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B-SPLINE REPRESENTATION OF INTERPOLATING AND SMOOTHING QUADRATIC SPLINE

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Abstract

The algorithms for computation of the coefficients of the B-spline representation of quadratic splines interpolating derivatives, local mean-values and quadratic smoothing splines are given. The algorithms for piecewise polynomial representation were studied in [2]–[6]; the relation between these representations is mentioned.

Key words: splines, quadratic splines, B-splines, interpolating and smoothing splines

MS Classification: 41A15, 65D05

1 Introduction

The problem of interpolation or smoothing of given values of the derivatives or mean-values by some quadratic spline was formulated and solved in [2]–[6]. It was proved there, that the solution of the smoothing problem is given through some "natural quadratic spline". The algorithm for computing of needed parameters for piecewise polynomial representation using the function values and the first derivatives at knots of spline was given in [5], [6].

The purpose of this contribution is to present the algorithm for computing the coefficients of the B-spline representation of interpolating and smoothing quadratic splines.

1.1 Quadratic B-splines

Let us have the mesh of spline knots

$$(\Delta x) \qquad a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$$

Denote $\mathcal{S}_{21}(\Delta x)$ the linear space of quadratic splines on the mesh (Δx). The dimension of $\mathcal{S}_{21}(\Delta x)$ is

$$\dim \mathcal{S}_{21}(\Delta x) = 3(n+1) - 2n = n + 3.$$

As the basis in $\mathcal{S}_{21}(\Delta x)$ we will use the system of quadratic B-splines, defined on the extended mesh (see [1])

$$(\Delta \bar{x}) \qquad x_{-2} \le x_{-1} \le x_0 < x_1 < \ldots < x_{n+1} \le x_{n+2} \le x_{n+3}.$$

To each knot x_i, $i = -2(1)n$ there corresponds some quadratic B-spline $B_i(x) \in C^1[x_{-2}, x_{n+3}]$ with support $[x_i, x_{i+3}]$ defined by divided difference (see [1]).

$$B_i(x) = (x_{i+3} - x_i)[x_i, x_{i+1}, x_{i+2}, x_{i+3}](t - x)^2_+ =$$

$$(1) \quad = \begin{cases} (h_{i+1} + h_{i+2})^{-1}\{h_{i+2}^{-1}[(x_{i+3} - x)^2 - (x_{i+2} - x)^2] - \\ \quad -h_{i+1}^{-1}[(x_{i+2} - x)^2 - (x_{i+1} - x)^2]\} - \\ \quad -(h_i + h_{i+1})^{-1}\{h_{i+1}^{-1}[(x_{i+2} - x)^2 - (x_{i+1} - x)^2] - \\ \quad -h_i^{-1}[(x_{i+1} - x)^2]\} & \text{for } x \in [x_i, x_{i+1}]; \\ (h_{i+1} + h_{i+2})^{-1}\{h_{i+2}^{-1}[(x_{i+3} - x)^2 - (x_{i+2} - x)^2] - \\ \quad -[(h_i + h_{i+1})]h_{i+1}]^{-1}(x_{i+2} - x)^2 & \text{for } x \in [x_{i+1}, x_{i+2}]; \\ [(h_{i+1} + h_{i+2})]h_{i+2}]^{-1}(x_{i+3} - x)^2 & \text{for } x \in [x_{i+2}, x_{i+3}]. \end{cases}$$

Let us mention that in practical computations the values of $B_i(x)$ are more frequently computed by some simple recursive formulas.

For every spline $s(x) \in \mathcal{S}_{21}(\Delta x)$ we have then representation

$$(2) \qquad s(x) = \sum_{j=-2}^{n} b_j B_j(x) \qquad \text{with some coefficients } b_j.$$

The values of B_i and B_i' at the knots x_j are given in Table 1.

mesh	equidistant		general	
knot	$B_i(x)$	$B_i'(x)$	$B_i(x)$	$B_i'(x)$
x_i	0	0	0	0
x_{i+1}	$\frac{1}{2}$	$\frac{1}{h}$	$\frac{h_i}{(h_i + h_{i+1})}$	$\frac{2}{(h_i + h_{i+1})}$
x_{i+2}	$\frac{1}{2}$	$-\frac{1}{h}$	$\frac{h_{i+2}}{(h_{i+1} + h_{i+2})}$	$-\frac{2}{(h_{i+1} + h_{i+2})}$
x_{i3}	0	0	0	0

The graph of $B_i(x)$ we can see on Fig. 1.

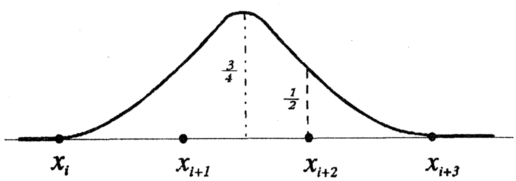

Fig. 1

2 Interpolation of function values and mean-values

2.1 Function values interpolation

Let us have the points of interpolation $t_i \in [a, b]$ with prescribed values y_i. The general problem of existence of interpolating spline $s(x) \in \mathcal{S}_{21}(\Delta x)$ with $s(t_i) = y_i$ is solved for example in ([1], Theorem of Schoenberg-Whitney). The coefficients of the B-spline representation (2) of such spline can be found through solution of the system

$$y_i = \sum_j b_j B_j(t_i) \qquad \text{with } n+3 \text{ points of interpolation (or some boundary conditions).}$$

The algorithms for computing local parameters of quadratic interpolating splines are studied in [2].

2.2 Mean–values interpolation

The problem of the existence of the quadratic spline interpolating the given local mean values

$$(3) \qquad g_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x)dx, \quad i = 0(1)n \qquad \text{on the mesh } (\Delta x)$$

is solved in [6], where also the algorithm for computing of parameters for the piecewise polynomial representation of $s(x)$ is given.

Now, let us search the solution of this problem in the B-spline representation (2). Denoting

$$(4) \qquad c_j^i = \int_{x_i}^{x_{i+1}} B_j(x)dx,$$

we have

$$(5) \qquad \begin{aligned} c_j^i &= 0 \quad \text{for } j \notin \{i-2, i-1, i\}, \\ c_{i-2}^i &= \textstyle\int_{x_i}^{x_{i+1}} B_{i-2}(x)dx = \frac{1}{3}h_i^2/(h_{i-1}+h_i) \\ c_{i-1}^i &= \textstyle\int_{x_i}^{x_{i+1}} B_{i-1}(x)dx = \\ &= \frac{1}{2}h_i[\frac{h_{i-1}}{h_{i-1}+h_i} + \frac{h_i}{h_i+h_{i+1}} + \frac{1}{3}h_i[\frac{1}{h_{i-1}+h_i} + \frac{1}{h_i+h_{i+1}})], \\ c_i^i &= \textstyle\int_{x_i}^{x_{i+1}} B_i(x)dx = \frac{1}{3}h_i^2/(h_i+h_{i+1}). \end{aligned}$$

The conditions of interpolation (3) can be written now as

$$(6) \qquad \begin{aligned} &\textstyle\int_{x_i}^{x_{i+1}} s(x)dx = \int_{x_i}^{x_{i+1}} \sum_{j=-2}^{n} b_j B_j(x) = \\ &= \textstyle\sum_{j=-2}^{n} b_j \int_{x_i}^{x_{i+1}} B_j(x)dx = b_{i-2}c_{i-2}^i + b_{i-1}c_{i-1}^i + b_i c_i^i. \end{aligned}$$

To define the spline $s(x)$ uniquely, we can prescribe e.g. the boundary conditions (see [6])

$$(7) \qquad s_0 = s(x_0), \qquad s_{n+1} = s(x_{n+1}).$$

The relations (6),(7) form the system of equations

$$(8) \qquad \begin{aligned} b_{-2}B_{-2}(x_0) + b_{-1}B_{-1}(x_0) \qquad\qquad &= s_0 \\ b_{-2}c_{-2}^0 + b_{-1}c_{-1}^0 + b_0 c_0^0 \qquad &= h_0 g_0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_{n-2}c_{n-2}^n + b_{n-1}c_{n-1}^n + b_n c_n^n &= h_n g_n \\ b_{n-1}B_{n-1}(x_{n+1}) + b_n B_n(x_{n+1}) &= s_{n+1}. \end{aligned}$$

Theorem 1 *For the mesh (Δx) with given mean values g_i and*

$$(9) \qquad h_{i+1} \geq h_i(2h_i - h_{i-1})/(6h_{i-1} + 3h_i), \quad h_{-1} \leq h_0, \quad h_n \geq h_{n+1},$$

the B-spline coefficients b_j of the unique quadratic spline interpolating this mean-values are determined by the system of equations (8) with the coefficients c_j^i given in (5).

The proof follows from the fact that the matrix of the system (8) is irreducibly diagonally dominant under conditions (9) and with respect to (5).

Remark 1 In the case of equidistant mesh (Δx) with $h_i = h$ the system (8) can be written as

$$(10) \qquad \begin{bmatrix} 1, & 1, & & & \\ 1, & 4, & 1, & & \\ & \cdots & \cdots & \cdots & \\ & & 1, & 4, & 1, \\ & & & 1, & 1, \end{bmatrix} \begin{bmatrix} b_{-2} \\ b_{-1} \\ \cdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} 2s_0 \\ g_0 \\ \cdots \\ g_n \\ 2s_{n+1} \end{bmatrix} .$$

3 Interpolation of the derivatives

The problem of interpolation of the first derivatives prescribed at the knots of the quadratic spline was discussed in [4] and solved with respect to the piecewise polynomial representation of the spline. For the extended knot set $(\Delta\bar{x})$ with prescribed values $m_i = s'(x_i)$ let us search now for the interpolating spline in its B-spline representation (2). The conditions of interpolation and the initial value $s_0 = s(x_0)$ determine uniquely the spline (see [4]). For the coefficients b_i we obtain the relations

$$(11) \qquad \begin{aligned} s_0 &= \sum_{j=-2}^{n} b_j B_j(x_0) \, , \\ m_i &= \sum_{j=-2}^{n} b_j B_j'(x_i), \qquad i = 0(1)n+1 \, . \end{aligned}$$

This system of $n+3$ linear equations with the special band matrix can be solved (using the values given in the Table 1) by recursion.

Theorem 2 *The B-spline coefficients of the quadratic spline $s(x)$ interpolating the given values $m_i = s'(x_i)$, $i = 0(1)n+1$ and $s_0 = s(x_0)$ are given uniquely by the recurrence formula*

$$(12) \qquad \begin{aligned} b_{-2} &= s_0 - \tfrac{1}{2}h_{-1}m_0 \\ b_{i-1} &= \tfrac{1}{2}m_i(h_{i-1} + h_i) + b_{i-2}, \qquad i = 0(1)n+1 \end{aligned}$$

Remark 2 More generally, we can determine uniquely the quadratic spline interpolating given values $m_i = s'(x_i), \quad i = 0(1)n+1$ and one function value s_k. By elimination of b_{k-1}, b_{k-2} from the equations with s_k, m_k we can write forward and backward recurrence relations for the calculation of remaining coefficients b_i.

4 Quadratic smoothing spline

4.1 Statement of the problem

Let us have the local mean-values

$$g_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} f(x)dx, \qquad h_i = x_{i+1} - x_i$$

prescribed for each interval $[x_i, x_{i+1}]$, $i = 0(1)n$ of the mesh (Δx). We can define the functional

$$J(f) = \int_a^b [f'(x)]^2 dx + \alpha \sum_{i=0}^{n} w_i [h_i g_i - \int_{x_i}^{x_{i+1}} f(x)dx] \tag{13}$$
$$\text{for } f \in V = W_2^1(a,b), \quad \alpha \geq 0$$

which represents some compromise between different quantitative measures of the function $f(x)$

a) the measure of the speed of the changes in the process described by the function f (the first part with f');

b) the least-square deviation between the data g_i and the local mean-values of the process described by the function $f(x)$ (with the weight coefficients w_i — the second part). The balance between these two parts can be regulated by the parameter α. It is proved in [5] that the function minimizing $J(f)$ on the class V is some quadratic spline $s(x)$ with "natural" boundary conditions

$$s'(x_0) = s'(x_{n+1}) = 0. \tag{14}$$

There is also given the algorithm for computation of the local parameters of such spline in piecewise polynomial representation.

4.2 B–spline representation of the smoothing spline

The aim of this contribution is to present the algorithm for computing the global coefficients of the smoothing spline in the B-spline representation (2) . Writting

$$s(x) = \sum_{j=-2}^{n} b_j B_j(x), \quad \text{we have} \quad s'(x) = \sum_{j=-2}^{n} b_j B'(x). \tag{15}$$

The boundary conditions (14) could be written now as

$$b_{-2}B'_{-2}(x_0) + b_{-1}B'_{-1}(x_0) = 0$$
$$b_{n-1}B'_{n-1}(x_{n+1}) + b_n B'_n(x_{n+1}) = 0.$$

When we use the values $B'(x_0), B'(x_{n+1}), j = -2, -1, n-1, n$ from the Table 1, we obtain

$$\begin{aligned} b_{-2} &= -[B'_1(x_0)/B'_{-2}(x_0)]b_{-1} = b_{-1}, \\ b_n &= -[B'_{n-1}(x_{n+1})/B'_n(x_{n+1})]b_{n-1} = b_{n-1}. \end{aligned} \tag{16}$$

We have

$$[s'(x)]^2 = \sum_{i=-2}^{n} b_i B'_i(x) \sum_{j=-2}^{n} b_j B'_j(x) = \sum_{i,j=-2}^{n} b_i b_j B'_i(x) B'_j(x) = \boldsymbol{b}^T \boldsymbol{D}_1 \boldsymbol{b}$$

with pentadiagonal matrix

$$\boldsymbol{D}_1 = [B'_i(x).B'_j(x)] = [d^1_{ij}(x)], \; d^1_{ij}(x) \equiv 0 \quad \text{for } |i-j| \geq 3.$$

For the first part of the functional (13) we have then

$$\int_a^b [s'(x)]^2 dx = \boldsymbol{b}^T \quad [\int_a^b B'_i(x) B'_j(x) dx]\boldsymbol{b} = \boldsymbol{b}^T \boldsymbol{D}\boldsymbol{b} \tag{17}$$

with

$$\boldsymbol{D} = [d_{ij}] = [\int_a^b B'_i(x) B'_j(x) dx]. \tag{18}$$

The value of the functional $J(s)$ can be written now as the function of the coefficients $[b_{-2}, b_{-1}, \ldots, b_n]$:

$$\begin{aligned} J(s) &= \int_a^b [s'(x)]^2 dx + \alpha \sum_{i=0}^{n} w_i [h_i g_i - \int_{x_i}^{x_{i+1}} s(x) dx] = \\ &= \sum_{i,j=-2}^{n} b_i d_{ij} b_j + \alpha \sum_{i=0}^{n} w_i [h_i g_i - (b_{i-2} c^i_{i-2} + b_{i-1} c^i_{i-1} + b_i c^i_i)]^2. \end{aligned} \tag{19}$$

The coefficients d_{ij} can be determined by the relations

$$\begin{aligned} d_{ij} &= \int_a^b B'_i(x) B'_j(x) dx = \int_{x_i}^{x_{i+1}} B'_i(x) B'_j(x) = \\ &= \sum_{k=i}^{i+2} [B'_k(x) B_j(x)|_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} B''_k(x_k+0) B_j(x)] = \\ &= -B''_i(x_i+0) \int_{x_i}^{x_{i+1}} B_j(x) dx - B''_{i+1}(x_{i+1}+0) \int_{x_{i+1}}^{x_{i+2}} B_j(x) - \\ &\quad - B''_{i+2}(x_{i+2}+0) \int_{x_{i+2}}^{x_{i+3}} B_j(x) dx = \beta_i c^i_j + \beta_{i+1} c^{i+1}_j + \beta_{i+2} c^{i+2}_j. \end{aligned} \tag{20}$$

The coefficients c_j^i are given by (8); for the coefficients β_j we obtain (using the values from Table 1)

$$
\begin{aligned}
&\beta_i = -2/[h_i(h_i + h_{i+1})], \\
(21) \qquad &\beta_{i+2} = -2/[h_{i+2}(h_{i+1} + h_{i+2})] \\
&\beta_{i+1} = -(2/h_{i+1})[(h_i + h_{i+1})^{-1} + (h_{i+1} + h_{i+2})^{-1}].
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
(22) \qquad &\boldsymbol{c}^i = [0, \ldots, c_{i-2}^i, c_{i-1}^i, c_i^i, 0, \ldots]^T, \quad i = 0(1)n, \\
&\boldsymbol{b} = [b_{-2}, b_{-1}, \ldots, b_n]^T = [b_{-1}, b_{-1}, b_0, \ldots, b_{n-1}, b_{n-1}]^T.
\end{aligned}
$$

We can write then (19) as

$$
\begin{aligned}
J(s) &:= F(\boldsymbol{b}) = \sum_{i,j=-2}^{n} b_i d_{ij} b_j + \\
(23) \qquad &+\alpha \sum_{i=0}^{n} w_i [h_i g_i - (b_{i-2} c_{i-2}^i + b_{i-1} c_{i-1}^i + b_i c_i^i)]^2 = \\
&= \boldsymbol{b}^T \boldsymbol{D} \boldsymbol{b} + \alpha \{ \sum_{i=0}^{n} w_i h_i^2 g_i^2 + \sum_{i=0}^{n} \{ (\boldsymbol{b}^T \boldsymbol{c}^i)^2 - 2 h_i g_i (\boldsymbol{b}^T \boldsymbol{c}^i)] \}
\end{aligned}
$$

with the function $F(\boldsymbol{b})$ of $n+1$ parameters b_j, $j = -1(1)n-1$. Following the chain rule of differentiation,

(24)

$$
\begin{aligned}
\frac{\partial F}{\partial b_{-1}} &= \sum_{j=-2}^{n} (d_{-2,j} + d_{-1,j} + d_{j,-1} + d_{j,-2}) b_j + \\
&\quad +2\alpha \sum_{j=-2}^{n} b_j \sum_{i=0}^{n} w_i c_j^i (c_{-1}^i + c_{-2}^i) - 2\alpha \sum_{i=0}^{n} w_i h_i g_i (c_{-1}^i + c_{-2}^i), \\
\frac{\partial F}{\partial b_k} &= \sum_{j=-2}^{n} b_j (d_{kj} + d_{jk}) + 2\alpha \sum_{j=-2}^{n} b_j \sum_{i=0}^{n} w_i c_k^i c_j^i - 2\alpha \sum_{i=0}^{n} w_i h_i g_i c_k^i, \\
&\qquad k = 0(1)n-2; \\
\frac{\partial F}{\partial b_{n-1}} &= \sum_{j=-2}^{n} b_j (d_{n-1,j} + d_{j,n-1} + d_{nj} + d_{jn}) + \\
&\quad +2\alpha \sum_{j=-2}^{n} b_j \sum_{i=0}^{n} w_i c_j^i (c_{n-1}^i + c_n^i) - 2\alpha \sum_{i=0}^{n} w_i h_i g_i (c_{n-1}^i + c_n^i).
\end{aligned}
$$

The necessary conditions for the extremum, $\frac{F}{b_i} = 0$, can be written now as the system of equations

$$\sum_{j=-1}^{n-1} a_{ij} b_j = r_i, \qquad i = -1(1)n-1, \tag{25}$$

where

$$a_{-1,-1} = 2(d_{-2,-2} + d_{-2,-1} + d_{-1,-2} + d_{-1,-1}) + 2\alpha \sum_{i=0}^{n} w_i (c_{-1}^i + c_{-2}^i)^2;$$

$$a_{-1,j} = a_{j,-1} = d_{-2,j} + d_{-1,j} + d_{j,-1} + d_{j,-2} + \\ +2\alpha \sum_{i=0}^{n} w_i c_j^i (c_{-2}^i + c_{-1}^i), \qquad j = 0(1)n-2;$$

$$a_{-1,n-1} = a_{n-1,-1} = d_{-2,n-1} + d_{-1,n-1} + d_{n-1,-2} + d_{n-1,-1} + \\ +d_{-2,n} + d_{-1,n} + d_{n,-2} + d_{n,-1} + 2\alpha \sum_{i=0}^{n} w_i (c_{-2}^i + c_{-1}^i)(c_{n-1}^i + c_n^i);$$

$$a_{kj} = a_{jk} = d_{kj} + d_{jk} + 2\alpha \sum_{i=0}^{n} w_i c_k^i c_j^i, \qquad j = 0(1)n-2, \quad k = 0(1)n-2;$$

$$a_{k,n-1} = a_{n-1,k} = d_{k,n-1} + d_{n-1,k} + d_{k,n} + d_{n,k} + \\ +2\alpha \sum_{i=0}^{n} w_i c_k^i (c_{n-1}^i + c_n^i), \qquad k = 0(1)n-2;$$

$$a_{n-1,n-1} = 2(d_{n-1,n-1} + d_{n-1,n} + d_{n,n-1} + d_{n,n}) + 2\alpha \sum_{i=0}^{n} w (c_{n-1}^i + c_n^i)^2;$$

$$\begin{aligned} r_{-1} &= 2\alpha \sum_{i=0}^{n} w_i h_i g_i (c_{-2}^i + c_{-1}^i) = 2\alpha \sum_{i=0}^{1} w_i h_i g_i (c_{-2}^i + c_{-1}^i); \\ r_j &= 2\alpha \sum_{i=0}^{n} w_i h_i g_i c_j^i = 2\alpha \sum_{i=j}^{j+2} w_i h_i g_i c_j^i, \qquad j = 0(1)n-2; \\ r_{n-1} &= 2\alpha \sum_{i=0}^{n} w_i h_i g_i (c_{n-1}^i + c_n^i) = 2\alpha \sum_{i=n-1}^{n} w_i h_i g_i (c_{n-1}^i + c_n^i). \end{aligned}$$

The matrix $\mathbf{A} = [a_{ij}]$ of the system (25) is symmetric ,

$$a_{ij} = 0 \qquad \text{for} \quad |j-i| > 2 \qquad \text{(pentadiagonal)},$$

as can be seen from its construction.

Theorem 3 *The B-spline coefficients of the quadratic spline minimizing the functional $J(f)$ given in (13) can be computed from the symmetric pentadiagonal system of linear equations (25).*

4.3 Conversion from local to B-spline representation

We have seen in part 4.2 that the computation of the B-spline coefficients of the smoothing spline from the system (25) requires considerable amount of operations(the calculation of the coefficients a_{ij}, d_i, solving pentadiagonal system) in comparision to the case of local piecewise polynomial (PP) representation (see [5]). In some cases it can be more advantegeous to calculate the parameters of PP representation and then use the transformation of PP — to B-spline representation described below (see also [1]).

Let us suppose that we have calculated the coefficients

$$s_k = s(x_k), \quad s'_k = s'(x_k), \qquad k = 0(1)n+1$$

of the PP-representation of some $s \in \mathcal{S}_{21}(\Delta \bar{x})$. To determine the coefficients b_k of the B-spline representation (2), we use the conditions

$$\begin{aligned} s(x_k) &= s_k = b_{k-2}B_{k-2}(x_k) + b_{k-1}B_{k-1}(x_k), \\ s'(x_k) &= s'_k = b_{k-2}B'_{k-2}(x_k) + b_{k-1}B'_{k-1}(x_k) \end{aligned} \tag{26}$$

with the values $B_i^{(j)}(x)$ given in the Table 1. We obtain

$$b_{k-2} = s_k - \frac{1}{2}h_{k-1}s'_k, \qquad b_{k-1} = s_k + \frac{1}{2}h_k s'_k. \tag{27}$$

The condition

$$s(x_{k+1}) = s_{k+1} = b_{k-1}B_{k-1}(x_{k+1}) + b_k B_k(x_{k+1})$$

enables us to calculate then

$$b_k = s_{k+1} + b_{k+1}[(s_{k+1} - s_k)/h_k - s'_k/2] \tag{28}$$

Theorem 4 *Given the local parameters s_k, s'_k, $k = 0(1)n+1$ of the spline $s \in \mathcal{S}_{21}(\Delta \bar{x})$, we can determine its B-spline coefficients b_k, $k = -2(1)n$ according to the relations (27), (28).*

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