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GOLDEN SECTION QUASIGROUPS AS SPECIAL IDEMPOTENT MEDIAL QUASIGROUPS

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Abstract

After some modification of Toyoda's representation theorem for idempotent medial quasigroups we characterize the role of golden section quasigroups according to this theorem.

Key words: quasigroup: medial, idempotent, linear over a commutative group.

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1 We deduce some modification of Toyoda's theorem, following [1], pp. 240–244

An element a of a groupoid $(G, \bullet)$ is said to be *idempotent* if $a \bullet a = a$.

A groupoid $(G, \bullet)$ is said to be *idempotent* if all elements of it are idempotent. It is called *medial* if it satisfies the identity $ab \bullet cd = ac \bullet bd$.

A groupoid $(G, \bullet)$ is called *linear over a commutative group* $(G, +)$, *with the dilatation* φ if there is an automorfism $\varphi \neq \mathrm{id}_G$ of $(G, +)$ such that it holds $x \bullet y = y + \varphi(x - y)$ for all $x, y \in G$.

If $(Q, \bullet)$ is a quasigroup and e some element of it, then we define the map $Q \to Q$, $x \mapsto {}^e x$ such that the image ${}^e x$ is the solution $e\backslash(x \bullet e)$ of the equation $e \bullet {}^e x = x \bullet e$.[1]

[1] $\backslash$ and $/$ are accompanying operations of $\bullet$, i.e. $x \bullet y = z \iff y = x\backslash z \iff x = z/y$ for all elements x, y, z of a quasigroup $(Q, \bullet)$.

Lemma 1 *Every medial quasigroup $(Q, \bullet)$ satisfies the "cross rule"*

$$\begin{array}{rcl} a_1 a & = & b b_1 \\ a_2 a & = & b b_2 \end{array} \Rightarrow a_1 b_2 = a_2 b_1 .$$

Proof We have

$$a_1 b_2 \bullet ab \underset{\text{by mediality}}{=} a_1 a \bullet b_2 b \underset{\substack{\text{by substitution} \\ a_1 a = b b_1}}{=} b b_1 \bullet b_2 b \underset{\text{by mediality}}{=} b b_2 \bullet b_1 b \underset{\substack{\text{by substitution} \\ b b_2 = a_2 a}}{=} a_2 a \bullet b_1 b \underset{\text{by mediality}}{=} a_2 b_1 \bullet ab.$$

Thus $a_1 b_2 \bullet ab = a_2 b_1 \bullet ab$ and by cancellation from right we obtain $a_1 b_2 = a_2 b_1$.

Lemma 2 *Every medial quasigroup $(Q, \bullet)$ satisfies the "pseudocommutativity rule" $a \bullet {}^e b = b \bullet {}^e a$ where e is an arbitrary idempotent element of Q and a, b are arbitrary elements of Q.*

Proof By cross rule applied to $ae = e\,{}^e a$, $be = e\,{}^e b$.

Lemma 3 *Let $(Q, \bullet)$ be a medial quasigroup with an idempotent element e. The binary operation $\circ_e$ defined by $(xe) \circ_e (ey) = xy$ for all $x, y \in Q$ leads to a commutative group $(Q, \circ_e)$ with neutral element e.*

Proof We see that $(Q, \circ_e)$ is a principal loop isotop of $(Q, \bullet)$ and that e is its neutral element. For all $a, b \in Q$ we have

$$a \circ_e b = (a/e) \bullet {}^e(b/e) = (b/e) \bullet {}^e(a/e) = b \circ_e a$$

(as $a = (a/e) \bullet e = e \bullet {}^e(a/e)$, $b = (b/e) \bullet e = e \bullet {}^e(b/e)$). Similarly we obtain for all $a, b, c \in Q$

$$(a/e) \bullet {}^e(b/e) = d_1 e = e\,{}^e d_1 = (b/e) \bullet {}^e(a/e),$$
$$(b/e)\,{}^e(c/e) = d_2 e = e\,{}^e d_2 = (b/e)\,{}^e(c/e)$$

so that $d_1\,{}^e(c/e) = d_2\,{}^e(a/e) = (a/e)\,{}^e d_2$ and finally

$$(a \circ_e b) \circ_e c = \left((a/e)^e(b/e)\right) \circ_e c = d_1{}^e(c/e),$$
$$a \circ_e (b \circ_e c) = a \circ_e \left((b/e)\,{}^e(c/e)\right) = (a/e)^e d_2.$$

Thus from $d_1\,{}^e(c/e) = (a/e)\,{}^e d_2$ we get

$$(a \circ_e b) \circ_e c = a \circ_e (b \circ_e c).$$

Lemma 4 *Let $(Q, \bullet)$ be a medial quasigroup with an idempotent element e. The "mixed mediality rule." holds*

$$(ab) \circ_e (cd) = (a \circ_e c)(b \circ_e d) \qquad \textit{for all } a, b, c, d \in Q.$$

Proof After a routine arrangement the expression $(ab) \circ_e (cd)$ goes over onto $({}^e a\, {}^e b)({}^e({}^e c){}^e({}^e d))$ and $(a \circ_e c)(b \circ_e d)$ onto $({}^e a^e({}^e c))({}^e b^e({}^e d))$. Thus using the mediality we obtain the requisted equality.

Lemma 5 *Let $(Q, \bullet)$ be a medial quasigroup with an idempotent element e. Then the translations $R_e : x \mapsto x \bullet e$, $L_e : x \mapsto e \bullet y$ are commuting automorphisms of the group $(Q, \circ_e)$.*

Proof According to mixed mediality rule we have

$$xe \circ_e ye = (x \circ_e y)\, e, \qquad ex \circ_e ey = e\,(x \circ_e y) \qquad \text{for all } x, y \in Q$$

so that R_e and L_e result as automorphisms of $(Q, \circ_e)$. Furthermore

$$R_e L_e(x) = ex \bullet e, \qquad L_e R_e(x) = e \bullet xe \qquad \text{for all } x \in Q$$

and, by mediality, $ex \bullet ee = ee \bullet xe$ and consenquently $ex \bullet e = e \bullet xe$.

Theorem 1a *If a quasigroup $(Q, \bullet)$ is a medial and contains an idempotent element e then there is a commutative group $(Q, \circ)$ which admits commuting automorphisms σ and τ such that*

$$x \bullet y = \sigma(x) \bullet \tau(y) \qquad \textit{for all } x, y \in Q.$$

Proof Let $(Q, \bullet)$ be a medial quasigroup with an idempotent element e. Then by Lemmas 3–5 for $\sigma = R_e$, $\tau = L_e$, $\circ = \circ_e$ we get the commutative group $(Q, \circ)$ having all the properties requested.

Theorem 1b *Let a groupoid $(Q, \bullet)$ arise from a commutative group $(Q, \circ)$, which admits commuting automorphisms σ and τ by the rule*

$$x \bullet y = \sigma(x) \circ \tau(y) \qquad \textit{for all } x, y \in Q.$$

Then $(Q, \bullet)$ is a medial quasigroup with an idempotent element e such that $\sigma = R_e$ and $\tau = L_e$.

Proof Let there exist a commutative group $(Q, \circ)$ with commuting automorphisms σ, τ satisfying $xy = \sigma(x) \circ \tau(y)$ for all $x, y \in Q$. Then the groupoid $(Q, \bullet)$ is an isotop of $(Q, \circ)$ and as such it is a quasigroup. On the other side $(Q, \circ)$ is a principal loop isotop of $(Q, \bullet)$ so that $\sigma = R_u$ and $\tau = L_v$ for convenient elements $u, v \in Q$. Since σ, τ are automorphisms of $(Q, \circ)$, the neutral element e of $(Q, \circ)$ satisfies equalities $eu = ve = e$, $ee = eu \circ ve = e \circ e = e$. Herefrom we obtain $u = v = e$. Thus for all $a, b, c, d \in Q$ we have

$$\begin{aligned} ab \bullet cd &= (ae \circ eb)(ce \circ ed) = \\ &= ((ae \circ eb)e) \circ (e(ce \circ ed)) = (ae \bullet e) \circ (eb \bullet e) \circ (e \bullet ce) \circ (e \bullet ed). \end{aligned}$$

Similarly we can deduce that

$$ac \bullet bd = (ae \bullet e) \circ (ec \bullet e) \circ (e \bullet be) \circ (e \bullet ed).$$

Since R_e and L_e commute, we obtain $e \bullet xe = ex \bullet e$ and consequently

$$(ae \bullet e) \circ (eb \bullet e) \circ (e \bullet ce) \circ (e \bullet ed) = (ae \bullet e) \circ (ec \bullet e) \circ (e \bullet be) \circ (e \bullet ed).$$

Thus $(Q, \bullet)$ must be medial.

Theorem 2a *If $(Q, \bullet)$ is a non-trivial idempotent medial quasigroup then for each $e \in Q$ there is a commutative group $(Q, +_e)$ ($xy = xe +_e ey$ for all $x, y \in Q$) such that R_e, L_e are commuting automorphisms of $(Q, +_e)$ and $x +_e ex = x$ for all $x \in Q$. Further, $(Q, \bullet)$ is linear over $(Q, +_e)$ with dilatation L_e. Various choices of the element e lead to mutually isomorphic groups $(Q, +_e)$.*

Proof Follows from Theorem 1a. If $(Q, \bullet)$ is a non-trivial idempotent medial quasigroup then $x = xx = xe +_e ex$ and

$$xy = xe +_e ey = xe +_e ex -_e ex +_e ey = x +_e L_e(y -_e x)$$

for all $x, y \in Q$. As $L_e \neq \mathrm{id}_Q$,[2] $(Q, \bullet)$ is linear over $(Q, +_e)$ with dilatation L_e. By Lemma 5, R_e and L_e are commuting automorphisms of $(Q, +_e)$. The groups for various $e \in Q$ are isotopic as isotops of the same quasigroup $(Q, \bullet)$, and isotopic groups are necessarily isomorphic as it is well known from the elements of the quasigroup theory.

Theorem 2b *Let $(Q, \bullet)$ be a groupoid linear over a commutative group $(Q, +)$ with a dilatation φ. Then $(Q, \bullet)$ is a non-trivial idempotent medial quasigroup and $\varphi = L_e$ for some element $e \in Q$.*

Proof The map $\psi = \mathrm{id}_Q - \varphi$ is an automorphism of $(Q, +)$ too, as

$$\psi(x + y) = x + y - \varphi(x + y) = x - \varphi(x) + y - \varphi(y) = \psi(x) + \psi(y)$$

for all $x, y \in Q$. Moreover,

$$\psi\varphi(x) = \varphi(x) - \varphi(\varphi(x)), \qquad \varphi\psi(x) = \varphi(x - \varphi(x)) = \varphi(x) - \varphi^2(x)$$

so that φ and ψ are commuting. All which remains is already the consequence of Theorem 1b ($\sigma = \psi$, $\tau = \varphi$).

[2] From $L_e = \mathrm{id}_Q$ it would follow that $ex = xx$ and consequently $e = x$ for all $x \in Q$ so that $(Q, \bullet)$ would be trivial.

2 We are prepared to find some specialization of Theorems 2a–b

Let $(Q, \bullet)$ be a quasigroup. We shall investigate an identity of the form

$$f(x, f(x, y, z), z) = y$$

namely the identity $a(ab \bullet c) \bullet c = b$ so that $f(x, y, z) = xy \bullet z$ called the *first golden section identity.* It is equivalent with the identity $a \bullet (a \bullet bc)c = b$, the *second golden section identity* (cf. [3], p. 307). The golden section identity implies the mediality. Mediality and idempotency imply autodistributivities ($x \bullet yz = xy \bullet xz$, $xy \bullet z = xz \bullet yz$) and elasticity ($x \bullet yx = xy \bullet x$).

A quasigroup satisfying golden section identities and idempotency is called *golden section quasigroup* (according to V. Volenec, [3], p. 307).

Example 1a Let $(\mathbb{C}, +, \bullet)$ be the field of all complex numbers and $\Box$ a binary operation on $\mathbb{C}$ such that $a \Box b = a + q(b - a)$ for all $a, b \in \mathbb{C}$, where

$$q = \frac{1}{2}(1 \pm \sqrt{5}), \qquad q^2 = q + 1.$$

Thus

$$\frac{a \Box b - a}{b - a} = q, \qquad (q - 1) : 1 = 1 : q, \qquad \frac{a \Box b - b}{b - a} = \frac{b - a}{a \Box b - a}.$$

Then $(\mathbb{C}, \Box)$ can be shown to be a golden section quasigroup. This example was a first inspiration for golden section quasigroups.

Example 2 Let $(F, +, \bullet)$ be a field, q an element of F satisfying the equation $q = q^2 - 1$ and $\Box$ a binary operation on F such that $a \Box b = a + q(b - a)$ for all $a, b \in F$. Hence $x \mapsto qx$ is a non-identical additive automorphism of $(F, +)$ and $(F, \Box)$ is linear over $(F, +)$ with dilatation $F \to F$, $x \mapsto qx$. It can be shown that $(F, \Box)$ is a golden section quasigroup.

Theorem 3a *Every quasigroup $(Q, \bullet)$ linear over a commutative group $(Q, +)$ with a dilatation $\varphi = \varphi^2 - \mathrm{id}_Q$ is a golden section quasigroup. (Cf. also [3], pp. 307–308.)*

Proof By definition of the binary operation $\bullet$, we have

$$xy = (\mathrm{id}_Q - \varphi)(x) + \varphi(y) \qquad \text{for all } x, y \in Q,$$

so that $(Q, +)$ is a principal isotop of $(Q, \bullet)$. For all $a, b \in Q$ we obtain successively

$$ab = a + \varphi(b - a),$$
$$ab \bullet c = (a + \varphi(b - a)) + \varphi(c - a - \varphi(b - a)) =$$
$$= a + \varphi(b + c - 2a) - \varphi^2(b - a) = 2a - b + \varphi(c - a),$$
$$a(ab \bullet c) \bullet c = 2a - (2a - b + \varphi(c - a)) + \varphi(c - a) = b.$$

Theorem 3b *Every golden section quasigroup* $(Q,\bullet)$ *is linear over a commutative group* $(Q,+)$ *with a dilatation* $\varphi = \varphi^2 - \mathrm{id}_Q$. *(Cf. also [3], Theorem 19 on p. 317.)*

Proof As $(Q,\bullet)$ is idempotent and medial we can use Theorem 2a and express $(Q,\bullet)$ as a linear quasigroup over a commutative group $(Q,+)$ with some dilatation $\varphi : ab = a + \varphi(b-a)$ for all $a,b \in Q$. We obtain succesively

$$ab\bullet c = (a+\varphi(b-a)) + \varphi(c-(a+\varphi(b-a))) = a+\varphi(b+c-2a) - \varphi^2(b-a),$$

$$a(ab\bullet c)\bullet c =$$

$$= a+\varphi(a+\varphi(b+c-2a)-\varphi^2(b-a)+c-a)-\varphi^2(a+\varphi(b+c-2a)-\varphi^2(b-a)-a) =$$

$$= a+\varphi(c-a)+\varphi^2(b+c-2a)-\varphi^3(2b+c-3a)+\varphi^4(b-a),$$

$$a+\varphi(c-a)+\varphi^2(b+c-2a)-\varphi^3(2b+c-3a)+\varphi^4(b-a) = b,$$

$$-(b-a)+\varphi(c-a)+\varphi^2((b-a)+(c-a))-\varphi^3(2(b-a)+(c-a))+\varphi^4(b-a) = 0.$$

If we put $b-a = x$, $c-a = y$ we get

$$-x+\varphi(y)+\varphi^2(x+y)-\varphi^3(2x+y)+\varphi^4(x) = 0.$$

For $x = 0$ it follows that

$$\varphi(y)+\varphi^2(y)-\varphi^3(y) = 0.$$

Thus the substitution $\varphi(y) = z$ gives

$$z+\varphi(z)-\varphi^2(z) = 0, \quad \text{i.e.} \quad \varphi(z) = \varphi^2(z) - \mathrm{id}_Q(z).$$

Since φ is a bijection, the preceding equality holds for every $z \in Q$ and we get $\varphi = \varphi^2 - \mathrm{id}_Q$.

V. Volenec obtained Theorem 3b without use of Toyoda's theorem as a final conclusion of his reasoning in the adjacent parallelogram space. But herein they occur some difficulties namely with the construction of the basic commutative group.

Now we utilize Theorems 3a–b and express the condition $\varphi = \varphi^2 - \mathrm{id}_Q$ in form of some identities over the quasigroup under consideration.

Theorem 4 *Let* $(Q,\bullet)$ *be a non-trivial idempotent medial quasigroup. It is a golden section quasigroup if and only if it satisfies the identity* $y\bullet yx = (x/y)\bullet y$ *or the identity* $y\bullet yx = x\bullet((y\backslash x)(x/y))$.

Proof Using Theorem 2b $(Q,\bullet)$ can be expressed as a linear quasigroup over a commutative group $(Q,+_e)$ with the dilatation $\varphi = L_e$ for some $e \in Q$. $(Q,\bullet)$ is then a golden section quasigroup if and only if $\varphi = \varphi^2 -_e \mathrm{id}_Q$ (Theorem 3a-b). Recall that $xy = x +_e \varphi(y -_e x)$ for all $x,y \in Q$. Thus $\varphi^2 = \mathrm{id}_Q +_e \varphi$ may be written as $e \bullet ex = x +_e ex$ and this is equal to $(x/e) \bullet e +_e ex = (x/e) \bullet x$ or, respectively, to

$$x +_e e \bullet (x +_e x -_e x) = x \bullet (x +_e x) = x \bullet ((e\backslash x)(x/e)).$$

This reasoning can be conversed. The element $e \in Q$ can be taken arbitrarily (as $(Q, \bullet)$ is idempotent) so that we have obtained the identity $y \bullet yx = (x/y) \bullet x$ or, respectively the identity $y \bullet yx = x \bullet ((y\backslash x)(x/y))$ as a necessary and sufficient condition for $(Q, \bullet)$ to be a golden section quasigroup.

Our final remark concerns finite nearfields. If $(F, +, \bullet)$ is a finite nearfield and q its element such that $\varphi : F \to F,\ x \mapsto qx$ is non-identifical additive automorphism then we shall speak of a *standard dilatation* φ with *slope* q. Every groupoid $(F, \circ)$ linear over the additive group of a finite nearfield $(F, +, \bullet)$, with a standard dilatation, is 2-homogeneous, i.e., the full automorphism group of $(F, \bullet)$ operates strongly doubly transitively on F. Conversely, every finite 2-homogeneous quasigroup $(Q, \circ)$ is linear over the additive group of some finite nearfield $(Q, +, \bullet)$ with a standard dilatation. (For proofs, cf. [2], pp. 1093–1098.) It can be proved that every quasigroup linear over the additive group of some finite nearfield with a standard dilatation is medial if and only if this nearfield is associative (i.e., a field). Thus a golden section quasigroup is linear over the additive group of a finite field, with standard dilatation, if and only if it is 2-homogeneus.

In GF(2), $1 = q = q^2 = q + 1$ implies $q = 0$ and similarly, in GF(3), $1 = q^2 = q + 1$ implies $q = 0$. On the other side, in GF(4),

$$1 = q^3 = q^2 \bullet q = (q+1)q = q^2 + q = q + 1 + q \iff q + q = 0.$$

Thus no slope q exists for a standard dilatation over GF(2), GF(3) and GF(4). On the other side, in GF(5)

$$\begin{aligned} 1 = q^4 &= q^3 \bullet q = (q^2 \bullet q) \bullet q = (q+1)q \bullet q = \\ &= (q^2 + q)q = (q + q + 1) \bullet q = q + 1 + q + 1 + q = (q + q + q) + 1 + 1. \end{aligned}$$

E.g. $q = 1 + 1 + 1$ satisfies the equation $q + q + q + 1 = 0$ and is a slope for a standard dilatation over GF(5).

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