, \mathcal{M})$ are antiisomorphic.

Proof (1) $\Rightarrow$ (2) If $A \in 2^G$, then $\varphi_1(A) = \varphi_1(\langle A\rangle) : A \subseteq \langle A\rangle$ implies $\varphi_1(\langle A\rangle) \subseteq \varphi_1(A)$. If $C \subseteq \varphi_1(A)$, then $\varphi_2\varphi_1(A) = \langle A\rangle \subseteq \varphi_2(C)$ and $C \subseteq \langle C\rangle = \varphi_1\varphi_2(C) \subseteq \varphi_1(\langle A\rangle)$, thus $\varphi_1(A) \subseteq \varphi_1(\langle A\rangle)$.

Hence $\xi : \langle A\rangle \to \varphi_1(A)$, $A \in 2^G$, is a map of $\mathcal{G}$ into $\mathcal{M}$. ξ is surjective: Take $B \in \mathcal{M}$, thus $B = \langle B\rangle$. Then $\varphi_2(B) \in \mathcal{G}$ and $\xi(\varphi_2(B)) = \varphi_1\varphi_2(B) = \langle B\rangle = B$.

ξ is injective: Let $\xi(\langle A\rangle) = \xi(\langle B\rangle)$ for $A, B \in 2^G$. Then $\varphi_1(A) = \varphi_1(B)$ and $\varphi_2\varphi_1(A) = \langle A\rangle = \varphi_2\varphi_1(B) = \langle B\rangle$. ξ is antitone: If $A_1, A_2 \in 2^G$, then $\langle A_1\rangle \subseteq \langle A_2\rangle$ implies $\varphi_1(\langle A_2\rangle) \subseteq \varphi_1(\langle A_1\rangle)$, from which $\varphi_1(A_2) \subseteq \varphi_1(A_1)$ immediately follows. Hence $\xi(\langle A_2\rangle) \subseteq \xi(\langle A_1\rangle)$.

The map ξ^{-1} is antitone too. Therefore, ξ is an antiisomorphism of the ordered sets $(\mathcal{G}, \subseteq), (\mathcal{M}, \subseteq)$ and also the antiisomorphism of the complete lattices $L(G, \mathcal{G}), L(M, \mathcal{M})$.

(2) $\Rightarrow$(1) Let ξ be an antiisomorphism of the complete lattices $L(G, \mathcal{G})$, $L(M, \mathcal{M})$. ξ is also the antiisomorphism of the ordered sets $(\mathcal{G}, \subseteq)$, $(\mathcal{M}, \subseteq)$.

If we put $\varphi_1(A) = \xi(\langle A\rangle)$ for $A \in 2^G$, then φ_1 is a map of 2^G onto $\mathcal{M}$.

If $\varphi_2(B) = \xi^{-1}(\langle B\rangle)$ for $B \in 2^M$, then φ_2 is a map of 2^M onto $\mathcal{G}$.

For $A_1, A_2 \in 2^G$ we obtain: $A_1 \subseteq A_2$ implies that $\langle A_1\rangle \subseteq \langle A_2\rangle$ and hence $\xi(\langle A_2\rangle) \subseteq \xi(\langle A_1\rangle)$. From this $\varphi_1(A_2) \subseteq \varphi_1)A_1)$ and φ_1 is antitone. Similarly for φ_2.

Obviously, $\varphi_2\varphi_1(A) = \varphi_2(\xi(\langle A\rangle))$. And since of $\xi(\langle A\rangle) \in \mathcal{M}$, we obtain $\langle\xi(\langle A\rangle)\rangle = \xi(\langle A\rangle)$ and $\varphi_2\langle\xi(\langle A\rangle)\rangle = \xi^{-1}\langle\xi(\langle A\rangle)\rangle = \langle A\rangle$ for $A \in 2^G$ and $\varphi_1\varphi_2(B) = \langle B\rangle$ for $B \in 2^M$.

Remark 5 If $(G, \mathcal{G}_\mathfrak{I})$, $(M, \mathcal{M}_\mathfrak{I})$ are closure spaces belonging to the incidence structure $\mathfrak{I} = (G, M, I)$, then the maps $\varphi_1 :=\uparrow$, $\varphi_2 :=\downarrow$ satisfy the condition (1) from Theorem 5.

Theorem 6 *Let $(G, \mathcal{G})$, $(M, \mathcal{M})$ be closure spaces, and let the complete lattices $L(G, \mathcal{G})$, $L(M, \mathcal{M})$ be antiisomorphic i.e. let the maps φ_1, φ_2 mentioned in Theorem 5 exist.*

If we consider the incidence structure $\mathfrak{I} = (G, M, I)$ in which gIm if and only if $m \in \varphi_1(\{g\})$, then $L(G, \mathcal{G}) = L(G, \mathcal{G}_\mathfrak{I})$ and $L(M, \mathcal{M}) = L(M, \mathcal{M}_\mathfrak{I})$.

Proof Obviously, $m \in \varphi_1(\{g\})$ iff $\langle\{m\}\rangle \subseteq \varphi_1(\{g\})$ iff $\varphi_2\varphi_1(\{g\}) \subseteq \varphi_2(\langle\{m\}\rangle)$ iff $\langle\{g\}\rangle \subseteq \varphi_2(\{m\})$ iff $g \in \varphi_2(\{m\})$.

By the assumption there exists an antiisomorphism ξ of the complete lattice $L(G, \mathcal{G})$ onto $L(M, \mathcal{M})$, which is described in the proof of Theorem 5. For an arbitrary set $A \in 2^G$ we obtain:

$$\varphi_1(A) = \xi(\langle A\rangle) = \xi\Big(\bigvee_{g\in A} \langle\{g\}\rangle\Big) = \bigwedge_{g\in A}{}' \xi(\langle\{g\}\rangle) = \bigcap_{g\in A} \varphi_1(\{g\})$$
$$= \{m \in M \mid m \in \varphi_1(\{g\} \ \forall g \in A\} = \{m \in M \mid gIm \ \forall g \in A\} = A^\uparrow.$$

Similarly, for $B \in 2^M$ we get

$$\varphi_2(B) = \xi^{-1}(\langle B\rangle) = \xi^{-1}\Big(\bigvee_{m\in B}{}' \langle\{m\}\rangle\Big) = \bigcap_{m\in B} \varphi_2(\{m\})$$
$$= \{g \in G \mid g \in \varphi_2(\{m\}) \ \forall m \in B\} = \{g \in G \mid gIm \ \forall m \in B\} = B^\downarrow.$$

If $B \in \mathcal{M}$, then $B = \langle B\rangle = \varphi_1\varphi_2(B) = B^{\downarrow\uparrow}$ and thus $B \in \mathcal{M}_\mathfrak{I}$.

Similarly, if $B \in \mathcal{M}_\mathfrak{I}$, then $B = B^{\downarrow\uparrow} = \varphi_1\varphi_2(B) = \langle B\rangle$ and thus $B \in \mathcal{M}$. Therefore $\mathcal{M} = \mathcal{M}_\mathfrak{I}$ and, obviously, $L(M, \mathcal{M}) = L(M, \mathcal{M}_\mathfrak{I})$.

In a similar way we can prove $L(G, \mathcal{G}) = L(G, \mathcal{G}_\mathfrak{I})$.

Remark 6 Let $(G, \mathcal{G})$ be a closure space and $P \subseteq \mathcal{G}$ be an infimally dense set in the complete lattice $L(G, \mathcal{G})$. According to Theorems 1 and 2, $(P, \mathcal{U}^P)$ is a closure space and the map $\xi : A \to U^P(A)$, where $A \in \mathcal{G}$, is an antiisomorphism of the complete lattices $L(G, \mathcal{G})$, $L(P, \mathcal{U}^P)$.

According to Theorem 5, let us define the maps $\varphi_1 : 2^G \to \mathcal{U}^P, \varphi_2 : 2^P \to \mathcal{G}$ by setting $\varphi_1(A) = \xi(\langle A \rangle) = U^P(\langle A \rangle)$ for $A \in 2^G$, and

$$\varphi_2(\mathcal{B}) = \xi^{-1}(\langle \mathcal{B} \rangle) = \xi^{-1}\left(U^P\left(\bigwedge \mathcal{B}\right)\right) = \bigwedge \mathcal{B}$$

for $\mathcal{B} \in 2^P$, where $\langle \mathcal{B} \rangle = U^P\left(\bigwedge \mathcal{B}\right)$ by Theorem 1.

Consider the incidence structure $\mathfrak{I}_P = (G, P, I)$, in which for $g \in G$, $B \in P$ it holds gIB iff $B \in \varphi_1(g)$, which is equivalent to $B \in U^P(\langle\{g\}\rangle)$ and $\langle\{g\}\rangle \subseteq B$ if and only if $g \in B$.

According to Theorem 6 we get $A^\uparrow = \varphi_1(A) = U^P(A)$ for $A \in 2^G$ and $\mathcal{B}^\downarrow = \varphi_2(\mathcal{B}) = \bigwedge \mathcal{B}$ for $\mathcal{B} \in 2^P$.

Particularly, $B^\downarrow = B$ for $B \in P$. It is also easy to prove that $A^{\uparrow\downarrow} = \langle A \rangle$ for $A \in 2^G$, and $\mathcal{B}^{\downarrow\uparrow} = \langle \mathcal{B} \rangle = U^p(\bigwedge \mathcal{B})$ for $\mathcal{B} \in 2^P$.

Definition 5 Consider incidence structures $\mathfrak{I} = (G, M, I)$, $\mathfrak{I}_1 = (G_1, M_1, I_1)$. A map $\varphi : G \cup M \to G_1 \cup M_1$ is called a *homomorphism* of the incidence structure $\mathfrak{I}$ onto the incidence structure $\mathfrak{I}_1$ if $\varphi(G) = G_1, \varphi(M) = M_1$ and gIm implies that $\varphi(g)I_1\varphi(m)$. A homomorphism φ is called an *I-homomorphism* if $\varphi(g)I_1\varphi(m)$ implies gIm. An I-homomorphism φ is called an *isomorphism* if restrictions $\varphi|G : G \to G_1$ and $\varphi|M : M \to M_1$ of the map φ are injective.

Theorem 7 *Let $(G, \mathcal{G})$ be a closure space and $\mathfrak{I} = (G, M, I)$ be an incidence structure. Then the following conditions are equivalent:*

(1) Let $P = \{P_i \subseteq G \mid i \in K\}$ be an infimally dense set in the complete lattice $L(G, \mathcal{G})$ and let J_i be an index set for each $i \in K$.

Then $M = \{m_j{}^i \mid i \in K, j \in J_i\}$, and for an arbitrary $i \in K$ $gIm_j{}^i$ if and only if $g \in P_i$ for all $j \in J_i$.

(2) $L(G, \mathcal{G}) = L(G, \mathcal{G}_\mathfrak{I})$

(3) There exist an infimally dense set P in the complete lattice $L(G, \mathcal{G})$ and an I-homomorphism φ of the incidence structure $\mathfrak{I}$ onto the incidence structure $\mathfrak{I}_P = (G, P, I_1)$ such that $\varphi(g) = g$ for all $g \in G$ (see Remark 6).

Proof (1) $\Rightarrow$ (2) Let us consider the incidence structure $\mathfrak{I}_P = (G, P, I_1)$. Then, by Remark 6, gI_1P_i if and only if $g \in P_i$ for $g \in G$, $P_i \in P$. For an arbitrary subset $A \subseteq G$ it holds $A^{\uparrow\downarrow} = \langle A \rangle$, where $\langle A \rangle$ is a closed set in $(G, \mathcal{G})$, and thus $L(G, \mathcal{G}) = L(G, \mathcal{G}_\mathfrak{I})$. We will write the operators $\uparrow$ and $\downarrow$ on the left hand side of a set symbol in the incidence structure $\mathfrak{I}$. Then $\mathcal{G}_\mathfrak{I} = \{A \subseteq G \mid A = {}^{\downarrow\uparrow}A\}$. For $A \subseteq G$ we get

$$\begin{aligned} A^\uparrow &= \{P_r \in P \mid gI_1P_r \ \forall g \in A\} \\ &= \{P_r \in P \mid g \in P_r \ \forall g \in A\} = \{P_r \in P \mid A \subseteq P_r\} \end{aligned}$$

and

$$\begin{aligned} {}^{\uparrow}\!A &= \{m_j{}^r \in M \mid gIm_j{}^r\ \forall g \in A\} \\ &= \{m_j{}^r \in M \mid g \in P_r\ \forall g \in A\} = \{m_j{}^r \in M \mid A \subseteq P_r\}. \end{aligned}$$

Hence $h \in A^{\uparrow\downarrow} \Leftrightarrow hI_1P_r$ for all $P_r \in A^{\uparrow} \Leftrightarrow h \in P_r$ for all P_r such that $A \subseteq P_r \Leftrightarrow hIm_j{}^r$ for all P_r such that $A \subseteq P_r$ and for all $j \in J_r \Leftrightarrow hIm_j{}^r$ for all $m_j{}^r \in {}^{\uparrow}\!A \Leftrightarrow h \in {}^{\downarrow\uparrow}\!A$.

Therefore, $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}\!A$ for all subsets $A \subseteq G$ and thus $\mathcal{G}_{\mathfrak{I}} = \mathcal{G}_{\mathfrak{I}_P}$ and $L(G, \mathcal{G}) = L(G, \mathcal{G}_{\mathfrak{I}})$.

(2)$\Rightarrow$ (3) If we put $\mathcal{B}_{\mathfrak{I}} = \{m^{\downarrow} \mid m \in M\} = P$, then, by Theorem 4, P is an infimally dense set in the complete lattice $L(G, \mathcal{G}_{\mathfrak{I}})$ and, by the assumption, also in $L(G, \mathcal{G})$.

If we consider the incidence structure $\mathfrak{I}_P = (G, P, I_1)$, then, by Remark 6, $gI_1m^{\downarrow}$ if and only if $g \in m^{\downarrow}$ for $g \in G, m \in M$.

Now, let us consider the mapping $\varphi : G \cup M \to G \cup P$ in which $g \mapsto g$ for $g \in G$ and $m \mapsto m^{\downarrow}$ for $m \in M$. Then gIm iff $g \in m^{\downarrow}$ which is equivalent to $gI_1m^{\downarrow}$ if and only if $\varphi(g)I_1\varphi(m)$ and φ is an I-homomorphism of the incidence structure $\mathfrak{I}$ onto $\mathfrak{I}_P$.

(3) $\Rightarrow$ (1) Let φ be an I-homomorphism of the incidence structure $\mathfrak{I}$ onto $\mathfrak{I}_P = (G, P, I_1)$, where P is an infimally dense set in $L(G, \mathcal{G})$ and $\varphi(g) = g$ for $g \in G$. Then gIm iff $gI_1\varphi(m)$ iff $g \in \varphi(m)$.

We put $P = \{P_i \subseteq G \mid i \in K\}$, $\overline{m}^i = \{n \in M \mid \varphi(n) = P_i\}$ and, with the index set J_i, $\overline{m}^i = \{m_j{}^i \mid j \in J_i\}$. Hence $gIm_j{}^i$ if and only if $g \in P_i$.

Remark 7 An incidence structure $\mathfrak{I} = (G, M, I)$ is called *M-simple* if $m^{\downarrow} = n^{\downarrow}$ implies $m = n$ for $m, n \in M$.

Let us consider a closure space $(G, \mathcal{G})$, an infimally dense set P in the complete lattice $L(G, \mathcal{G})$ and the incidence structure $\mathfrak{I}_P = (G, P, I)$.

By Remark 6, $B^{\downarrow} = B$ for $B \in P$, hence $B^{\downarrow} = C^{\downarrow}$ implies $B = C$ and the incidence structure $\mathfrak{I}_P$ is M-simple. Let φ be an I-homomorphism of the incidence structure $\mathfrak{I} = (G, M, I)$ onto $\mathfrak{I}_P$ described in Theorem 7. Then φ is an isomorphism if and only if the incidence structure $\mathfrak{I}$ is M-simple. In this case we obtain $|J_i| = 1$ for all $i \in K$ in Theorem 7.

References

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