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An Application of the Leray–Schauder Degree Theory to Boundary Value Problem for Third and Fourth Order Differential Equations Depending on the Parameter

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Abstract

The Leray–Schauder degree theory is used to obtain sufficient conditions for the existence and uniqueness of solutions of the boundary value problem $x''' = f(t, x, x', x'', \lambda)$, $x(0) = x(1) = 0$, $x'(0) - x'(1) = 0$, $x''(0) - x''(1) = 0$, depending on the parameter λ. The application is given to boundary value problems for one-parameter fourth order functional differential equations.

Key words: Leray–Schauder degree theory, Schauder fixed point theorem, quasi-linearization technique, boundary value problems depending on the parameter.

MS Classification: 34B15, 34K10

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1 Introduction

Consider the one-parameter boundary value problem (BVP for short)

$$x''' = f(t, x, x', x'', \lambda), \tag{1}$$

$$x(0) = x(1) = 0, \quad x'(0) - x'(1) = 0, \quad x''(0) - x''(1) = 0, \tag{2}$$

where $f \in C^0(\langle 0,1\rangle \times \mathbf{R}^4)$.

We say that (x, λ_0) is a *solution* of BVP (1), (2) if $(x, \lambda_0) \in C^3(\langle 0,1\rangle) \times \mathbf{R}$ and x is a solution of (1) for $\lambda = \lambda_0$ satisfying (2).

This paper establishes sufficient conditions for the existence and uniqueness of solutions of BVP (1), (2). The proof of the existence theorem is based on the theory of completely continuous mapping and on the invariance of the Leray–Schauder degree with respect to a homotopy. More precisely, we apply the following theorem.

Theorem 1 [1, Theorem 1] *Let $\mathbf{X}$ be a Banach space, $A : \mathbf{X} \to \mathbf{X}$ be a completely continuous mapping such that $I - A$ is one to one, and let Ω be an open bounded set such that $0 \in (I - A)(\Omega)$. Then the completely continuous mapping $T : \bar{\Omega} \to \mathbf{X}$ has a fixed point in $\bar{\Omega}$ if for any $c \in (0,1)$, the equation*

$$x = cTx + (1-c)Ax$$

has no solution x on the boundary $\partial\Omega$ of Ω.

The application of the obtained results for the existence and uniqueness of solutions for BVP (1), (2) leads in Section 4 to the investigation of functional BVPs for fourth order one-parameter functional differential equations using the quasi-linearization technique and the Schauder fixed point theorem.

We observe that BVPs for third order differential and functional differential equations depending on the parameter were studied by Pachpatte [3] using the technique of Green's functions and the Banach fixed point theorem and by the author [4]–[6]. In [4] using the Schauder linearization technique and the Schauder fixed point theorem, in [5] using a method based on a combination of the quasi-linearization technique, the Schauder fixed point theorem and a surjectivity result in $\mathbf{R}^n$ and, finally, in [6] using a combination of the Leray–Schauder degree theory, the quasi-linearization technique and the Schauder fixed point theorem.

2 Existence theorem

Let $\mathbf{X}$ be the Banach space of C^0-functions x on $\langle 0,1\rangle$ with the norm

$$\|x\| = \max\{|x(t)|;\ 0 \le t \le 1\}.$$

The proof of the existence theorem for BVP (1), (2) is based on the following lemma.

Lemma 1 *Let $h \in C^0(\langle 0,1\rangle \times \mathbf{R}^4)$. Assume there exist constants $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $T > 0$ and a nondecreasing function $w : \langle 0,\infty) \to (0,\infty)$ such that*

(3′) $h(t,p-y,y,0,\lambda_2) > 0$ *for* $(t,p,y) \in \langle 0,1\rangle \times (0,M\rangle \times \langle 0,M\rangle$,

(3″) $h(t,p-y,y,0,\lambda_1) < 0$ *for* $(t,p,y) \in \langle 0,1\rangle \times \langle -M,0) \times \langle -M,0\rangle$,

(4) $h(t,x,-M,0,\lambda) < 0 < h(t,x,M,0,\lambda)$ *for* $(t,x,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle \lambda_1,\lambda_2\rangle$,

(5) $|h(t,x,y,z,\lambda)| \le w(|z|)$ *for* $(t,x,y,z,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle -M,M\rangle \times \mathbf{R} \times \langle \lambda_1,\lambda_2\rangle$ *and*

$$\int_0^T \frac{s\,ds}{w(s)} > 2M.$$

Let (x,λ_0) be a solution of BVP

$$x''' = h(t,x,x',x'',\lambda), \qquad (2), \tag{6}$$

such that $\lambda_1 \le \lambda_0 \le \lambda_2$, $\|x\| \le M$, $\|x'\| \le M$, $\|x''\| \le T$. Then

$$\lambda_1 < \lambda_0 < \lambda_2, \quad \|x\| < M, \quad \|x'\| < M, \quad \|x''\| < T.$$

Proof Assume $\lambda_0 = \lambda_2$. Since $x(0)-x(1) = 0$ and $x'(0)-x'(1) = 0$, we have $0 \le \max\{x'(t);\ 0 \le t \le 1\} = x'(\xi)$ for a $\xi \in \langle 0,1)$.

a) If $\xi = 0$, then $x'(t) \le x'(0)\ (= x'(1))$ on $\langle 0,1\rangle$ and therefore $x''(0) \le 0$, $x''(1) \ge 0$ which implies $x''(0) = 0\ (= x''(1))$ since $x''(0) = x''(1)$ by (2). Thus $x(0) = 0$, $x'(0) \ge 0$, $x''(0) = 0$, $x'''(0) \le 0$ which contradicts $x'''(0) = h(0,0,x'(0),0,\lambda_2) > 0$ by (3′) with $y = p = x'(0)$.

b) If $\xi \in (0,1)$, then $x'(\xi) \ge 0$, $x''(\xi) = 0$, $x'''(\xi) \le 0$, and consequently $(x'''(\xi) =)\ h(\xi,x(\xi),x'(\xi),0,\lambda_2) \le 0$. By (3′), $h(\xi,p-x'(\xi),x'(\xi),0,\lambda_2) > 0$ for all $p \in (0,M\rangle$ and therefore $x(\xi) = p_0 - x'(\xi)$ for a $p_0 < 0$. Thus $x(\xi) < 0$ and $x(\xi) < -x'(\xi) \le -x'(t)$ for $t \in \langle 0,1\rangle$. Integrating the inequality $x(\xi) < -x'(t)$, $t \in \langle 0,1\rangle$, from ξ to 1 we obtain

$$x(\xi)(1-\xi) < -x(1) + x(\xi) = x(\xi)$$

and then $\xi < 0$, a contradiction.

Assume $\lambda_0 = \lambda_1$. Then $(0 \ge) \min\{x'(t);\ 0 \le t \le 1\} = x'(\tau)$ for a $\tau \in \langle 0,1)$.

a) If $\tau = 0$, then $x'(t) \ge x'(0)\ (= x'(1))$ and therefore $x''(0) \ge 0$, $x''(1) \le 0$ which implies $x''(0) = x''(1) = 0$ since $x''(0) = x''(1)$ by (2). Hence $x(0) = 0$, $x'(0) \le 0$, $x''(0) = 0$, $x'''(0) \ge 0$ which contradicts $x'''(0) = h(0,0,x'(0),0,\lambda_1) < 0$ by (3″) with $y = p = x'(0)$.

b) If $\tau \in (0,1)$, then $x'(\tau) \le 0$, $x''(\tau) = 0$, $x'''(\tau) \ge 0$, and consequently $(x'''(\tau) =)\ h(\tau,x(\tau),x'(\tau),0,\lambda_1) \ge 0$. On the other hand

$$h(\tau,p-x'(\tau),x'(\tau),0,\lambda_1) < 0 \qquad \text{for all } p \in \langle -M,0)$$

by $(3'')$ and therefore $x(\tau) = p_0 - x'(\tau)$ for a $p_0 > 0$. Thus $x(\tau) > 0$ and $x(\tau) > -x'(\tau) \geq -x'(t)$ for $t \in \langle 0,1\rangle$. Integrating the inequality $x(\tau) > -x'(t)$, $t \in \langle 0,1\rangle$, from τ to 1 we obtain $x(\tau)(1-\tau) > -x(1) + x(\tau) = x(\tau)$, and then $\tau < 0$, a contradiction.

This proves $\lambda_0 \in (\lambda_1, \lambda_2)$. Assume $\|x'\| = M$, say for example $x'(\varepsilon) = M$ with an $\varepsilon \in \langle 0,1\rangle$ (the case where $x(\varepsilon) = -M$ treats similarly using the second inequality in (4)). Then $x''(\varepsilon) = 0$ (cf. the first part of the proof) and $x'''(\varepsilon) \leq 0$ which contradicts $x'''(\varepsilon) = h(\varepsilon, x(\varepsilon), M, 0, \lambda_0) > 0$ by (4). Thus $\|x'\| < M$ and then

$$|x(t)| = \left|\int_0^t x'(s)\,ds\right| < M \qquad \text{for } t \in \langle 0,1\rangle.$$

Finally, by (5), we have

$$|x'''(t)| = |h(t, x(t), x'(t), x''(t), \lambda_0)| \leq w(|x''(t)|) \qquad \text{for } t \in \langle 0,1\rangle$$

and since $x''(\nu) = 0$ for a $\nu \in (0,1)$, we can prove $\|x''\| < T$ using the standard procedure (see e.g. [2]) and assumption (5). □

We shall assume that for constants $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $T > 0$ and a nondecreasing function $w : \langle 0,\infty) \to (0,\infty)$ the function f satisfies the following assumptions:

(H_1) $f(t, p-y, y, 0, \lambda_2) \geq 0$ for $(t,p,y) \in \langle 0,1\rangle \times \langle 0,M\rangle \times \langle 0,M\rangle$,
$f(t, p-y, y, 0, \lambda_1) \leq 0$ for $(t,p,y) \in \langle 0,1\rangle \times \langle -M,0\rangle \times \langle -M,0\rangle$,

(H_2) $f(t,x,-M,0,\lambda) \leq 0 \leq f(t,x,M,0,\lambda)$ for $(t,x,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle \lambda_1, \lambda_2\rangle$,

(H_3) $|f(t,x,y,z,\lambda)| \leq w(|z|)$ for $(t,x,y,z,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle -M,M\rangle \times \mathbf{R} \times \langle \lambda_1, \lambda_2\rangle$ and

$$\int_0^T \frac{s\,ds}{w(s)} > 2M.$$

Theorem 2 *Assume assumptions (H_1)–(H_3) are satisfied with constants $\lambda_1 < 0 < \lambda_2$, $M > 0$, $T > 0$ and a nondecreasing function $w : \langle 0,\infty) \to (0,\infty)$. Then BVP (1), (2) has at least one solution.*

Proof We denote by $\mathbf{Y}$ the Banach space of C^2-functions on $\langle 0,1\rangle$ with the norm $\|x\|_2 = \|x\| + \|x'\| + \|x''\|$, $\mathbf{Z}$ the Banach space of C^3-functions with the norm $\|x\|_3 = \|x\|_2 + \|x'''\|$, $\mathbf{X} \times \mathbf{R} = \{(x,\lambda);\ x \in \mathbf{X},\ \lambda \in \mathbf{R}\}$ the Banach space with the norm $\|(x,\lambda)\| = \|x\| + |\lambda|$, $\mathbf{Y}_0 = \{x;\ x \in \mathbf{Y},\ x \text{ satisfies } (2)\}$, $\mathbf{Z}_0 = \mathbf{Z} \cap \mathbf{Y}_0$, $\mathbf{Y}_0 \times \mathbf{R} = \{(x,\lambda);\ x \in \mathbf{Y}_0,\ \lambda \in \mathbf{R}\}$ the Banach space with the norm $\|(x,\lambda)\|_2 = \|x\|_2 + |\lambda|$, $\mathbf{Z}_0 \times \mathbf{R} = \{(x,\lambda);\ x \in \mathbf{Z}_0,\ \lambda \in \mathbf{R}\}$ the Banach space with the norm $\|(x,\lambda)\|_3 = \|x\|_3 + |\lambda|$ and $\mathbf{S} = \{(x, \int_0^1 x(s)ds);\ x \in \mathbf{X}\} \subset \mathbf{X} \times \mathbf{R}$. Clearly, $\mathbf{S}$ is a Banach space. Set

$$\varepsilon = \min\left\{\left(\frac{2w(0)}{3M}\right)^{\frac{1}{2}}, \pi\right\}, \qquad k = \frac{\varepsilon^2 M}{2\max\{-\lambda_1, \lambda_2\}}.$$

Define the operators $L, F, K : \mathbf{Z}_0 \times \mathbf{R} \to \mathbf{S}$ by

$$\begin{aligned}
(L(x,\lambda))(t) &= (x'''(t) + \varepsilon^2 x'(t) + k\lambda, k\lambda), \\
(F(x,\lambda))(t) &= \left(f(t,x(t),x'(t),x''(t),\lambda), \int_0^1 f(s,x(s),x'(s),x''(s),\lambda)\,ds\right), \\
(K(x,\lambda))(t) &= (\varepsilon^2 x'(t) + k\lambda, k\lambda).
\end{aligned}$$

Consider the operator equation

$$(6_c) \qquad L(x,\lambda) = c(F(x,\lambda) + K(x,\lambda)) + 2(1-c)K(x,\lambda), \qquad c \in \langle 0,1\rangle.$$

We see that BVP (1), (2) has a solution (x,λ_0) if and only if (x,λ_0) is a solution of (6_1). The existence of a solution of (6_1) will be proved using Theorem 1.

We will show that $L : \mathbf{Z}_0 \times \mathbf{R} \to \mathbf{S}$ is one to one and onto. Let $(u, \int_0^1 u(s)ds) \in \mathbf{S}$ and consider the equation

$$L(x,\lambda) = \left(u, \int_0^1 u(s)\,ds\right),$$

that is, the equations

$$(7) \qquad \begin{aligned} x''' + \varepsilon^2 x' + k\lambda &= u(t), \\ k\lambda &= \int_0^1 u(s)\,ds, \end{aligned}$$

where $x \in \mathbf{Z}_0$ and $\lambda \in \mathbf{R}$. The function

$$x(t) = \frac{c_1}{\varepsilon}\sin(\varepsilon t) - \frac{c_2}{\varepsilon}\cos(\varepsilon t) - \frac{k\lambda t}{\varepsilon^2} + \frac{1}{\varepsilon}\int_0^t\int_0^s u(\tau)\sin(\varepsilon(s-\tau))\,d\tau ds + c_3$$

is the general solution of (7), where c_1, c_2, c_3 are integration constants. We can easily check that there exists a unique solution $x_0(t)$ of (7) with $\lambda = \frac{1}{k}\int_0^1 u(s)ds$ satisfying the boundary conditions $x_0(0) = 0$, $x_0'(0) - x_0'(1) = 0$ and $x_0''(0) - x_0''(1) = 0$. Integrating the both sides of the equality

$$x_0'''(t) + \varepsilon^2 x_0'(t) + k\lambda = u(t), \qquad t \in \langle 0,1\rangle$$

from 0 to 1 we obtain $x_0(1) = 0$ since $x_0''(1) - x_0''(0) = 0$, $x_0(0) = 0$ and $k\lambda = \int_0^1 u(s)ds$. Hence $L^{-1} : \mathbf{S} \to \mathbf{Z}_0 \times \mathbf{R}$ exists, L^{-1} is a linear bounded operator by the Banach theorem and (6_c) can be written in the equivalent form

$$(8_c) \quad (x,\lambda) = c(L^{-1}Fj(x,\lambda) + L^{-1}Kj(x,\lambda)) + 2(1-c)L^{-1}Kj(x,\lambda), \quad c \in \langle 0,1\rangle,$$

where $j : \mathbf{Z}_0 \times \mathbf{R} \to \mathbf{Y}_0 \times \mathbf{R}$ is the natural embedding, which is completely continuous by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem. Define

$$\Omega = \{(x,\lambda);\ x \in \mathbf{Z}_0,\ \lambda \in \mathbf{R},\ \|x\| < M,\ \|x'\| < M,\ \|x''\| < T,\ \|x'''\| < w(T) + 1,\ \lambda_1 < \lambda < \lambda_2\}.$$

Then Ω is a bounded open subset of $\mathbf{Z}_0 \times \mathbf{R}$. $L^{-1}Fj + L^{-1}Kj$ is a compact operator on $\bar{\Omega}$ and $2L^{-1}Kj$ is a completely continuous operator on $\mathbf{Z}_0 \times \mathbf{R}$. In order to prove that (8_1) has a solution, that is, $L^{-1}Fj + L^{-1}Kj$ has a fixed point, we have to show (cf. Theorem 1) that
(a) $(x,\lambda) - 2L^{-1}Kj(x,\lambda) = (0,0)$ implies $(x,\lambda) = (0,0)$, and
(b) for any $c \in (0,1)$ equation (8_c) has no solution on the boundary $\partial\Omega$ of Ω.

To prove (a) consider the equation

$$(x,\lambda) - 2L^{-1}Kj(x,\lambda) = (0,0),$$

which is equivalent to the equation

$$L(x,\lambda) = 2K(x,\lambda). \tag{9}$$

A couple $(x,\lambda) \in \mathbf{Z}_0 \times \mathbf{R}$ is a solution of (9) if and only if $\lambda = 0$ and x is a solution of the equation

$$y''' = \varepsilon^2 y' \tag{10}$$

satisfying

$$y(0) = 0, \quad y'(0) - y'(1) = 0, \quad y''(0) - y''(1) = 0. \tag{11}$$

Since

$$y(t) = \frac{c_1}{\varepsilon}e^{\varepsilon t} + \frac{c_2}{\varepsilon}e^{-\varepsilon t} + c_3$$

is the general solution of (10), where c_1, c_2, c_3 are integration constants, we can easily verily that $y = 0$ is the unique solution of BVP (10), (11); hence $x = 0$.

We shall now prove (b). Consider the differential equation

$$x''' = cf(t,x,x',x'',\lambda) + (1-c)(\varepsilon^2 x' + k\lambda), \qquad c \in (0,1). \tag{12_c}$$

Assume (x_c, λ_c) is a solution of BVP (12_c), (2). We show that $(x_c,\lambda_c) \notin \partial\Omega$, that is, for any $c \in (0,1)$ equation (8_c) has no solution on $\partial\Omega$. Set $p_c(t,x,y,z,\lambda) = cf(t,x,y,z,\lambda) + (1-c)(\varepsilon^2 y + k\lambda)$ for $(t,x,y,z,\lambda) \in \langle 0,1\rangle \times \mathbf{R}^4$ and $c \in (0,1)$. Then ($c \in (0,1)$)

$$p_c(t,p-y,y,0,\lambda_2) = cf(t,p-y,y,0,\lambda_2) + (1-c)(\varepsilon^2 y + k\lambda_2) > 0$$

for $(t,p,y) \in \langle 0,1\rangle \times \langle 0,M\rangle \times \langle 0,M\rangle$,

$$p_c(t,p-y,y,0,\lambda_1) = cf(t,p-y,y,0,\lambda_1) + (1-c)(\varepsilon^2 y + k\lambda_1) < 0$$

for $(t,p,y) \in \langle 0,1\rangle \times \langle -M,0\rangle \times \langle -M,0\rangle$,

$$p_c(t,x,-M,0,\lambda) \le cf(t,x,-M,0,\lambda) + (1-c)(-\varepsilon^2 M + k\lambda_2) \le$$
$$\le (1-c)\Big(-\varepsilon^2 M + \frac{\varepsilon^2 M\lambda_2}{2\max\{-\lambda_1,\lambda_2\}}\Big) < 0$$

for $(t,x,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle \lambda_1,\lambda_2\rangle$,

$$p_c(t,x,M,0,\lambda) \geq cf(t,x,M,0,\lambda) + (1-c)(\varepsilon^2 M + k\lambda_1) \geq$$
$$\geq (1-c)\left(\varepsilon^2 M + \frac{\varepsilon^2 M \lambda_1}{2\max\{-\lambda_1,\lambda_2\}}\right) > 0$$

for $(t,x,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle \lambda_1,\lambda_2\rangle$,

$$|p_c(t,x,y,z,\lambda)| \leq cw(|z|) + (1-c)\left(\varepsilon^2 M + \frac{\varepsilon^2 M}{2}\right) \leq$$
$$\leq cw(|z|) + \frac{2w(0)}{3M}(M + (M/2)) = cw(|z|) + (1-c)w(0) \leq w(|z|)$$

for $(t,x,y,z,\lambda) \in \langle 0,1\rangle \times \langle -M,M\rangle \times \langle -M,M\rangle \times \mathbf{R} \times \langle \lambda_1,\lambda_2\rangle$.

Thus $\lambda_1 < \lambda_c < \lambda_2$, $\|x_c\| < M$, $\|x_c'\| < M$, $\|x_c''\| < T$ by Lemma 1, and

$$|x_c'''(t)| = |p_c(x_c(t), x_c'(t), x_c''(t), \lambda_c)| \leq w(|x_c''(t)|) \leq w(T)$$

for $t \in \langle 0,1\rangle$. This proves $(x_c,\lambda_c) \notin \partial\Omega$. The proof is finished. □

Corollary 1 *Let* $f \in C^0(\mathbf{R}^5)$,

$$f(t+1,x,y,z,\lambda) \equiv f(t,x,y,z,\lambda)$$

for all $(t,x,y,z,\lambda) \in \mathbf{R}^5$. *Assume assumptions of Theorem 2 are satisfied. Then there exists a* $\lambda_0 \in \langle \lambda_1,\lambda_2\rangle$ *such that equation (1) for* $\lambda = \lambda_0$ *has a 1-periodic solution* x *satisfying* $x(0) = 0$.

Proof Corollary 1 follows immediately from Theorem 2 since f is a 1-periodic function by the assumption. □

Example 1 Let $a,b \in C^0(\langle 0,1\rangle)$, $q \in C^0(\mathbf{R})$, $b(t) > a(t) \geq 0$ for $t \in \langle 0,1\rangle$, $q(0) \neq 0$, $\limsup_{|x|\to\infty} |x^{-2}q(x)| < \infty$. Consider the differential equation

$$x''' = a(t)x + b(t)x' + q(x'') + (1+|x''|)\lambda. \tag{13}$$

The assumptions of Theorem 2 are satisfied with $\lambda_2 = -\lambda_1 = |q(0)|$,

$$M = \frac{2|q(0)|}{\min\{b(t)-a(t); 0 \leq t \leq 1\}}$$

and $w(z) = A + Bz^2$, where A, B are sufficiently large positive constants.

3 Uniqueness theorem

Unless otherwise stated, we shall assume that f satisfies the assumptions:

(H_4) $f(t,.,y,z,\lambda)$ is increasing on $\mathbf{R}$ for each fixed $(t,y,z,\lambda) \in \langle 0,1\rangle \times \mathbf{R}^3$,

(H_5) $f(t,x,.,z,\lambda)$ is increasing on $\mathbf{R}$ for each fixed $(t,x,z,\lambda) \in \langle 0,1\rangle \times \mathbf{R}^3$,

(H_6) $f(t,x,y,z,.)$ is increasing on $\mathbf{R}$ for each fixed $(t,x,y,z) \in \langle 0,1\rangle \times \mathbf{R}^3$,

(H_7) $f(t,x_2,y_2,z,\lambda) - f(t,x_1,y_1,z,\lambda) > 0$ for all $(t,x_i,y_i,z,\lambda) \in \langle 0,1\rangle \times \mathbf{R}^4$ $(i=1,2)$ and $-(x_2-x_1) < y_2 - y_1 > 0$.

Lemma 2 *Let $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in C^0(\langle 0,1\rangle \times \mathbf{R}^4)$. Assume*

$$(14) \qquad \frac{\partial f}{\partial y}(t,x,y,z,\lambda) \geq \frac{\partial f}{\partial x}(t,x,y,z,\lambda) > 0 \qquad \forall (t,x,y,z,\lambda) \in \langle 0,1\rangle \times \mathbf{R}^4.$$

Then f satisfies assumptions (H_4), (H_5) and (H_7).

Proof First, $\frac{\partial f}{\partial x} > 0$ (resp. $\frac{\partial f}{\partial y} > 0$) implies that assumption (H_4) (resp. (H_5)) is satisfied. Further, let $-(x_2 - x_1) < y_2 - y_1 > 0$. Then the Taylor formula and (14) imply

$$\begin{aligned} f(t,x_2,y_2,z,\lambda) - f(t,x_1,y_1,z,\lambda) &= \\ &= \frac{\partial f}{\partial x}(t,\xi,\nu,z,\lambda)(x_2 - x_1) + \frac{\partial f}{\partial y}(t,\xi,\nu,z,\lambda)(y_2 - y_1) = \\ &= \frac{\partial f}{\partial x}(t,\xi,\nu,z,\lambda)(x_2 - x_1 + y_2 - y_1) + \\ &+ \left(\frac{\partial f}{\partial y}(t,\xi,\nu,z,\lambda) - \frac{\partial f}{\partial x}(t,\xi,\nu,z,\lambda)\right)(y_2 - y_1) > 0, \end{aligned}$$

where ξ and ν lie between x_1, x_2 and y_1, y_2, respectively. Thus (H_7) is satisfied. □

Theorem 3 *Let assumptions (H_1)–(H_7) be satisfied with constants $\lambda_1 < 0 < \lambda_2$, $M > 0$, $T > 0$ and a nondecreasing function $w : \langle 0,\infty) \to (0,\infty)$. Then BVP (1), (2) has a unique solution.*

Proof By Theorem 2 there exists at least one solution (x_1, μ_1) of BVP (1), (2). Assume (x_2, μ_2) is another solution of BVP (1), (2). Without loss of generality we may assume $\mu_2 \geq \mu_1$. Set $w = x_2 - x_1$. Then $w(0) = w(1) = 0$, $w'(0) - w'(1) = 0$, $w''(0) - w''(1) = 0$. Assume $w \not\equiv 0$. Let $\max\{w'(t); 0 \leq t \leq 1\} = w'(\tau)$ for a $\tau \in \langle 0,1)$. Since $w(0) = w(1) = 0$ and $w \not\equiv 0$, it is necessarily $w'(\tau) > 0$. If $\tau = 0$, then $w''(0) \leq 0$, $w''(1) \geq 0$ and with respect to $w''(0) = w''(1)$ we have $w''(0) = 0$. Thus

$$w'''(0) = f(0,0,x_2'(0),x_2''(0),\mu_2) - f(0,0,x_1'(0),x_2''(0),\mu_1) > 0,$$

a contradiction. This proves $\tau \in (0,1)$ and then $w''(\tau) = 0$, $w'''(\tau) \leq 0$ which imply $w(\tau) < 0$ (cf. (H_4)–(H_6)). Since $w(1) = 0$, there exists a $\xi \in (\tau, 1\rangle$ such that $w(t) < 0$ on $\langle \tau, \xi)$ while $w(\xi) = 0$, and consequently

$$-w(\tau) = \int_\tau^\xi w'(s)\,ds \leq w'(\tau)(\xi - \tau) < w'(\tau).$$

Hence $-(x_2(\tau) - x_1(\tau)) < x_2'(\tau) - x_1'(\tau)\,(> 0)$ and by (H_6) and (H_7) we have

$$\begin{aligned} w'''(\tau) = f(\tau,x_2(\tau),x_2'(\tau),x_2''(\tau),\mu_2) - f(\tau,x_1(\tau),x_1'(\tau),x_2''(\tau),\mu_1) \geq \\ \geq f(\tau,x_2(\tau),x_2'(\tau),x_2''(\tau),\mu_1) - f(\tau,x_1(\tau),x_1'(\tau),x_2''(\tau),\mu_1) > 0, \end{aligned}$$

which contradicts $w'''(\tau) \le 0$. Thus $x_2 = x_1$. If $\lambda_2 > \lambda_1$ then

$$0 = x_2'''(t) - x_1'''(t) = f(\tau, x_2(t), x_2'(t), x_2''(t), \mu_2) - f(\tau, x_2(t), x_2'(t), x_2''(t), \mu_1) > 0,$$

a contradiction. This completes the proof. □

Example 2 Let $a(t), b(t)$ and $q(t)$ be as in Example 1 and, moreover, $a(t) > 0$ on $\langle 0, 1\rangle$. Then Theorem 3 can be applied to equation (13).

Using Corollary 1 and Theorem 3, we obtain

Corollary 2 *Let* $f \in C^0(\mathbf{R}^5)$,

$$f(t+1, x, y, z, \lambda) \equiv f(t, x, y, z, \lambda)$$

for all $(t, x, y, z, \lambda) \in \mathbf{R}^5$. *If assumptions of Theorem 2 and assumptions* (H_4)–(H_7) *are satisfied, then there exists a unique* $\lambda_0 \in \mathbf{R}$ *such that equation (1) for* $\lambda = \lambda_0$ *has a 1-periodic solution* x, $x(0) = 0$ *and, moreover, this solution is unique.*

4 An application

We give the application of the above results for BVP

$$x^{(4)} = (g(x, x'(t), x''(t), x'''(t), \lambda))(t), \tag{15}$$

$$\alpha(x) = 0,\ x'(0) = x'(1) = 0,\ x''(0) - x''(1) = 0,\ x'''(0) - x'''(1) = 0, \tag{16}$$

depending on the parameter λ. Here $g : \mathbf{X} \times \mathbf{R}^4 \to \mathbf{X}$ is a locally bounded continuous operator, $\alpha : \mathbf{X} \to \mathbf{R}$ is a continuous increasing functional (i.e., $x, y \in \mathbf{X}$, $x(t) < y(t)$ on $\langle 0, 1\rangle \Rightarrow \alpha(x) < \alpha(y)$) mapping $\mathbf{X}$ onto $\mathbf{R}$, $\alpha(0) = 0$. We observe that $\alpha(x) = 0$ for an $x \in \mathbf{X}$ implies $x(\xi) = 0$ for a $\xi \in \langle 0, 1\rangle$ (see, e.g., Remark 1 in [5]).

The special case of (15) is the differential equation

$$x^{(4)} = r(t, x, x', x'', x''', \lambda), \qquad r \in C^0(\langle 0, 1\rangle \times \mathbf{R}^5).$$

We say that (x, λ_0) is a *solution* of BVP (15), (16) if $(x, \lambda_0) \in C^4(\langle 0, 1\rangle) \times \mathbf{R}$ and x is a solution of (15) for $\lambda = \lambda_0$ satisfying (16).

We shall assume that there exist constants $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $T > 0$ and a nondecreasing function $w : \langle 0, \infty) \to (0, \infty)$ such that the operator g satisfies the following assumptions $\big(t \in \langle 0, 1\rangle$, $\mathbf{X}_M = \{x; x \in \mathbf{X}, \|x\| \le M\}\big)$

(A_1) $\big(g(\varphi, p-y, y, 0, \lambda_2)\big)(t) \ge 0$ for all $(\varphi, p, y) \in \mathbf{X}_M \times \langle 0, M\rangle \times \langle 0, M\rangle$,
$\big(g(\varphi, p-y, y, 0, \lambda_1)\big)(t) \le 0$ for all $(\varphi, p, y) \in \mathbf{X}_M \times \langle -M, 0\rangle \times \langle -M, 0\rangle$;

(A_2) $\big(g(\varphi, x, -M, 0, \lambda)\big)(t) \le 0 \le \big(g(\varphi, x, M, 0, \lambda)\big)(t)$ for all $(\varphi, x, \lambda) \in \mathbf{X}_M \times \langle -M, M\rangle \times \langle \lambda_1, \lambda_2\rangle$;

(A_3) $|(g(\varphi, x, y, z, \lambda))(t)| \le w(|z|)$ for all $(\varphi, x, y, z, \lambda) \in \mathbf{X}_M \times \langle -M, M\rangle \times \langle -M, M\rangle \times \mathbf{R} \times \langle \lambda_1, \lambda_2\rangle$ and

$$\int_0^T \frac{s\,ds}{w(s)} > 2M;$$

(A_4) $(g(\varphi, ., y, z, \lambda))(t)$ is increasing on $\mathbf{R}$ for each fixed $(\varphi, y, z, \lambda) \in \mathbf{X}_M \times \mathbf{R}^3$;

(A_5) $(g(\varphi, x, ., z, \lambda))(t)$ is increasing on $\mathbf{R}$ for each fixed $(\varphi, x, z, \lambda) \in \mathbf{X}_M \times \mathbf{R}^3$;

(A_6) $(g(\varphi, x, y, z, .))(t)$ is increasing on $\mathbf{R}$ for each fixed $(\varphi, x, y, z) \in \mathbf{X}_M \times \mathbf{R}^3$;

(A_7) $(g(\varphi, x_2, y_2, z, \lambda))(t) - (g(\varphi, x_1, y_1, z, \lambda))(t) > 0$ for each $(\varphi, x_i, y_i, z, \lambda) \in \mathbf{X}_M \times \mathbf{R}^4$ $(i = 1, 2)$ and $-(x_2 - x_1) < y_2 - y_1 > 0$.

Theorem 4 *Let assumptions (A_1)–(A_7) be satisfied with constants $\lambda_1 < 0 < \lambda_2$, $M > 0$, $T > 0$ and a nondecreasing function $w : (0, \infty) \to (0, \infty)$. Then BVP (15), (16) has at least one solution.*

Proof Set $\mathbf{S} = \{x;\ x \in \mathbf{Z}, \|x\| \le M, \|x'\| \le M, \|x''\| \le M, \|x'''\| \le T\}$, where $\mathbf{Z}$ is defined in the proof of Theorem 2. Then $\mathbf{S}$ is a bounded closed convex subset of $\mathbf{Z}$. Let $\varphi \in \mathbf{X}_M$ and $f_\varphi(t, x, y, z, \lambda) = (g(\varphi, x, y, z, \lambda))(t)$ for $(t, x, y, z, \lambda) \in \langle 0, 1\rangle \times \mathbf{R}^4$. Then f_φ satisfies assumptions (H_1)–(H_7) with the constants $\lambda_1 < 0$, $\lambda_2 > 0$, $M > 0$, $T > 0$ and the function w. Therefore there exists the unique solution $(v_\varphi, \lambda_\varphi)$ of BVP (1), (2) (with $f = f_\varphi$) by Theorem 3. Moreover, $\|v_\varphi^{(i)}\| \le M$ $(i = 0, 1)$, $\|v_\varphi''\| \le T$, $\lambda_1 \le \lambda_\varphi \le \lambda_2$ by Theorem 1 and the proof of Theorem 2. The function $k : \mathbf{R} \to \mathbf{R}$ defined by

$$k(c) = \alpha\left(\int_0^t v_\varphi(s)\,ds + c\right)$$

is continuous increasing and maps $\mathbf{R}$ onto $\mathbf{R}$, and so a unique $c_\varphi \in \mathbf{R}$ exists such that $k(c_\varphi) = 0$. Set

$$x_\varphi(t) = \int_0^t v_\varphi(s)\,ds + c_\varphi \qquad \text{for } t \in \langle 0, 1\rangle.$$

Then $\alpha(x_\varphi) = 0$ which implies $x_\varphi(\xi) = 0$ for a $\xi \in \langle 0, 1\rangle$. Hence

$$x_\varphi(t) = \int_\xi^t v_\varphi(s)\,ds \qquad \text{on } \langle 0, 1\rangle,$$

and consequently $|x_\varphi(t)| \le M|t - \xi| \le M$. This proves $x_\varphi \in \mathbf{S}$. Define the operator $T : \mathbf{S} \to \mathbf{S}$ by $T(\varphi) = x_\varphi$. We see that to prove of Theorem 4 it is sufficient to show that T has a fixed point.

Let $\{x_n\} \subset \mathbf{S}$ be a convergent sequence, $\lim_{n\to\infty} x_n = x$ and let $z_n = T(x_n)$, $z = T(x)$. Then, by the definition of the operator T, we have

$$\alpha(z_n) = 0,\ z_n'(0) = z_n'(1) = 0,\ z_n''(0) - z_n''(1) = 0,\ z_n'''(0) - z_n'''(1) = 0, \quad n \in \mathbf{N},$$

$$\alpha(z) = 0,\ z'(0) = z'(1) = 0,\ z''(0) - z''(1) = 0,\ z'''(0) - z'''(1) = 0$$

and there exist sequence $\{\mu_n\} \subset \langle\lambda_1, \lambda_2\rangle$ and $\mu_0 \in \langle\lambda_1, \lambda_2\rangle$ such that the equalities

$$z_n^{(4)}(t) = (g(x_n, z_n'(t), z_n''(t), z_n'''(t), \mu_n))(t), \quad n \in \mathbf{N},$$
$$z^{(4)}(t) = (g(x, z'(t), z''(t), z'''(t), \mu_0))(t)$$

hold for $t \in \langle 0,1\rangle$. Moreover $\|z_n^{(4)}\| \le L$ $(n \in \mathbf{N})$, where

$$L = \sup\Big\{\|(g(\varphi, x, y, z, \lambda))\|;\ \varphi \in \mathbf{X}_M,$$
$$x, y, \in \langle -M, M\rangle,\ z \in \langle -T, T\rangle,\ \lambda \in \langle\lambda_1, \lambda_2\rangle\Big\}\ (< \infty).$$

In order to prove that $\{z_n\}$ is convergent let $\{\tilde{z}_n\}$ be a subsequence of $\{z_n\}$ and $\{\tilde{\mu}_n\}$ be the corresponding subsequence of $\{\mu_n\}$. By the Arzelà–Ascoli theorem, we can select a convergent subsequence $\{\tilde{z}_{k_n}\}$ of $\{\tilde{z}_n\}$, $\lim_{n\to\infty} \tilde{z}_{k_n} = u$, and since $\{\tilde{\mu}_n\}$ is bounded without loss of generality we may assume that $\{\tilde{\mu}_{k_n}\}$ is convergent, $\lim_{n\to\infty} \tilde{\mu}_{k_n} = \tau$. Thus, taking the limit in the equalities

$$\tilde{z}_{k_n}'''(t) = \tilde{z}_{k_n}'''(0) + \int_0^t (g(\tilde{x}_{k_n}, \tilde{z}_{k_n}'(s), \tilde{z}_{k_n}''(s), \tilde{z}_{k_n}'''(s), \tilde{\mu}_{k_n}))(s)\, ds, \quad n \in N$$

as $n \to \infty$, we have

$$u'''(t) = u'''(0) + \int_0^t (g(x, u'(s), u''(s), u'''(s), \tau))\, ds, \quad t \in \langle 0, 1\rangle.$$

Therefore (u, τ) is a solution of BVP

$$r^{(4)}(t) = (g(x, r'(t), r''(t), r'''(t), \tau))(t), \quad (16).$$

Since this BVP has a unique solution (z, μ_0) we have $(u, \tau) = (z, \mu_0)$. Hence every subsequence of $\{z_n\}$ has in turn a subsequence that converges to z, and we conclude that $\{z_n\}$ is convergent $\lim_{u\to\infty} z_n = z$ and T is a continuous operator. Since

$$T(\mathbf{S}) \subset \Big\{x;\ x \in \mathbf{S} \cap C^4(\langle 0,1\rangle),\ \|x^{(4)}\| \le L\Big\}\ (=: \mathscr{L})$$

and $\mathscr{L}$ is a compact subset of $\mathbf{Z}$, $T(\mathbf{S})$ is relatively compact subset of $\mathbf{Z}$. Now, the existence of a fixed point of T follows by the Schauder fixed point theorem. □

Example 3 Consider the functional differential equation

$$(17) \qquad x^{(4)}(t) = \int_0^t |x(s)|\, ds + a(t)x'(t) + b(t)x''(t) + q(x'''(t)) + (1 + t^2)\lambda,$$

where $a, b \in C^0(\langle 0,1\rangle)$, $q \in C^0(\mathbf{R})$, $b(t) - 2 > a(t) > 0$ on $\langle 0,1\rangle$, $q(0) \ne 0$ and $\limsup_{|x|\to\infty} |x^{-2}q(x)| < \infty$. The assumptions of Theorem 4 are satisfied with constants

$$\lambda_1 = -|q(0)|\frac{D+1}{D-2}, \quad \lambda_2 = |q(0)|\frac{2D-1}{2(D-2)}, \quad M = \frac{3|q(0)|}{D-2},$$

$T > 0$ sufficiently large and $w(u) = A + Bu^2$, where $D = \min\{b(t) - a(t); t \in \langle 0,1\rangle\}$ and A, B are suitable positive constants. Hence, there exists a solution (x, λ_0) of BVP (17), (16) by Theorem 4. Note that for example the functionals

$$\int_0^1 x(s)\,ds, \quad \max\{x(t); t \in \langle 0,1\rangle\}, \quad \min\{x(t); t \in \langle 0,1\rangle\} \quad \text{and} \quad x(\xi)\ (\xi \in \langle 0,1\rangle)$$

satisfy the assumptions imposed upon α in (16).

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