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On Automorphisms of the Lattice of Quasivarieties of Lattice-ordered Groups *

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Abstract

In this work the existence of an automorphism of the lattice (semigroup) of quasivarieties of lattice-ordered groups $\mathbf{\Lambda}$ is established.

Key words: Quasivariety, lattice-ordered group, quasiidentity.

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In the paper of Huss M. E. and Reilly N. R. [1] a non-trivial automorphism θ of the lattice (semigroup) of varieties of lattice-ordered groups (l-groups) $\mathbf{L}$ of order 2 was discovered. Also it is known ([2], [3]) that the lattice (semigroup) $\mathbf{L}$ is a sublattice (subsemigroup) of the lattice (semigroup) of quasivarieties of l-groups $\mathbf{\Lambda}$.

The purpose of this work is to prove that this automorphism θ of the lattice (semigroup) $\mathbf{L}$ can be extended to an automorphism of the lattice (semigroup) of quasivarieties of l-groups $\mathbf{\Lambda}$.

For the background necessary for this paper, the reader is referred to [4], [5].

For any l-group $G = (G, \leq)$, let $G^R = (G^R, \leq^R)$ denote the l-group obtained from G by reversing the order; thus $a \leq^R b$ in G^R if and only if $b \leq a$ in G.

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As usual, $\bar{\prod\limits_{i\in I}} G_i$ denotes the Cartesian product of l-groups $\{G_i \mid i \in I\}$. If $\mathcal{F}$ is an ultrafilter over I then by $\bar{\prod\limits_{i\in I}} G_i/\mathcal{F}$ we denote the ultraproduct of l-groups $\{G_i \mid i \in I\}$ by the ultrafilter $\mathcal{F}$.

Lemma 1

$$(1) \quad \left(\bar{\prod_{i\in I}} G_i\right)^R = \bar{\prod_{i\in I}} G_i^R, \qquad (2) \quad \left(\bar{\prod_{i\in I}} G_i/\mathcal{F}\right)^R = \bar{\prod_{i\in I}} G_i^R/\mathcal{F}.$$

Proof is straightforward.

Now for any l-group word

$$w(x_1, \dots, x_n) = \bigvee_{i\in I} \bigwedge_{j\in J} \prod_{k\in K} x_{ijk}^{\varepsilon(ijk)},$$

where index sets I, J, K are finite and $\varepsilon(ijk) = \pm 1$ for all $i \in I$, $j \in J$, $k \in K$ let

$$w^R(x_1, \dots, x_n) = \bigvee_{i\in I} \bigwedge_{j\in J} \left(\prod_{k\in K} x_{ijk}^{\varepsilon(ijk)}\right)^{-1} = \bigvee_{i\in I} \bigwedge_{j\in J} {\prod_{k\in K}}' x_{ijk}^{-\varepsilon(ijk)},$$

where $\prod'_{k\in K} y_k$ denotes the product taken in the reverse order.

For any quasiidentity

$$\begin{aligned} &\varphi = \varphi(x_1, \dots, x_n) \\ &= (\forall x_1, \dots, x_n)\ (w_1(x_1, \dots, x_n) = e \& \dots \& w_m(x_1, \dots, x_n) = e \\ &\Rightarrow w_0(x_1, \dots, x_n) = e) \end{aligned}$$

let

$$\begin{aligned} &\varphi^R = \varphi^R(x_1, \dots, x_n) \\ &= (\forall x_1, \dots, x_n)\ (w_1^R(x_1, \dots, x_n) = e \& \dots \& w_m^R(x_1, \dots, x_n) = e \\ &\Rightarrow w_0^R(x_1, \dots, x_n) = e) \end{aligned}$$

As in [1] we denote the lattice operations in G^R by $\vee^R, \wedge^R$ and write for $x \in G$, as usual

$$x^+ = x \vee e, \quad x^{+R} = x \vee^R e, \quad x^- = (x \wedge e)^{-1}, \quad x^{-R} = (x \wedge^R e)^{-1}.$$

It is clear that for all $x, y \in G$ is valid $x \vee^R y = x \wedge y$ and $x \wedge^R y = x \vee y$.

Lemma 2 *The equality*

$$w(g_1,\dots,g_n) = \bigvee_{i\in I}\bigwedge_{j\in J}\prod_{k\in K} g_{ijk}^{\varepsilon(ijk)} = e$$

is valid in the group G if and only if the equality

$$w^R(g_1,\dots,g_n) = \bigvee_{i\in I}^{R}\bigwedge_{j\in J}^{R}\prod_{k\in K}{}' g_{ijk}^{-\varepsilon(ijk)} = e$$

is valid in G^R.

Proof It is clear that

$$w(g_1,\dots,g_n) = \bigvee_{i\in I}\bigwedge_{j\in J}\prod_{k\in K} g_{ijk}^{\varepsilon(ijk)} = e \Longleftrightarrow$$

$$w(g_1,\dots,g_n)^{-1} = \left(\bigvee_{i\in I}\bigwedge_{j\in J}\prod_{k\in K} g_{ijk}^{\varepsilon(ijk)}\right)^{-1} = e =$$

$$= \bigwedge_{i\in I}\bigvee_{j\in J}\prod_{k\in K}{}' g_{ijk}^{-\varepsilon(ijk)} = \bigvee_{i\in I}^{R}\bigwedge_{j\in J}^{R}\prod_{k\in K}{}' g_{ijk}^{-\varepsilon(ijk)} = w^R(g_1,\dots,g_n) = e.$$

Lemma 3 *For any l-group G and any quasiidentity $\varphi = \varphi(x_1,\dots,x_n)$ the following statements are equivalent.*

(1) *The quasiidentity φ holds in G.*

(2) *The quasiidentity φ^R holds in G^R.*

Proof Let us assume that φ^R holds in G^R and φ is violated in G. Then there are elements $g_1,\dots,g_n \in G$ such that $w_1(g_1,\dots,g_n) = e,\dots,w_m(g_1,\dots,g_n) = e$ and $w_0(g_1,\dots,g_n) \neq e$ in G. Then by Lemma 2 in l-group G^R is valid $w_1^R(g_1,\dots,g_n) = e,\dots,w_m^R(g_1,\dots,g_n) = e$ and $w_0^R(g_1,\dots,g_n) \neq e$. A contradiction with our assumption. The converse statement is proved by similar arguments.

Now for any quasivariety of l-groups $\mathcal{K}$ we will write $\mathcal{K}^R = \{G^R \mid G \in \mathcal{K}\}$.

Corollary 1 *For any quasivariety of l-groups $\mathcal{K}$, $\mathcal{K}^R$ is a quasivariety. Moreover, the following are equivalent.*

(1) *$\mathcal{K}$ has a basis of quasiidentities $\{\varphi_t \mid t \in A\}$.*

(2) *$\mathcal{K}^R$ has a basis of quasiidentities $\{\varphi_t^R \mid t \in A\}$.*

In [1] it is shown that there exist varieties of l-groups $\mathcal{V}$ such that $\mathcal{V} \neq \mathcal{V}^{\mathcal{R}}$. Thus, the mapping $\theta : \mathbf{\Lambda} \longrightarrow \mathbf{\Lambda}$ defined by the rule $\mathcal{K}\theta = \mathcal{K}^R$ is not identical.

Theorem 1 *The mapping θ is a lattice automorphism with the following properties:*

(1) *θ^2 is the identity mapping;*

(2) *θ preserves arbitrary joins and meets.*

Proof For any l-group word w it is clear that $(w^R)^R = w$, so by Corollary 1 of Lemma 3 we have $\mathcal{K}\theta^2 = \mathcal{K}$ for any $\mathcal{K} \in \mathbf{\Lambda}$. Therefore, the property (1) holds and hence θ is a one-to-one mapping.

Clearly, for any l-group G it is true $G \in \mathcal{K}\theta \iff G^R \in \mathcal{K}\theta^2 = \mathcal{K}$.

Hence, for any family $\{\mathcal{K}_\alpha \mid \alpha \in A\} \subseteq \mathbf{\Lambda}$ the following relations hold:

$$G \in \Big(\bigwedge_{\alpha\in A} \mathcal{K}_\alpha\Big)\theta \iff G^R \in \bigwedge_{\alpha\in A} \mathcal{K}_\alpha \iff G^R \in \mathcal{K}_\alpha \text{ for all } \alpha \in A$$

$$\iff G \in \mathcal{K}_\alpha^R \text{ for all } \alpha \in A \iff G \in \bigwedge_{\alpha\in A} \mathcal{K}_\alpha^R = \bigwedge_{\alpha\in A} \mathcal{K}_\alpha\theta.$$

Hence, θ preserves arbitrary meets.

Now suppose that $G \in (\bigvee_{\alpha\in A} \mathcal{K}_\alpha)\theta$. Then $G^R \in \bigvee_{\alpha\in A} \mathcal{K}_\alpha$ and by Theorem 2 of Chapter 14 from the book [4] $G^R \leq \bar{\prod}_{\beta\in B} V_\beta$ where V_β is an ultraproduct of l-groups $\{X_i \mid i \in I(\beta)\}$ from quasivarieties $\mathcal{K}_\alpha$ $(\alpha \in A)$. Then by Lemma 1

$$G \leq \bar{\prod_{\beta\in B}} V_\beta^R \quad \text{and} \quad V_\beta^R = \bar{\prod_{i\in I(\beta)}} X_i^R/\mathcal{F}_\beta \in \bigvee_{\alpha\in A} \mathcal{K}_\alpha\theta\,.$$

The converse statement is similar. Thus, θ is an automorphism of the lattice Λ.

Now let us consider in $\mathbf{\Lambda}$ the subset $\mathbf{\Upsilon} = \{\mathcal{K} \in \mathbf{\Lambda} \mid \mathcal{K}^R = \mathcal{K}\}$. It is obvious that if any quasivariety $\mathcal{Q}$ is defined by quasiidentities of the signature of the group theory then $\mathcal{Q} \in \mathbf{\Upsilon}$.

The proof of the following statement follows immediately from Theorem 1.

Corollary 2

(1) *$\mathbf{\Upsilon}$ is a complete sublattice of Λ.*

(2) *For any quasivariety $\mathcal{K} \in \Lambda$, $\mathcal{K} \vee \mathcal{K}^R \in \mathbf{\Upsilon}$.*

As usual (cf. [3]), for $\mathcal{K}$, $\mathcal{P} \in \mathbf{\Lambda}$ let $\mathcal{K} \cdot \mathcal{P}$ be the class of all l-groups G for which there exists an l-ideal H such that $H \in \mathcal{K}$ and $G/H \in \mathcal{P}$. It is known (cf. [3], [5]) that $\mathcal{K} \cdot \mathcal{P}$ is a quasivariety. This quasivariety is called a product of quasivarieties $\mathcal{K}$ and $\mathcal{P}$. In [3] it is shown that $\mathbf{\Lambda}$ is a semigroup with respect to the above-defined product of quasivarieties.

Theorem 2 *The mapping θ is an automorphism of the semigroup $\mathbf{\Lambda}$.*

Proof Since θ is one-to-one, it suffices to show that θ is a semigroup homomorphism. Let $\mathcal{K}, \mathcal{P} \in \mathbf{\Lambda}$. Then $G \in (\mathcal{K} \cdot \mathcal{P})\theta \iff G^R \in \mathcal{K} \cdot \mathcal{P} \iff$ there exists an l-ideal H of G^R with $H \in \mathcal{K}$ such that $G^R/H \in \mathcal{P} \iff$ there exists an l-ideal K of G $(K = H^R)$ with $K \in \mathcal{K}\theta$ such that G/K $(\cong (G^R/H)^R$ by Lemma 2.7 from [1]) $\in \mathcal{P}\theta \iff G \in (\mathcal{K})\theta \cdot (\mathcal{P})\theta$.

References

[1] Huss, M. E., Reilly, N. R.: *On reversing the order of a lattice-ordered group.* J. Algebra **91** (1984), 176–191.

[2] Arora, A. K.: *Quasi-varieties of lattice-ordered groups.* Algebra Univ. **20** (1985), 34–50.

[3] Mal'cev, A. I.: *Multiplication of classes of algebraic systems.* Sibirsk. Mat. Ž., 8 (1967), 346–365, (Russian).

[4] Kopytov, V. M., Medvedev, N. Ya.: *The theory of lattice-ordered groups.* Kluwer Academic Publ., Dordrecht, 1994.

[5] Mal'cev, A. I.: *Algebraic systems.* Springer, Berlin, 1973.