Svatoslav Staněk

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Solvability of Nonlinear Functional Boundary Value Problems

SVATOSLAV STANĚK

Department of Mathematical Analysis, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: stanek@risc.upol.cz

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Abstract

New boundary value problems for the functional differential equation $x'' = f(t, x, x', x_t, x'_t)$ are considered. By the Leray–Schauder degree method, existence results are proved under assumption that f is the Carathéodory operator.

Key words: Existence, Carathéodory solution, functional boundary conditions, functional differential equation, Leray–Schauder degree, Borsuk theorem.

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1 Introduction

Let C_r $(r > 0)$ be the Banach space of C^0-functions on $[-r, 0]$ with the norm $\|x\|_* = \max\{|x(t)| : t \in [-r, 0]\}$. For any continuous function $x : [-r, 1] \to \mathbf{R}$ and each $t \in [0, 1] =: J$ denote by x_t the element of C_r defined by

$$x_t(s) = x(t + s) \quad \text{for } s \in [-r, 0].$$

Let $\mathbf{X}$ be the Banach space of C^0-functions on J with the norm $\|x\|_\infty = \max\{|x(t)| : t \in J\}$ and $L_k(J)$ $(k \in \mathbf{N})$ be the Banach space of measurable functions $x : J \to \mathbf{R}$ such that

$$\|x\|_k = \left[\int_0^1 |x(t)|^k \, dt\right]^{\frac{1}{k}} < \infty .$$

For each interval $I \subset J$ denote by $\mathcal{D}_I$ the set of surjective functionals $\gamma : \mathbf{X} \to \mathbf{R}$ which are

(i) continuous, $\gamma(0) = 0$, and

(ii) increasing (i.e. $x, y \in \mathbf{X}$, $x(t) < y(t)$ for $t \in I \Rightarrow \gamma(x) < \gamma(y)$)

and set $\mathcal{D}_I^* = \{\gamma : \gamma \in \mathcal{D}_I, \lim_{n\to\infty} \gamma(\varepsilon x_n) = \varepsilon\infty$ for each $\varepsilon \in \{-1, 1\}$ and any $\{x_n\} \subset \mathbf{X}$, $\lim_{n\to\infty} x_n(t) = \infty$ locally uniformly on $I\}$ (see [11] and [12] where also some examples of functionals belonging to $\mathcal{D}_I$ are given).

From the following Example 1 follows that $\mathcal{D}_J - \mathcal{D}^*_{[0,1)} \neq \emptyset$.

Example 1 Consider the functional $\gamma : \mathbf{X} \to \mathbf{R}$ defined by

$$\gamma(x) = x(1) + \arctan x\left(\frac{1}{2}\right).$$

Obviously, $\gamma(0) = 0$, $\gamma(\mathbf{R}) = \mathbf{R}$ and γ is continuous increasing; hence $\gamma \in \mathcal{D}_J$. Set $x_n(t) = n\cos(\frac{t\pi}{2})$ for $t \in J$ and $n \in \mathbf{N}$. Then $\lim_{n\to\infty} x_n(t) = \infty$ locally uniformly on $[0, 1)$ and

$$\begin{aligned}\lim_{n\to\infty} \gamma(\varepsilon x_n) &= \lim_{n\to\infty} \left(\varepsilon x_n(1) + \arctan\left(\varepsilon x_n\left(\frac{1}{2}\right)\right)\right) \\ &= \lim_{n\to\infty} \arctan\left(\varepsilon n \cos\left(\frac{\pi}{4}\right)\right) = \frac{\varepsilon\pi}{2}\end{aligned}$$

for $\varepsilon \in \{-1, 1\}$. Thus $\gamma \notin \mathcal{D}^*_{[0,1)}$.

We say that $f : J \times \mathbf{R}^2 \times C_r \times C_r \to \mathbf{R}$ satisfies assumption (H) if

(H): (a) $f(\cdot, x, y, \varrho, \psi)$ is measurable on J for each $(x, y, \varrho, \psi) \in \mathbf{R}^2 \times C_r \times C_r$,

(b) $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous on $\mathbf{R}^2 \times C_r \times C_r$ for a.e. $t \in J$, and

(c) there exist $k, l, p, q, r \in L_1(J)$ such that

$$|f(t, x, y, \varrho, \psi)| \le k(t)|x| + l(t)|y| + p(t)\|\varrho\|_* + q(t)\|\psi\|_* + r(t) \quad (1)$$

for a.e. $t \in J$ and each $(x, y, \varrho, \psi) \in \mathbf{R}^2 \times C_r \times C_r$.

Let f satisfy assumption (H). In the paper we consider the functional differential equation

$$x'' = f(t, x, x', x_t, x'_t) \quad (2)$$

together with the functional boundary conditions

$$(x_0, x'_0) \in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbf{R}\}, \quad \alpha(x|_J) = A, \quad \beta(x'|_J) = B \quad (3)$$

or

$$(x_0, x'_0) \in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbf{R}\}, \quad \alpha(x|_J) = A, \quad \beta_1(x(1) - x|_J) = B. \quad (4)$$

Here $\varphi, \chi \in C_r$, $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$, $A, B \in \mathbf{R}$ and $x|_J$ is the restriction of x to J.

By a *solution* of BVP (2), (i) ($i = 3, 4$) we mean a continuous function $x : [-r, 1] \to \mathbf{R}$ having the absolutely continuous first derivative on J (i.e. $x|_J \in AC^1(J)$), $(x_0, x'_0) = (\varphi - \varphi(0) + x(0), \chi - \chi(0) + x'(0))$ and satisfying the last two boundary conditions of (i) (see [11]).

This definition of a solution of BVP (2), (i) ($i = 3, 4$) is motivated by the papers of Haščák ([5]–[7]) where some formulations of BVPs for the n-th order linear differential equations with delays were given. We observe that for any solution x of BVP (2), (i) ($i = 3, 4$) the functions x_1, x_2 defined by

$$x_1(t) = \begin{cases} \varphi(t) + c_1 & \text{for } t \in [-r, 0] \\ x(t) & \text{for } t \in (0, 1], \end{cases}$$

$$x_2(t) = \begin{cases} \chi(t) + c_2 & \text{for } t \in [-r, 0] \\ x'(t) & \text{for } t \in (0, 1] \end{cases}$$

with $c_1 = -\varphi(0) + x(0)$, $c_2 = -\chi(0) + x'(0)$ are continuous on $[-r, 1]$.

Remark 1 If $f(t, x, y, \varrho, \psi) = f_1(t, x, y)$ is independent of ϱ, ψ and if we set $\alpha(x) = x(0)$, $\beta(x) = x(1)$, $\beta_1(x) = x(\eta)$ for $x \in \mathbf{X}$ with an $\eta \in (0, 1)$, then (2)–(4) (with $A = B = 0$) imply

$$x'' = f_1(t, x, y), \tag{5}$$

$$x(0) = 0, \quad x'(1) = 0 \tag{6}$$

and

$$x(0) = 0, \quad x(1) - x(\eta) = 0. \tag{7}$$

This paper was motivated by the recently papers of Marano [8] and Gupta [3] where sufficient conditions for the existence of BVP (5), (i) ($i = 6, 7$) where given. In [8] the results are proved by an existence theorem for operator inclusions by O. N. Ricceri and B. Ricceri [10]. In [3] it is given a simple proof of Theorem 1 of [8] using a Leray–Schauder continuation theorem by Mawhin [9] and the author also obtained a better analogue of Theorem 3 of [8]. The results of [3] and [8] improve those of [2].

In this paper we generalize results of [3] and [8] especially in the following directions:

(i) there are considered functional differential equations, and

(ii) boundary conditions have a nonlinear functional form.

The existence theorems are proved by the Leray–Schauder degree method and by the Borsuk theorem (see e.g. [1], [9]).

2 Lemmas, notation

Lemma 1 *Let $I \subset J$ be an interval, $u \in \mathbf{X}$, $\alpha \in \mathcal{D}_I$ and $c \in [0,1]$. Let the equality*

$$\alpha(x+u) + (c-1)\alpha(-x+u) = c\alpha(u)$$

be satisfied for an $x \in \mathbf{X}$. Then there exists a $\xi \in I$ such that

$$x(\xi) = 0.$$

Proof Set $\gamma(z) = \alpha(z+u) + (c-1)\alpha(-z+u) - c\alpha(u)$ for $z \in \mathbf{X}$. Then $\gamma \in \mathcal{D}_I$ and $\gamma(x) = 0$. If $x(t) \neq 0$ on I we obtain $\gamma(x) \neq 0$, a contradiction. □

Lemma 2 *Let α, $\beta \in \mathcal{D}_J$ and A, $B \in \mathbf{R}$. Then the system*

$$\alpha(a+bt) = A, \quad \beta(b) = B \tag{8}$$

has a unique solution $(a_0, b_0) \in \mathbf{R}^2$.

Proof Define the continuous functions $p : \mathbf{R}^2 \to \mathbf{R}$, $q : \mathbf{R} \to \mathbf{R}$ by

$$p(a,b) = \alpha(a+bt), \quad q(b) = \beta(b).$$

Since q is increasing on $\mathbf{R}$ and $\lim_{b\to\pm\infty} q(b) = \pm\infty$, there exists a unique $b_0 \in \mathbf{R}$ such that $q(b_0) = B$. The function $p(\cdot, b_0)$ is increasing on $\mathbf{R}$ and $\lim_{a\to\pm\infty} p(a, b_0) = \pm\infty$, and consequently $p(a_0, b_0) = A$ for a unique $a_0 \in \mathbf{R}$. We see that $(a_0, b_0) \in \mathbf{R}^2$ is the unique solution of (8). □

Lemma 3 *Let $\alpha \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$ and A, $B \in \mathbf{R}$. Then the system*

$$\alpha(a+bt) = A, \quad \beta_1(b(1-t)) = B \tag{9}$$

has a unique solution $(a_0, b_0) \in \mathbf{R}^2$.

Proof Since the proof is very similar to that of Lemma 2, it is omitted. □

Let $u, v \in \mathbf{X}$, $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$, $\varphi, \chi \in C_r$ and let h satisfy assumption (H) (with $f = h$). To prove the main existence results we consider the auxiliary BVPs (10), (i) ($i = 11, 12$) where

$$x'' = h(t, x, x', x_t, x'_t), \tag{10}$$

$$\begin{aligned} (x_0, x'_0) &\in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbf{R}\}, \\ &\alpha(x|_J + u) = \alpha(u), \quad \beta(x'|_J + v) = \beta(v), \end{aligned} \tag{11}$$

$$\begin{aligned} (x_0, x'_0) &\in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbf{R}\}, \\ &\alpha(x|_J + u) = \alpha(u), \quad \beta_1(x(1) - x|_J + v) = \beta_1(v). \end{aligned} \tag{12}$$

Let $\mathbf{Y}$ be the Banach space of AC^1-functions on J endowed with the norm $\|x\|_A = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_1\}$. For each $c \in [0,1]$ define the operators

$$H_c, V_c : \mathbf{Y} \times \mathbf{R}^2 \to \mathbf{Y} \times \mathbf{R}^2$$

by

$$H_c(x,A,B) = \Big(A + Bt + c\int_0^t\int_0^s h(\tau, x(\tau), x'(\tau), x_\tau, x'_\tau)\,d\tau\,ds,$$
$$A + \alpha(x+u) + (c-1)\alpha(-x+u) - c\alpha(u),$$
$$B + \beta(x'+v) + (c-1)\beta(-x'+v) - c\beta(v)\Big),$$

$$V_c(x,A,B) = \Big(A + Bt + c\int_0^t\int_0^s h(\tau, x(\tau), x'(\tau), x_\tau, x'_\tau)\,d\tau\,ds,$$
$$A + \alpha(x+u) + (c-1)\alpha(-x+u) - c\alpha(u),$$
$$B + \beta_1(x(1) - x + v) + (c-1)\beta_1(-x(1) + x + v) - c\beta(v)\Big),$$

where

$$x_t(s) = \begin{cases} \varphi(t+s) - \varphi(0) + x(0) & \text{for } t+s \in [-r,0] \\ x(t+s) & \text{for } t+s \in (0,1], \end{cases} \tag{13'}$$

$$x'_t(s) = \begin{cases} \chi(t+s) - \chi(0) + x'(0) & \text{for } t+s \in [-r,0] \\ x'(t+s) & \text{for } t+s \in (0,1]. \end{cases} \tag{13''}$$

Consider the operator equations

$$H_c(x,A,B) = (x,A,B), \quad c \in [0,1] \tag{14_c}$$

and

$$V_c(x,A,B) = (x,A,B), \quad c \in [0,1]. \tag{15_c}$$

Remark 2 We see that x is a solution of BVP (10), (11) (resp. (10), (12)) if $(x|_J, x(0), x'(0))$ is a solution of (14_1) (resp. (15_1)). And conversely, let (x,A,B) be a solution of (14_1) (resp. (15_1)) and $\bar{x} : [-r,1] \to \mathbf{R}$ be given by $\bar{x}(t) = \varphi(t) - \varphi(0) + x(0)$ for $t \in [-r,0]$ and $\bar{x}|_J = x$. Then $(\bar{x}.A,B)$ is a solution of BVP (10), (11) (resp. (10),(12)). So to prove existence results for BVP (10), (11) and BVP (10), (12) it is enough to show ones for functional equations (14_1) and (15_1), respectively.

Lemma 4 *Let h satisfy assumption (H) (with $f = h$). Let*

$$\lambda := \|k\|_1 + \|l\|_1 + \|p\|_1 + \|q\|_1 < 1 \tag{16}$$

and set

$$\Lambda = \frac{1}{1-\lambda}\Big[2(\|p\|_1\|\varphi\|_* + \|q\|_1\|\chi\|_*) + \|r\|_1\Big] + 1,$$

$$\Omega = \Big\{(x,A,B) : (x,A,B) \in \mathbf{Y} \times \mathbf{R}^2, \|x\|_A < \Lambda,\ |A| < \Lambda,\ |B| < \Lambda\Big\}.$$

If (x,A,B) is a solution of (14_c) or (15_c) for a $c \in [0,1]$, then $(x,A,B) \in \Omega$.

Proof Let (x, A, B) be a solution of (14_c) for a $c \in [0,1]$. Then the following equalities

$$x(t) = A + Bt + c\int_0^t\int_0^s h(\tau, x(\tau), x'(\tau), x_\tau, x'_\tau)\,d\tau\,ds, \quad t \in [0,1], \tag{17}$$

$$\alpha(x+u) + (c-1)\alpha(-x+u) = c\alpha(u), \tag{18}$$

$$\beta(x'+v) + (c-1)\beta(-x'+v) = c\beta(v) \tag{19}$$

hold, where x_t and x'_t are defined by $(13')$ and $(13'')$, respectively. By (18), (19) and Lemma 1, there exist some $\xi, \eta \in J$ such that $x(\xi) = 0$, $x'(\eta) = 0$; hence (cf. (17))

$$\begin{aligned} x(t) &= c\int_\xi^t\int_\eta^s h(\tau, x(\tau), x'(\tau), x_\tau, x'_\tau)\,d\tau\,ds, \\ x'(t) &= c\int_\eta^t h(s, x(s), x'(s), x_s, x'_s)\,ds \end{aligned}$$

for $t \in J$. Thus (cf. (1) with $f = h$)

$$\begin{aligned} |x'(t)| &\le c\Big(\|k\|_1\|x\|_\infty + \|l\|_1\|x'\|_\infty + \|p\|_1 \max\{\|x_t\|_* : t \in J\} \\ &\quad + \|q\|_1 \max\{\|x'_t\|_* : t \in J\} + \|r\|_1\Big) \\ &\le (\|k\|_1 + \|p\|_1)\|x\|_\infty + (\|l\|_1 + \|q\|_1)\|x'\|_\infty \\ &\quad + 2(\|p\|_1\|\varphi\|_* + \|q\|_1\|\chi\|_*) + \|r\|_1, \quad t \in J \end{aligned}$$

since $\|x_t\|_* \le \|x\|_\infty + 2\|\varphi\|_*$, $\|x'_t\|_* \le \|x'\|_\infty + 2\|\chi\|_*$ for $t \in J$. Consequently,

$$\begin{aligned} \|x'\|_\infty &\le (\|k\|_1 + \|p\|_1)\|x\|_\infty + (\|l\|_1 + \|q\|_1)\|x'\|_\infty \\ &\quad + 2(\|p\|_1\|\varphi\|_* + \|q\|_1\|\chi\|_*) + \|r\|_1. \end{aligned} \tag{20}$$

We next have $|x(t)| = \left|\int_\xi^t x'(s)\,ds\right| \le \|x'\|_\infty$ for $t \in J$ and therefore

$$\|x\|_\infty \le \|x'\|_\infty \tag{21}$$

which implies (cf. (20))

$$\|x'\|_\infty \le \lambda\|x'\|_\infty + 2(\|p\|_1\|\varphi\|_* + \|q\|_1\|\chi\|_*) + \|r\|_1$$

and $\|x'\|_\infty < \Lambda$. Then $\|x\|_\infty < \Lambda$ and since $A = x(0)$, $B = x'(0)$ we obtain $|A| < \Lambda$, $|B| < \Lambda$. Finally,

$$\begin{aligned} \|x''\|_1 &= c\textstyle\int_0^t |h(s, x(s), x'(s), x_s, x'_s)|\,ds \\ &\le \lambda\|x'\|_\infty + 2(\|p\|_1\|\varphi\|_* + \|q\|_1\|\chi\|_*) + \|r\|_1 \\ &\le \lambda\Lambda + (\Lambda - 1)(1 - \lambda) < \Lambda. \end{aligned}$$

Hence $(x, A, B) \in \Omega$.

Let (x, A, B) be a solution of (15_c) for a $c \in [0,1]$. Then the equalities (17), (18) and

$$\beta_1(x(1) - x + v) + (c-1)\beta_1(-x(1) + x + v) = c\beta(v) \tag{22}$$

are satisfied. By (18), (22) and Lemma 1, there exist a $\xi \in J$ and an $\varepsilon \in [0,1)$ such that $x(\xi) = 0$, $x(1) - x(\varepsilon) = 0$. Thus $x'(\eta) = 0$ for an $\eta \in (\varepsilon, 1)$ and, in the same manner as in the first part of our proof, we obtain $(x, A, B) \in \Omega$. □

Lemma 5 *Let h satisfy assumption (H) (with $f = h$). Assume $k \in L_2(J)$, $l \in L_i(J)$, $p \in L_j(J)$ and $q \in L_m(J)$ where i, j, $m \in \{1, 2\}$ and*

$$\lambda^* := \frac{2}{\pi}||k||_2 + ||l||_i + ||p||_j + ||q||_m < 1.$$

Let (x, A, B) be a solution of (14_c) or (15_c) for a $c \in [0,1]$. Then

$$||x||_\infty < \Lambda_1, \quad ||x'||_\infty < \Lambda_1, \quad ||x''||_1 < \Lambda_1, \quad |A| < \Lambda_1, \quad |B| < \Lambda_1, \tag{23}$$

where

$$\Lambda_1 = \frac{1}{1-\lambda^*}\Big[2(||p||_j||\varphi||_* + ||q||_m||\chi||_*) + ||r||_1\Big] + 1.$$

Proof By the proof of Lemma 4, $A = x(0)$, $B = x'(0)$ and there exist some $\xi, \eta \in J$ such that $x(\xi) = 0$, $x'(\eta) = 0$. Hence

$$||x||_\infty \le ||x'||_\infty \le ||x''||_1, \quad ||x'||_2 \le ||x'||_\infty$$

and

$$||x||_2 \le \frac{2}{\pi}||x'||_2$$

by the Wintinger inequality (see e.g. [4], Theorem 256). Using (1) (with $f = h$) we get

$$\begin{aligned}
||x''||_1 &= c\int_0^1 |h(t, x(t), x'(t), x_t, x'_t)|\,dt \\
&\le ||k||_2||x||_2 + ||l||_i||x'||_\infty + ||p||_j(||x||_\infty + 2||\varphi||_*) \\
&\quad + ||q||_m(||x'||_\infty + 2||\chi||_*) + ||r||_1 \\
&\le \Big(\frac{2}{\pi}||k||_2 + ||l||_i + ||p||_j + ||q||_m\Big)||x'||_\infty \\
&\quad + 2(||p||_j||\varphi||_* + ||q||_m||\chi||_*) + ||r||_1 \\
&\le \lambda^*||x''||_1 + 2(||p||_j||\varphi||_* + ||q||_m||\chi||_*) + ||r||_1,
\end{aligned}$$

and consequently $||x''||_1 < \Lambda_1$ which implies that (23) holds. □

3 Existence theorems

Proposition 1 *Let h satisfy assumption (H) (with $f = h$) and $\Omega \subset \mathbf{Y} \times \mathbf{R}^2$ be open bounded and symmetric with respect to $0 \in \Omega$. Then operator equation (14_1) and (15_1) has a solution in Ω provided $H_c(x, A, B) \neq (x, A, B)$ and $V_c(x, A, B) \neq (x, A, B)$ on $\partial\Omega$ for any $c \in [0, 1]$, respectively.*

Proof Assume $(x, A, B) \neq \partial\Omega$ for any solution (x, A, B) of the family od equations (14_c) (resp. (15_c)) with $c \in [0, 1]$. Set $W(c, x, A, B) = H_c(x, A, B)$ (resp. $W(c, x, A, B) = V_c(x, A, B)$) for $(c, x, A, B) \in [0, 1] \times \mathbf{Y} \times \mathbf{R}^2$. Then W is a compact operator on the closure $\bar{\Omega}$ of Ω by the Arzelà–Ascoli theorem, the Bolzano–Weierstrass theorem and the Lebesgue theorem, and $W(c, x, A, B) \neq (x, A, B)$ for any $(x, A, B) \in \partial\Omega$ and each $c \in [0, 1]$ by our assumption. Thus

$$D(I - W(1, \cdot, \cdot, \cdot), \Omega, 0) = D(I - W(0, \cdot, \cdot, \cdot), \Omega, 0)$$

where "D" denotes the Leray–Schauder degree (see e.g. [1]). To prove the existence of a solution for equation $W(1, x, A, B) = (x, A, B)$ (that is (14_1) resp. (15_1)) we have to show that

$$D(I - W(0, \cdot, \cdot, \cdot), \Omega, 0) \neq 0.$$

Since

$$\begin{gathered} H_0(-x, -A, -B) = \\ = \Big(-A - Bt, -A + \alpha(-x + u) - \alpha(x + u), -B + \beta(-x' + v) - \beta(x' + v)\Big) \\ = -H_0(x, A, B) \end{gathered}$$

and

$$\begin{gathered} V_0(-x, -A, -B) = \\ = \Big(-A-Bt, -A+\alpha(-x+u)-\alpha(x+u), -B+\beta_1(-x(1)+x+v)-\beta_1(x(1)-x+v)\Big) \\ = -V_0(x, A, B) \end{gathered}$$

for $(x, A, B) \in \mathbf{Y} \times \mathbf{R}^2$, $W(0, \cdot, \cdot, \cdot)$ is an odd operator and then

$$D(I - W(0, \cdot, \cdot, \cdot), \Omega, 0) \neq 0$$

by the Borsuk Theorem (see [1], Theorem 8.3). □

Theorem 1 *Let h satisfy assumption (H) (with $f = h$). Then BVP* (10), (i) *($i = 11, 12$) has at least one solution for each u, $v \in \mathbf{X}$, $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$ and $\varphi, \chi \in C_r$ provided*

$$||k||_1 + ||l||_1 + ||p||_1 + ||q||_1 < 1. \tag{24}$$

Proof Let $u, v \in \mathbf{X}$, $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$ and $\varphi, \chi \in C_r$ and let (24) be satisfied. By Remark 2, it is sufficient to show that operator equations (14_1) and (15_1) have solutions. By Lemma 4, there exists an open bounded subset Ω of $\mathbf{Y} \times \mathbf{R}^2$ which is symmetric with respect to $0 \in \Omega$ such that $(x, A, B) \notin \partial\Omega$ for any solution (x, A, B) of the family of equations (14_c) and (15_c) with $c \in [0, 1]$. The conclusion of Theorem 1 follows immediately from Proposition 1. □

Using Proposition 1 and Lemma 5 we can prove the following theorem.

Theorem 2 *Let h satisfy assumption (H) (with $f = h$). Assume $k \in L_2(J)$, $l \in L_i(J)$, $p \in L_j(J)$ and $q \in L_m(J)$ where $i, j, m \in \{1, 2\}$. Then BVP* (10), (i) $(i = 11, 12)$ *has at least one solution for each $u, v \in \mathbf{X}$, $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$ and $\varphi, \chi \in C_r$ provided*

$$\frac{2}{\pi}\|k\|_2 + \|l\|_i + \|p\|_j + \|q\|_m < 1. \tag{25}$$

The main existence results for BVP (2), (i) $(i = 3, 4)$ are given in the following two theorems.

Theorem 3 *Let f satisfy assumption (H). Assume that* (24) *is satisfied. Then BVP* (2), (i) $(i = 3, 4)$ *has at least one solution for each $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$, $\varphi, \chi \in C_r$ and $A, B \in \mathbf{R}$.*

Proof Fix $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$, $\varphi, \chi \in C_r$ and $A, B \in \mathbf{R}$. By Lemma 2 (resp. Lemma 3) there exist (unique) $a_0, b_0 \in \mathbf{R}$ such that $\alpha(a_0 + b_0 t) = A$, $\beta(b_0) = B$ (resp. $\alpha(a_0 + b_0 t) = A$, $\beta(b_0(1-t)) = B$). Set

$$h(t, x, y, \varrho, \psi) = f(t, x + a_0 + b_0 t, y + b_0, \varrho + w_t, \psi + b_0)$$

for $(t, x, y, \varrho, \psi) \in J \times \mathbf{R}^2 \times C_r \times C_r$ where

$$w_t(s) = \begin{cases} a_0 & \text{for } t + s \in [-r, 0] \\ a_0 + b_0(t + s) & \text{for } t + s \in (0, 1]. \end{cases}$$

We see that x is a solution of BVP (10), (11) with $u = a_0 + b_0 t$ and $v = b_0$ if and only if $x + a_0 + b_0 t$ is a solution of BVP (2), (3), and x is a solution of BVP (10), (12) with $u = a_0 + b_0 t$ and $v = b_0(1-t)$ if and only if $x + a_0 + b_0 t$ is a solution of BVP (2), (4). Since (cf. (1))

$$\begin{aligned} |h(t, x, y, \varrho, \psi)| &= k(t)|x + a_0 + b_0 t| + l(t)|y + b_0| + p(t)\|\varrho + w_t\|_* + q(t)\|\psi + b_0\|_* + r(t) \\ &\le k(t)|x| + l(t)|y| + p(t)\|\varrho\|_* + q(t)\|\psi\|_* + r_1(t) \end{aligned}$$

for $(t, x, y, \varrho, \psi) \in J \times \mathbf{R}^2 \times C_r \times C_r$, where

$$r_1(t) = (k(t) + p(t))(|a_0| + |b_0|) + (l(t) + q(t))|b_0| + r(t),$$

there exists a solution of BVP (10), (i) $(i = 11,\ 12)$ by Theorem 1. This completes the proof. □

Theorem 4 *Let f satisfy assumption (H). Assume $k \in L_2(J)$, $l \in L_i(J)$, $p \in L_j(J)$ and $q \in L_m(J)$ where $i, j, m \in \{1, 2\}$ and (25) is satisfied. Then BVP (2), (i) ($i = 3, 4$) has at least one solution for each $\alpha, \beta \in \mathcal{D}_J$, $\beta_1 \in \mathcal{D}^*_{[0,1)}$, $\varphi, \chi \in C_r$ and $A, B \in \mathbf{R}$.*

Proof We proceed exactly as in the proof of Theorem 3 but instead of Theorem 1 we now use Theorem 2. □

Remark 3 Note that analogously existence results as above can be shown for the functional differential equation of the form

$$x''(t) = f(t, x(t), x'(t), x(a(t)), x'(b(t)), x_t, x'_t)$$

with $a : J \to J$, $b : J \to J$ continuous.

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