Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Pavel Calábek

A four-point problem for second-order differential systems

*Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, Vol. 33 (1994), No. 1, 7--15

Persistent URL: `http://dml.cz/dmlcz/120301`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM
1994 Mathematica XXXIII Vol. 114

# A FOUR–POINT PROBLEM FOR SECOND–ORDER DIFFERENTIAL SYSTEMS

PAVEL CALÁBEK

(Received January 31, 1994)

**Abstract**

Sufficient conditions for the existence of a solution to four-point boundary value problems for the second order ordinary differential systems are established by means of the topological transversality method.

**Key words:** four-point boundary value problem, a priori estimate, topological transversality method, Carathéodory conditions

**MS Classification:** 34B10

# 1 Introduction

An $l$-point boundary value problem for a $k$-order ordinary differential equation, with $l > k$, appears very seldom in the literature. And a special case the questions of existence and uniqueness of solutions of the four-point boundary value problem for scalar differential equations of the second order was studied in [2]–[7]. There was studied problem

$$x'' = f(t, x, x')$$

with boundary conditions of a type

$$\begin{aligned} e_{11}x(a) + e_{12}x(b) &= g_1 \\ e_{21}x(c) + e_{22}x(d) &= g_2 \end{aligned}$$

where $e_{ij}, g_i \in \mathbb{R}$, and $f$ is a continuous function or satisfies Carathéodory conditions. This problems were studied as like as Neumann problem because

Neumann problem may be presented as a limit case of the four-point boundary value problem.

Questions of existence to the differential systems of the $k$-order for $k \geq 2$, where $f$ is a continuous or Carathéodory function, were studied in [1] by the method of topological transversality. There were explicitly solved $l$-point boundary value problems, where $l \leq 2$.

In this paper there are proved theorems of existence of a solution to the differential system

$$x'' = f(t, x, x') \tag{1.1}$$

satisfying four point boundary conditions

$$x(0) = x(a), \qquad x(b) = x(1), \tag{1.2}$$

where $0 < a \leq b < 1$.

We use the method of topological transversality based on the paper [1], especially on the theorem (3.2) part $(\tilde{C})$. This theorem is introduced in the end of this part.

Let $I = [0,1]$, $0 < a \leq b < 1$, $\mathbb{R} = (-\infty, \infty)$, $n$, $k$ natural numbers. $\mathbb{R}^n$ denotes as usual Euclidean n-space and $||x||$ denotes the Euclidean norm. $C_n^k = C^k([0,1], \mathbb{R}^n)$, $C_n = C_n^0$ is the Banach space of functions $u$ such that $u^{(k)}$ is continuous on $I$ with the norm

$$||u||_k = \max\left\{||u||, ||u'||, ||u''||, \ldots, ||u^{(k)}||\right\},$$

where

$$||u|| = \max\{||u(t)||, t \in I\}.$$

**Definition 1.1** A function $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function provided: if $f = f(t, u, p)$

(i) the map $(u, p) \mapsto f(t, u, p)$ is continuous for almost every $t \in I$,

(ii) the map $t \mapsto f(t, u, p)$ is measurable for all $(u, p) \in \mathbb{R}^n \times \mathbb{R}^n$,

(iii) for each bounded subset $B \subset \mathbb{R}^n \times \mathbb{R}^n$ the function $\sup\{||f(t, u, p)||, (u, p) \in B\} \in L(I)$.

In the whole paper assume $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous or a Carathéodory function.

If $f$ is continuous, by a solutions to the equation (1.1) we mean a classical solution with a continuous $2^{nd}$ derivative, while if $f$ is a Carathéodory function, a solution will mean a function $x$ which has an absolutely continuous $1^{st}$ derivative such that $x$ fulfills the equality $x''(t) = f(t, x(t), x'(t))$ for almost every $t \in I$.

By $xy$ in $\mathbb{R}^n$ we mean a scalar product of two vectors from $\mathbb{R}^n$.

**Topological transversality theorem** *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be continuous or a Carathéodory function. Assume furthermore that $\varepsilon \neq 0$ is not an eigenvalue of the differential operator*

$$\Lambda : C^2_{n0} \to C_n, \quad \Lambda y = y'',$$

*where $C^2_{n0}$ is a space of all functions from $C^2_n$ which fulfil boundary conditions (1.2), and that $\tilde{g}$ is the unique solution to $\tilde{L}(\tilde{g}) = 0$, where*

$$\tilde{L} : C^1_{n0} \to C_n, \quad (\tilde{L}x)(t) = x'(t) - x'(0) - \varepsilon \int_0^t x(s)\,ds.$$

*Then this statement is valid*

($\tilde{C}$) *Suppose there exists a constant* **M** *such that for any $\lambda \in (0,1)$ and any solution $x$ to the problem*

$$x'' - \varepsilon x = \lambda(f(t,x,x') - \varepsilon x)$$

*with boundary conditions (1.2) we have $||x||_1 <$* **M**. *Then the problem (1.1), (1.2) has a solution.*

**Proof** This theorem is a special case of the [1] theorem (3.2) part ($\tilde{C}$) for $k = 2$. □

# 2 A priori bounds on solutions

In this paragraph we present lemmas on a priori bounds on solutions of the differential equation (1.1) with the boundary conditions (1.2). Let $x$ be solution to (1.1) and set

$$h(t) = \frac{1}{2}x(t)^2.$$

**Lemma 2.1** *Let $f$ be a continuous function. Suppose there is a constant $M \geq 0$ such that $uf(t,u,p) > 0$ for $\forall t \in I$, $\forall u \in \mathbb{R}^n, ||u|| > M$ and $\forall p \in \mathbb{R}^n$, $pu = 0$.*

*If $x$ is a solution to the problem (1.1), (1.2), then $||x|| \leq M$.*

**Proof** If $||x(t)||$ achieves its maximum at $t = 0$ or $t = 1$, then $||x(t)||$ achieves this maximum at the points $t = a$ or $t = b$, and therefore we may assume that $||x(t)||$ achieves its maximum in a point $t_0 \in (0,1)$.

Suppose $||x(t_0)|| > M$. Let $h$ be the function introduced above. Since $x$ is the solution to the problem (1.1), $h$ is twice continuously differentiable function and $h$ has its maximum at $t_0 \in (0,1)$. Therefore

$$0 = h'(t_0) = x(t_0)x'(t_0)$$

and

$$0 \geq h''(t_0) = x(t_0)f(t_0, x(t_0), x'(t_0)) + x'(t_0)^2 \geq x(t_0)f(t_0, x(t_0), x'(t_0)).$$

This is a contradiction, which proves the lemma. □

Similar lemma may be proved for Carathéodory functions.

**Lemma 2.2** *Let $f$ be a Carathéodory function. Suppose there are constants $M \geq 0$, $\gamma > 0$ such that $uf(t,u,p) > 0$ for almost every $t \in I$, $\forall u \in \mathbb{R}^n$, $||u|| > M$ and $\forall p \in \mathbb{R}^n$, $|pu| < \gamma$.*

*If $x$ is a solution to the problem (1.1), (1.2), then $||x|| \leq M$.*

**Proof** This proof is very similar to the proof of lemma 2.1 By the same way as there we may prove that without loss of generality we may assume that $||x||$ achieves its maximum at internal point of interval $I$.

Suppose $||x(t_0)|| > M$. Let $h$ be the function introduced above. Since $x$ is the solution to the problem (1.1), $h$ has absolutely continuous first derivative. Assume $h$ has its maximum at $t_0 \in (0,1)$. Therefore there exists positive $\varepsilon$ such that

$$h(t_0) \geq h(t) \quad \text{for} \quad t \in (t_0 - \varepsilon, t_0 + \varepsilon)$$

$$h'(t_0) = x(t_0)x'(t_0) = 0$$

$$|h'(t)| < \gamma \quad \text{for} \quad t \in (t_0 - \varepsilon, t_0 + \varepsilon)$$

and

$$h''(t) = x(t)f(t, x(t), x'(t)) + x'(t)^2 \geq x(t)f(t, x(t), x'(t)) > 0$$

for almost every $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and integration from $t_0$ to $t \in (t_0, t_0 + \varepsilon)$ yields

$$h'(t) = \int_{t_0}^{t} h''(s)\, ds \ > 0$$

Therefore next integration on interval $(t_0, t_0 + \varepsilon)$ gives

$$h(t) = h(t_0) + \int_{t_0}^{t} h'(s)\, ds \ > h(t_0).$$

This is a contradiction, which proves the lemma. □

The next lemma is used to derive a priori bounds for the derivatives of solutions to (1.1).

**Lemma 2.3** *Let $f$ be continuous or a Carathéodory function.*

*(i) Let $M$ be fixed and $x(t)$ a solution to (1.1), (1.2) for which $||x|| \leq M$*

*(ii) Suppose there exists bounded on bounded sets (for example continuous) positive functions $A_j, B_j$, $j \in \{1, 2, \ldots, n\}$*

$$A_j : I \times \mathbb{R}^{n+j-1} \longrightarrow \mathbb{R}, \qquad B_j : I \times \mathbb{R}^{n+j-1} \longrightarrow \mathbb{R}$$

*such that for almost every $t \in I$*

$$|f_j(t, u, p)| \leq A_j(t, u, p_1, p_2, \ldots, p_{j-1})p_j^2 + B_j(t, u, p_1, p_2, \ldots, p_{j-1}),$$

*where $f = (f_1, f_2, \ldots, f_n)$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ $p = (p_1, p_2, \ldots, p_n)$ and for $j = 1$, $A_1$ and $B_1$ are independent of $p$ functions.*

*Then there is a constant $\tilde{M}$ depending only on $M, A_j, B_j$ such that*

$$||x'|| \leq \tilde{M}.$$

**Proof** Since by boundary conditions (1.2) $x(0) = x(a)$ there exist such points $t_j \in I$ that for $\forall j \in \{1, 2, \ldots, n\}$ derivative $x_j$ vanishes in them, it is $x_j'(t_j) = 0$, where $x = (x_1, x_2, \ldots, x_n)$.

Since $x_j'$ vanishes at least once on $I$, each point $t$ in $I$ for which $x_j'(t) \neq 0$ belongs to an interval $[\mu, \nu]$ such that $x_j'$ maintains a fixed sign on $[\mu, \nu]$ and $x_j'(\mu)$ or $x_j'(\nu)$ is zero. To be definite, assume that $x_1'(\mu) = 0$ and that $x_1' \geq 0$ on $[\mu, \nu]$. Since $A_1, B_1$ are bounded on bounded sets functions independent on $x'$ and $x$ is a bounded solution to the equation (1.1), there exist positive constants $\tilde{A}_1$, $\tilde{B}_1$ such that for almost every $t \in I$

$$|f_1(t, x, x')| \leq \tilde{A}_1 {x_1'}^2 + \tilde{B}_1.$$

From this

$$\frac{2\tilde{A}_1 x_1' x_1''}{\tilde{A}_1 {x_1'}^2 + \tilde{B}_1} \leq 2\tilde{A}_1 x_1',$$

and integration from $\mu$ to $t$ yields

$$\log \frac{\tilde{A}_1 {x_1'}^2(t) + \tilde{B}_1}{\tilde{B}_1} \leq 4\tilde{A}_1 M,$$

and from this

$$|x_1'(t)| \leq \left\{ \frac{\tilde{B}_1}{\tilde{A}_1} (e^{4\tilde{A}_1 M} - 1) \right\}^{\frac{1}{2}}.$$

By the same way we may treat the other possibilities that might occur. Arguing inductively we deduce componentwise bounds on each $x_j'$. From this it is easy to see that there exists a constant $\tilde{M}$ depending only on $M, A_j, B_j$ such that $||x'|| \leq \tilde{M}$. $\square$

## 3 Main results

In this paragraph we apply theory of topological transversality to our problem.

**Theorem 3.1** *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a continuous function and consider the problem (1.1), (1.2). Assume*

*(i) there is a constant $M \geq 0$ such that $uf(t,u,p) \geq 0$, $\forall t \in I$, $\forall u \in \mathbb{R}^n$, $||u|| > M$ and $\forall p \in \mathbb{R}^n$, $pu = 0$.*

*(ii) Suppose there exist continuous positive functions $A_j$, $B_j$, $j \in \{1, 2, \dots, n\}$*

$$A_j : I \times \mathbb{R}^{n+j-1} \to \mathbb{R}, \qquad B_j : I \times \mathbb{R}^{n+j-1} \to \mathbb{R}$$

*such that*

$$|f_j(t,u,p)| \leq A_j(t,u,p_1,p_2,\dots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\dots,p_{j-1}),$$

*where $f = (f_1, f_2, \dots, f_n)$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $p = (p_1, p_2, \dots, p_n)$ and for $j = 1$, $A_1$ and $B_1$ are independent of $p$ functions.*

*Then the problem (1.1), (1.2) has a solution.*

**Proof** Instead of a problem (1.1), (1.2) we will solve a problem

$$x'' - x = \lambda(f(t,x,x') - x) \tag{3.1}$$

with boundary conditions

$$x(0) = x(a), \qquad x(b) = x(1), \tag{3.2}$$

It is easy to see, that a trivial solution is the unique solution of a problem

$$x'' - x = 0,$$

with the boundary conditions (3.2). Therefore for $\varepsilon = 1$ and for $\tilde{g} = 0$ there the assumptions of the topological transversality theorem with exception to the statement $\tilde{C}$ are satisfied. Now we try to find a priori bounds of the solutions to problem (3.1), (3.2)

Equality (3.1) may be written in the form

$$x'' = \lambda f(t,x,x') + (1-\lambda)x = F(t,x,x',\lambda). \tag{3.3}$$

Let $M > 0$ and $f$ satisfy (i), then $uF(t,u,p,\lambda) > 0$ $\forall t \in I$ $\forall u \in \mathbb{R}^n$, $||u|| > M$, $\forall p \in \mathbb{R}^n$, $pu = 0$ and $\forall \lambda \in (0,1)$. And therefore for any solution to (3.1), (3.2) the inequality $||x|| \leq M$ is valid according to the lemma 2.1.

If in addition $f$ satisfies (ii) of this theorem, then

$$|F_j(t,u,p,\lambda)| \leq A_j(t,u,p_1,p_2,\dots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\dots,p_{j-1}) + M,$$

$\forall(t,u,p,\lambda) \in I \times \mathbb{R}^{2n} \times (0,1)$. Then lemma 2.3 implies that there is a constant $\tilde{M}$ independent on $\lambda$ such that $\|x'\| \le \tilde{M}$ is valid for solution $x$. Therefore there exists a positive constant $\mathbf{M}$ dependent only on $M$, $\tilde{M}$ such that for any solution $x$ to the problem (3.1), (3.2) $\|x\|_1 < \mathbf{M}$. Now all assumptions of topological transversality theorem are satisfied and therefore there exists at least one solution of the problem (1.1), (1.2). This proves the theorem. □

**Example 3.1** It follows from theorem 3.1 that the system

$$x_1'' = x_1 + {x_1'}^2 x_1 + \arctan(e^t + x_1 x_2)$$

$$x_2'' = \sin(2\pi t) + x_2 + e^{x_1'} x_2 + {x_2'}^2 x_2$$

has a solution satisfying boundary value conditions (3.2).

**Example 3.2** It follows from theorem 3.1 that the system

$$x_1'' = t + e^{x_1} x_1^2 \sin \frac{x_1}{1 + x_1^2 + x_2^2} - {x_1'}^2$$

$$x_2'' = \frac{1}{1 + |x_2| + \tan t} + x_2 + x_2 {x_1'}^{20} - x_1' x_2'$$

has a solution satisfying boundary value conditions (3.2).

**Theorem 3.2** *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function and consider the problem (1.1), (1.2). Assume*

*(i) there are constants $M \ge 0$, $\gamma > 0$ such that $uf(t,u,p) \ge 0$ for almost every $t \in I$, $\forall u \in \mathbb{R}^n$, $\|u\| > M$ and $\forall p \in \mathbb{R}^n$, $|pu| < \gamma$.*

*(ii) Suppose there exist bounded on bounded sets positive functions $A_j, B_j$, $j \in \{1, 2, \ldots, n\}$*

$$A_j : I \times \mathbb{R}^{n+j-1} \to \mathbb{R}, \qquad B_j : I \times \mathbb{R}^{n+j-1} \to \mathbb{R}$$

*such that for almost every $t \in I$*

$$|f_j(t,u,p)| \le A_j(t,u,p_1,p_2,\ldots,p_{j-1}) p_j^2 + B_j(t,u,p_1,p_2,\ldots,p_{j-1}),$$

*where $f = (f_1, f_2, \ldots, f_n)$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $p = (p_1, p_2, \ldots, p_n)$ and for $j = 1$, $A_1$ and $B_1$ are independent of $p$ functions.*

*Then the problem (1.1), (1.2) has a solution.*

**Proof** This proof is based as like as the proof of the theorem 3.1 on the application of lemmas 2.2 and 2.3. Similarly to the proof above instead of the problem (1.1), (1.2) we will solve the problem (3.1), (3.2)

By the same way as in the proof above we prove that there are satisfied assumptions of the topological transversality theorem with exception to the statement $\tilde{C}$. Similarly to the proof above equality (3.1) may be written in the form

$$x'' = \lambda f(t,x,x') + (1-\lambda)x = F(t,x,x',\lambda). \tag{3.4}$$

Let $M > 0$ and $f$ satisfy (i), then $uF(t,u,p,\lambda) > 0$ for almost every $t \in I$, $\forall u \in \mathbb{R}^n$, $||u|| > M$ and $\forall p \in \mathbb{R}^n$, $|pu| < \gamma$, $\forall \lambda \in (0,1)$. And according to the lemma 2.2 for any solution to (3.1), (3.2) the inequality $||x|| \leq M$ is valid .

If in addition $f$ satisfies (ii) of this theorem, then

$$|F_j(t,u,p,\lambda)| \leq A_j(t,u,p_1,p_2,\ldots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\ldots,p_{j-1}) + M,$$

for almost every $t \in I$ and $\forall (u,p,\lambda) \in \mathbb{R}^{2n} \times (0,1)$. Then lemma 2.3 implies that there is a constant $\tilde{M}$ independent on $\lambda$ such that $||x'|| \leq \tilde{M}$. Therefore there exists a positive constant $\mathbf{M}$ dependent only on $M$, $\tilde{M}$ such that for any solution $x$ to the problem (3.1), (3.2) $||x||_1 < \mathbf{M}$. Now all assumptions of topological transversality theorem are satisfied and therefore there exists at least one solution of the problem (1.1), (1.2). This proves the theorem. □

**Example 3.3** *Let $\zeta \in L^\infty(I)$ be a positive function, it follows from theorem 3.2 that the system*

$$x_1'' = \zeta(t)\left(x_1 + x_1' {x_2}^2 + t\right)$$
$$x_2'' = \zeta(t)\left(x_2 + x_2 e^{x_1'} + x_2' {x_2}^2\right)$$

*with boundary conditions (3.2) has a solution.*

# References


[1] Granas, A., Guenther, R., Lee, J.: *Some existence principles in the the Carathéodory theory of nonlinear differential systems,* J. Math. Pures Appl., (9) 70 (1991), no 2, 153–196.

[2] Granas, A., Guenther, R., Lee, J.: *Nonlinear boundary value problems for ordinary differential equations,* Dissertationes Math., 244, Warszaw, 1985.

[3] Andres, J.: *A four-point boundary value problem for the second-order ordinary differential equations,* Arch. Math., Basel, 53 (1989), 384–389.

[4] Andres, J., Vlček, V.: *On four-point regular BVPs for second-order quasi-linear ODEs,* Acta UP Olomucensis, Fac. rer. nat., 105 (1992), 37–44.

[5] Rachůnková, I.: *A four point problem for differential equations of the second order,* Arch. Math., Brno, 25 (1989), 175–184.

[6] Rachůnková, I.: *Existence and uniqueness of solutions of four-point boundary value problems for 2nd order differential equations,* Czech. Math. J., Praha, 39 (1989), 692–700.

[7] Šeda, V.: *A correct problem at a resonance,* Diff. and Int. Eq., 2 (1989), 389–396.



**Author's address:** Department of Algebra and Geometry,
Faculty of Science,
Palacký university,
Tomkova 40, Hejčín
771 46 Olomouc,
Czech Republic
[calabek@risc.upol.cz]