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ON FOUR-POINT REGULAR BVPs FOR SECOND-ORDER QUASI-LINEAR ODEs

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Abstract. Sufficient conditions for the existence of a solution to four-point boundary value problems for the second-order quasi-linear ordinary differential equations are given by means of the Schauder fixed point theorem.

Key words: Four-point problem, Green's function.

MS Classification: 34B10

1. In this note we will consider the boundary value problem (BVP):

(1) $x''=f(t,x,x')$, $f\in C(\langle\alpha,\beta\rangle\times R^2)$,

(2) $x(a)+px(b)=A$, $x(c)+qx(d)=B$,

where $A,B,a,b,c,d\in R^1$, $\alpha=\min\{a,b,c,d\}$, $\beta=\max\{a,b,c,d\}$; $p,q\in\{-1,0,1\}$.

So far, only multi-point problems for $p=1$, $q=0$ and $b\in(a,c)$ (see [4]) or for $p=q=1$ (see [6],[7]) and for the special case, when $a=0$, $d=b+c$, (see [1]) have been studied.

2. Besides (1)-(2), consider still the linear homogeneous BVP:

(3) $x''+kx=0,\ k\in R^1$,

(4) $x(a)= -px(b),\quad x(c)= -qx(d)$,

where $p,q\in\{-1,0,1\}$.

It is well-known (cf. e.g.[8]) that (1)-(4) is equivalent to the integral equation

(5) $$x(t)= \int_{\alpha}^{\beta} G(t,s)[kx(s)+f(s,x(s),x'(s))]ds:=F(x(t)),$$

as far as Green's function $G(t,s)$ [related to (3)-(4)] exists. This is true (see e. g. [8] again) if BVP (3)-(4) has only the trivial solution. Furthermore, since integral operators originated from solving the BVPs to ODEs are totally continuous, because Green's functions involved in these problems are continuous (see [3,p.123] and [5,p.25]), it is sufficient to verify that a closed convex subset S of the Banach space E of all continuously differentiable functions $x(t)$ on the interval $\langle\alpha,\beta\rangle$ with the norm

$$\|x(t)\| := \max_{t\in\langle\alpha,\beta\rangle} [|x(t)|+|x'(t)|]$$

exists such that [cf.(5)]

(6) $$F(S)\subset S,$$

in order to apply the well-known Schauder fixed point theorem (see e.g. [3,p.322]).

3. As we have just pointed out, our problem reduces to the question of

(i) the nonexistence of any nontrivial solution to (3)-(4), and (ii) the verification of (6).

Hence, let us begin with the first requirement.

For $k=0$ or $k<0$ or $k>0$ in (3), substituting

$$x(t)=C_1t+C_2 \quad \text{or} \quad x(t)=C_1\operatorname{ch}\sqrt{-k}t + C_2\operatorname{sh}\sqrt{-k}t$$

or $\quad x(t)=C_1\cos t\sqrt{k}+C_2\sin t\sqrt{k}\quad ,\quad C_{1,2}\in R^1,\quad$ into (4),

we obtain the system the determinant of which differs from zero iff

$$(a+pb)(1+q)\neq(c+qd)(1+p),$$

or

$$(\mathrm{ch}\sqrt{-k}a+p\,\mathrm{ch}\sqrt{-k}b)(\mathrm{sh}\sqrt{-k}c+q\,\mathrm{sh}\sqrt{-k}d)\neq$$
$$\neq(\mathrm{sh}\sqrt{-k}a+p\,\mathrm{sh}\sqrt{-k}b)(\mathrm{ch}\sqrt{-k}c+q\,\mathrm{ch}\sqrt{-k}d),$$

or

$$(\cos\sqrt{k}a+p\cos\sqrt{k}b)(\sin\sqrt{k}c+q\sin\sqrt{k}d)\neq$$
$$\neq(\sin\sqrt{k}a+p\sin\sqrt{k}b)(\cos\sqrt{k}c+q\cos\sqrt{k}d),$$

respectively.

Taking into account the values of $p\in\{-1,0,1\}$, we can easily arrive at the conditions stated below in the form of the following three lemmas.

Lemma 1. *Problem (3)-(4) has for $p=-1(a\neq b)$ only a trivial solution, provided*

$k=0,\ q\neq-1,$
or $k<0,\ q\neq-1,\ d=c,$
or $k<0,\ q\neq-1,\ a+b=c+d,$
or $k<0,\ q=-1,\ c\neq d,\ a+b\neq c+d.$

Lemma 2. *Problem (3)-(4) has for $k\leq0$ only a trivial solution, provided*

1) $p=0$ and
$q\neq-1,\ c=d,\ a\neq c,$
or $q\neq0,\ a=c,\ a\neq d,$
or $q\neq1,\ a=(c+d)/2,\ a\neq c,$
or $q\neq1,\ a=d,\ a\neq c,$
or $q=-1,\ c\neq d,$
or $q=0,\ a\neq c,$
or $q=1,\ a\neq(c+d)/2;$

2) $p=1$ and
$q\neq-1,\ c=d,\ c\neq(a+b)/2,$
or $q\neq0,\ c=(a+b)/2,\ d\neq(a+b)/2,$
or $q\neq1,\ c+d=a+b,\ c\neq(a+b)/2,$
or $q\neq1,\ d=(a+b)/2,\ c\neq(a+b)/2,$
or $q=-1,\ c\neq d,$

or $q=0$, $c\neq(a+b)/2$,

or $q=1$, $a+b\neq c+d$.

Lemma 3. *Problem (3)-(4) has for $k>0$ only a trivial solution, provided*

1) $p=-1$, $b\neq a+2m\pi/\sqrt{k}$ and

$q\neq-1$, $d=c+2j\pi/\sqrt{k}$, $c\neq(a+b)/2+(2n+1)\pi/\sqrt{k}$,

or $q\neq0$, $c=(a+b)/2+(2j+1)\pi/2\sqrt{k}$, $d\neq(a+b)/2+(2n+1)\pi/2\sqrt{k}$,

or $q\neq1$, $d=c+(2j+1)\pi/\sqrt{k}$, $c\neq(a+b)/2+(2n+1)\pi/2\sqrt{k}$,

or $q\neq1$, $c\neq(a+b)/2+(2j+1)\pi/2\sqrt{k}$, $d=(a+b)/2+(2n+1)\pi/2\sqrt{k}$,

or $q=-1$, $c+d\neq a+b+2j\pi/\sqrt{k}$, $d\neq c+2n\pi/\sqrt{k}$,

or $q=0$, $c\neq(a+b)/2+(2j+1)\pi/2\sqrt{k}$,

or $q=1$, $c+d\neq a+b+(2j+1)\pi/\sqrt{k}$, $d\neq c+(2n+1)\pi/\sqrt{k}$;

2) $p=0$ and

$q\neq1$, $d=c+2j\pi/\sqrt{k}$, $c\neq a+n\pi/\sqrt{k}$,

or $q\neq0$, $c=a+j\pi/\sqrt{k}$, $d\neq a+n\pi/\sqrt{k}$,

or $q\neq1$, $a=(c+d)/2+j\pi/\sqrt{k}$, $d\neq a+n\pi/\sqrt{k}$,

or $q\neq1$, $c=d+(2j+1)\pi/\sqrt{k}$, $d\neq a+n\pi/\sqrt{k}$,

or $q\neq1$, $a=(c+d)/2+2j\pi/\sqrt{k}$, $c\neq d+2n\pi/\sqrt{k}$

or $q=-1$, $a=(c+d)/2+(2j+1)\pi/\sqrt{k}$, $d\neq c+2n\pi/\sqrt{k}$,

or $q=0$, $c\neq a+j\pi/\sqrt{k}$,

or $q=1$, $a\neq(c+d)/2+j\pi/\sqrt{k}$, $d\neq c+(2n+1)\pi/\sqrt{k}$;

3) $p=1$, $b\neq a+(2m+1)\pi/\sqrt{k}$ and

$q\neq-1$, $d=c+2j\pi/\sqrt{k}$, $c\neq(a+b)/2+n\pi/\sqrt{k}$,

or $q\neq0$, $c=(a+b)/2+j\pi/\sqrt{k}$, $d\neq(a+b)/2+n\pi/\sqrt{k}$,

or $q\neq1$, $c+d=a+b+2j\pi/\sqrt{k}$, $d\neq(a+b)/2+n\pi/\sqrt{k}$,

or $q\neq1$, $c=d+(2j+1)\pi/\sqrt{k}$, $d\neq(a+b)/2+n\pi/\sqrt{k}$,

or $q\neq1$, $(a+b)/2=(c+d)/2+2j\pi/\sqrt{k}$, $d\neq c+n\pi/\sqrt{k}$,

or $q=1$, $c+d\neq a+b+(2j+1)\pi/\sqrt{k}$, $d\neq c+2n\pi/\sqrt{k}$,

or $q=0$, $c\neq(a+b)/2+j\pi/\sqrt{k}$,

or $q=1$, $c+d\neq a+b+2j\pi/\sqrt{k}$, $d\neq c+(2n+1)\pi/\sqrt{k}$,
where $j,m,n\in\{0,\pm1,\pm2,\dots\}$.

4. Denoting (see Section 2)

$$S:=\{\mathbf{x}(t)\in B:\ \|\mathbf{x}(t)\|\le D,\ D\in R^{+}\},$$

it is obvious that S is closed convex set. Thus, it is sufficient to prove that $\|F(\mathbf{x}(t))\|\le D$ with a suitable D for all $\mathbf{x}(t)\in S$ in order to satisfy (ii) (see Section 3).
Assuming that suitable function $F(t,r)$ exists which is piece-wise continuous in $t\in\langle\alpha,\beta\rangle$, $r\ge0$, and nondecreasing (for fixed t) with respect to r such that

(7) $|kx+f(t,x,y)|\le F(t,|x|+|y|)$ for $t\in\langle\alpha,\beta\rangle$, $(x,y)\in R^2$,

we can give the following

Lemma 4. *Let the assumptions of Lemma 1 or Lemma 2 or Lemma 3 be satisfied. If a nonnegative constant D exists such that*

(8) $\max\limits_{t\in\langle\alpha,\beta\rangle} F(t,D)\le D/(\beta-\alpha)G \quad (\alpha\neq\beta)$,

where $G:=\max\limits_{t\in\langle\alpha,\beta\rangle}\{\max\limits_{s\in\langle\alpha,\beta\rangle}[\,|G(t,s)|+\left|\frac{\partial G(t,s)}{\partial t}\right|\,]\}\quad(>0)$,

$G(t,s)$ is Green's function related to (3)-(4), then

$$\|F(x(t))\|\le D \text{ for all } x(t)\in S.$$

Proof. Let $\mathbf{x}(t)$ be a continuously differentiable function from S. Applying (7), (8), we obtain that

$$\|F(\mathbf{x}(t))\|=\|\int_\alpha^\beta G(t,s)[k\mathbf{x}(s)+f(s,\mathbf{x}(s),\mathbf{x}'(s))]ds\|\le$$

$$\le\max_{t\in\langle\alpha,\beta\rangle}\int_\alpha^\beta\Big\{|G(t,s)[k\mathbf{x}(s)+f(s,\mathbf{x}(s),\mathbf{x}'(s))]|+$$

$$+\left|\frac{\partial G(t,s)}{\partial t}[k\mathbf{x}(s)+f(s,\mathbf{x}(s),\mathbf{x}'(s))]\right|\Big\}ds\le\max_{t\in\langle\alpha,\beta\rangle}F(t,D)(\beta-\alpha)G\le D.$$

This completes the proof.

Remark 1. The same can be proved, when applying directly Bihari's theorem in [2].

Remark 2. Conditions (7), (8) are evidently fulfilled, provided the existence of nonnegative constants M_0, M such that

$$|kx+f(t,x,y)|\le M_0+M(|x|+|y|) \text{ for } t\in\langle\alpha,\beta\rangle,\ (x,y)\in R^2,$$

where $M<(\beta-\alpha)^{-1}G^{-1}$.

Remark 3. One can already easily deduce that under the assumptions of Lemma 1 or Lemma 2 or Lemma 3 problem (1)-(4) admits a solution, provided

$$(9)\quad \lim_{\|(x,y)\|\to\infty} \frac{\|kx + f(t,x,y)\|}{\|(x,y)\|}=0 \quad \text{uniformly to } t\in\langle\alpha,\beta\rangle$$

with the appropriate norm $\|\ .\ \|$.

5. It can be readily checked that $x_0(t)$ satisfies equation

$$x''=f(t,x-P(t),x'-P'(t))$$

with conditions (4) iff $x(t)=x_0(t)+P(t)$, where $P(t)$ is a suitable polynomial, is a solution of (1)-(2). Therefore, (1)-(2) is certainly solvable under the restrictions of Lemma 4, but (7), which reads here

$$(10)\qquad |k[x+P(t)]+f(t,x-P(t),y-P'(t))|\le F(t,|x|+|y|),$$

where $t\in\langle\alpha,\beta\rangle$, $(x,y)\in R^2$, and $P(t)$ is such that

$$P(a)+pP(b)=A, \qquad P(c)+qP(d)=B.$$

It is evident that (10) is satisfied, when

$$(10)\qquad |k(x+\varepsilon_1)+f(t,x+\varepsilon_1,y+\varepsilon_2)|\le F(t,|x|+|y|)$$

holds for all $t\in\langle\alpha,\beta\rangle$, $(x,y)\in R^2$, $\varepsilon_1\in\langle -P,P\rangle$ and $\varepsilon_2\in\langle -P',P'\rangle$, where

$$P:=\max_{t\in\langle\alpha,\beta\rangle} |P(t)|, \quad P':=\max_{t\in\langle\alpha,\beta\rangle} |P'(t)|.$$

According to monotonicity of $F(t,r)$ in r, it is, furthermore, obvious that (11) can be still replaced by

$$(12)\qquad |kx+f(t,x,y)|\le F(t,|x|+|y|-P-P'),$$

where $t\in\langle\alpha,\beta\rangle$, $(x,y)\in R^2$.

Remark 4. For the function $[kx+f(t,x,y)]$ bounded in a linear way (see Remark 2), and all the better in a sublinear way (see Remark 3), the same conclusion can be certainly done (i. e. without any modification of the growth restrictions) with respect to (1)-(2).
Therefore, we can give the main result.

Theorem. *Let the assumptions of Lemma 1 or Lemma 2 or Lemma 3 be satisfied. If condition (9) is still fulfilled, then problem (1)-(2) admits a solution.*

Remark 5. Knowing the explicid form of Green's function to (3)-(4), we can qualitatively improve the above assertion by means of (9) replaced by (12) [cf.(8)].

Remark 6. Another improvement consists of the application of the a priori estimates technique.

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