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STATISTICAL ANALYSIS OF PROBABILISTIC MODEL - VARIANT I OF THE SCHOOL-ACHIEVEMENT TEST WITH DOUBLE-CHOICE RESPONSE

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Abstract. The purpose of the present paper is to estimate statistically the parameter τ of the probabilistic model of school-achievement test with double choice response as described in article [1], and to evaluate its properties. Numerical results of point estimation and confidence intervals are summarized in tables printed by the personal computer PP06.

Key words: Statistical analysis of probalistic model, school-achievement test with double choice response, point estimation, confidence interval

MS Clasification: 62P10, 62P15

Introduction

The paper follows the study summarized in the previous article [1] describing the construction of Variant I of the probabilistic model of school-achievement tests with double-choice response where the probabilistic structure of this test was defined including certain simplifying assumptions.

The described probabilistic model suggests derivation of multinomic distribution of the random vector $M=(M_0,M_1,M_2)$ in the form

$$P(M_0=m_0,M_1=m_1,M_2=m_2) = \tag{1}$$
$$= \frac{n!}{m_1!m_2!m_3!} \left(\tau \frac{(q-2)(q-3)}{q\ (q-1)}\right)^{m_0} \left(\tau \frac{4(q-2)}{q(q-1)}\right)^{m_1} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{m_2}$$

where M_0 is a random variable expressing the number of questions of the test to which no correct response was given by the tested person, M_1 is a random variable expressing the number of questions to which only one correct response was given, M_2 is a random variable expressing the number of questions of the test to which both correct responses were given. The extend of the test is n questions, $q>3$ is a number of offered alternatives.

The distribution of the random vector M in variant I of the probabilistic model of school-achievement tests with the double-choice response is dependent on one parameter τ representing a part of the real unfamiliarity with the topic from the whole extent of the tested theme.

The purpose of the present paper is to estimate the parameter of this model and to evaluate its properties.

Point estimation of parameter τ

The point estimator of parameter τ representing a part of the tested person's unfamiliarity with the examined topic will be performed at first by the moment method using the theoretical and empirical moments of the variable M_2 or the variable $X=M_0+M_1$ on the basis of a single application of the test. Comparing theoretical and empirical mean values and using the formula (11) of [1] the following equation

$$E(M_2)=n\left(1-\tau \frac{q(q-1)-2}{q(q-1)}\right)=m_2$$

is derived. This provides the estimator of the parameter τ in the form

$$\hat{\tau} = \frac{n-m_2}{n} \frac{q(q-1)}{q(q-1)-2} = \frac{x}{n} \frac{q(q-1)}{q(q-1)-2} \tag{2}$$

The same solution is reached by the **maximum** likelihood method on the basis of a single application of the test, i.e. on the basis of values (m_0, m_1, m_2) of the random vector M.

The probability distribution (1) of the random vector M will be selected for the likelihood function

$$L(\tau,M)= \frac{n!}{m_1!m_2!m_3!} \left(\tau \frac{(q-2)(q-3)}{q(q-1)}\right)^{m_0} \left(\tau \frac{4(q-2)}{q(q-1)}\right)^{m_1} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{m_2} \tag{3}$$

The maximum likelihood estimator of the parameter τ, i.e. the point at which the likelihood function reaches its maximum, is solved by the likelihood equation

$$\frac{\partial \ ln \ L(\tau,M)}{\partial \ \tau} = 0 \tag{4}$$

After the expression of the logarithm of likelihood function

$$ln \ L(\tau,M) = ln \frac{n!}{m_0!m_1!m_2!} + m_0 ln\left(\tau \frac{(q-2)(q-3)}{q(q-1)}\right) + m_1 ln\left(\tau \frac{4(q-2)}{q(q-1)}\right) + $$
$$+ m_2 ln\left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)$$

and its partial derivative we reach the likelihood equation in the form

$$\frac{m_0}{\tau} + \frac{m_1}{\tau} - \frac{m_2 \frac{q(q-1)-2}{q(q-1)}}{1 - \tau \frac{q(q-1)-2}{q(q-1)}} = 0 \tag{5}$$

modified as follows

$$(m_0+m_1)q(q-1) - (m_0+m_1) \ \tau \ (q(q-1)-2) = (n-m_0-m_1) \ \tau \ (q(q-1)-2)$$

From this equation the expression of the maximum likelihood estimator $\hat{\tau}$ of parameter τ

$$\hat{\tau} = \frac{m_0+m_1}{n} \frac{q(q-1)}{q(q-1)-2} \qquad (6)$$

can be derived. This result equals the result (2) obtained by the moment method. The same form of the estimator will be obteined also by calculation of the maximum of the likelihood function expressed by the marginal distribution of the random variable M_2 or X.

However, the expression (2) or (6) is a solution only under the condition that the value of the estimator does not exceed the rank of realy possible values of parameter τ, i.e. under the condition that

$$\frac{x}{n} \frac{q(q-1)}{q(q-1)-2} \leq 1 , \qquad \text{i.e.} \quad x \leq n \frac{q(q-1)-2}{q(q-1)} .$$

For $x>n(q(q-1)-2)/(q(q-1))$ the likelihood function reaches its maximal value of interval $<0,1>$ on its upper boundary, i.e. at the poit $\tau=1$. This is due to the fact, that the likelihood function or its logarithm is, for a given value x, an increasing function of the argument τ in the interval from 0 to $\tau =xq(q-1)/(n(q(q-1)-2))$ and a decreasing function of the argument τ in the interval from $\tau =xq(q-1)/(n(q(q-1)-2))$ to 1. If the argument of total extreme of the function $L(\tau)$ is out of the interval $<0,1>$ the right-side boundary is exceeded by an increasing branch of function $L(\tau)$, so that its local extreme must be located at this point.

Therefore, for the maximal likelihood estimator of the parameter τ holds

$$\hat{\tau} = \begin{cases} \frac{x}{n} \frac{q(q-1)}{q(q-1)-2} & \text{for} \quad x \leq n \frac{q(q-1)-2}{q(q-1)} \\ 1 & \text{for} \quad x > n \frac{q(q-1)-2}{q(q-1)} \end{cases} \qquad (7)$$

The relations (2),(6),(7) for the point estimator of parametr τ in this variant of the probabilistic model suggest that the estimator of the real part of unfamiliarity with the tested topic depends at a given n and q only on the values of the random variable $M_2=n-X$ or the variable $X=M_0+M_1$, i.e. on the number of questions to which both correct responses were or were not given, respectively. The values of random variables M_0 or M_1 need not be followed up. This reality, however, corresponds to the pressumption of this variant of probabilistic model mentioned in [1].

Properties of the point estimator of parametr τ

Properties of the point estimator $\hat{\tau}$ related to the real value of the estimated parameter τ are given by the basic characteristics of estimator $\hat{\tau}$ calculated under presumption that the real unknown value of parameter is τ. To derive them the relation (7) is used, where the point estimator $\hat{\tau}$ is a function of random variable X and the characteristics of this function can be derived by means the distribution of this random variable.

For expression of the expectation holds the following relation

$$E(\hat{\tau}|\tau) = \sum_{x=0}^{c} \frac{x}{n} \frac{q(q-1)}{q(q-1)-2} \binom{n}{x} \left(\tau \frac{q(q-1)-2}{q(q-1)}\right)^{x} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{n-x} + \tag{8}$$

$$+ \sum_{x=c+1}^{n} \binom{n}{x} \left(\tau \frac{q(q-1)-2}{q(q-1)}\right)^{x} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{n-x} =$$

$$= \tau \sum_{x=1}^{c} \binom{n-1}{x-1} \left(\tau \frac{q(q-1)-2}{q(q-1)}\right)^{x-1} \left(1- \tau \frac{q(q-1)-2}{q(q-1)}\right)^{n-x} +$$

$$+ \sum_{x=c+1}^{n} \binom{n}{x} \left(\tau \frac{q(q-1)-2}{q(q-1)}\right)^{x} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{n-x} =$$

$$= \sum_{x=0}^{c-1} \binom{n-1}{x} \left(\tau \frac{q(q-1)-2}{q(q-1)}\right)^{x} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{n-1-x} +$$

$$+\sum_{x=c+1}^{n}\binom{n}{x}\left(\tau\frac{q(q-1)-2}{q(q-1)}\right)^{x}\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)^{n-x}=$$

$$=\tau B\left(c-1|n-1,\tau\frac{q(q-1)-2}{q(q-1)}\right)+1-B\left(c|n,\tau\frac{q(q-1)-2}{q(q-1)}\right)$$

where c is the integer part of the **expression** $n(q(q-1)-2)/q(q-1)$ **and** the function B is a distribution function defined by the formula

$$B(z|n,p)=\sum_{x=0}^{z}\binom{n}{x}p^{x}(1-p)^{n-x}$$

From the expression of expectation of the estimator $\hat{\tau}$ it is evident that the estimator is not an unbiased estimator of parameter τ when $E(\hat{\tau}|\tau)\neq\tau$ is valid. The value of the bias is defined by the relation

$$\tau-E(\hat{\tau}|\tau)=\tau\left\{1-B\left(c-1|n-1,\tau\frac{q(q-1)-2}{q(q-1)}\right)\right\}-\qquad(9)$$

$$-\left\{1-B\left(c|n,\tau\frac{q(q-1)-2}{q(q-1)}\right)\right\}$$

According to probabilistic theory the estimators obtained by the **method** of maximal likelihood are asymptotically unbiased. **In case** of estimator $\hat{\tau}$ it can be easy verified as follows. **With** regard to the validity of the relation

$$B(z|n,p)=\phi\left(\frac{z+0,5-np}{\sqrt{np(1-p)}}\right)$$

reported by means of standardised normal distribution function **the equalities**

$$B\left(\left[n\frac{q(q-1)-2}{q(q-1)}\right]-1\mid n-1,\tau\frac{q(q-1)-2}{q(q-1)}\right)=\qquad(10)$$

$$=\phi\left(\frac{n\frac{q(q-1)-2}{q(q-1)}-\frac{1}{2}-(n-1)\tau\frac{q(q-1)-2}{q(q-1)}}{\sqrt{(n-1)\tau\frac{q(q-1)-2}{q(q-1)}\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)}}\right)$$

and

$$B\left(\left[n \frac{q(q-1)-2}{q(q-1)}\right] \mid n, \tau \frac{q(q-1)-2}{q(q-1)} \right) = \qquad (11)$$

$$= \phi \left(\frac{n \frac{q(q-1)-2}{q(q-1)} + \frac{1}{2} - n \tau \frac{q(q-1)-2}{q(q-1)}}{\sqrt{n \tau \frac{q(q-1)-2}{q(q-1)} \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)}} \right)$$

hold aproximately for the increasing n, where the square bracket denotes the integer part of this relation.

In both relations the argument of the function ϕ tends with increasing n to infinity for all $\tau \in \langle 0,1 \rangle$ and for all the considered q , and thus the distribution function tends to one. To one there are tending also both expressions on the left-hand sides of relations (10),(11). Comparison to the expression (9) thus verifies the tendence of bias of the estimator $(\tau - E(\hat{\tau}|\tau))$ towards 0 for n increasing to infinity. In case of $\tau = 1$ or $\tau = 0$ follows the proved property directly from the properties of binomial distribution.

Important also is the characteristic variance by which the accuracy of estimation and also the effectiveness of the school-achievement test can be eveluated. The variance of estimator $\hat{\tau}$ of parameter τ can be derived by the relation

$$D(\hat{\tau}|\tau) = E(\hat{\tau}^2|\tau) - E(\hat{\tau}|\tau)^2$$

where the expectation of the square of the estimator can again be calculated by the probability distribution of random variable X. There holds the following relation

$$E(\hat{\tau}^2|\tau) = \sum_{x=0}^{c} \left(\frac{x}{n} \frac{q(q-1)}{q(q-1)-2} \right)^2 \binom{n}{x} \left(\tau \frac{q(q-1)-2}{q(q-1)}\right)^x \left(1 - \tau \frac{q(q-1)-2}{q(q-1)}\right)^{n-x} +$$

$$+ \sum_{x=c+1}^{n} \binom{n}{x} \left(\tau \frac{q(q-1)-2}{q(q-1)} \right)^x \left(1 - \tau \frac{q(q-1)-2}{q(q-1)} \right)^{n-x}$$

The first member of this relation can be adjusted in the form

$$\frac{1}{n^2}\left(\frac{q(q-1)}{q(q-1)-2}\right)^2 \sum_{x=0}^{c} (x(x-1)+x)\binom{n}{x}\left(\tau\frac{q(q-1)-2}{q(q-1)}\right)^x\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)^{n-x} =$$

$$= \frac{n-1}{n}\tau^2\sum_{x=2}^{c}\binom{n-2}{x-2}\left(\tau\frac{q(q-1)-2}{q(q-1)}\right)^{x-2}\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)^{n-x} +$$

$$+ \frac{\tau}{n}\frac{q(q-1)}{q(q-1)-2}\sum_{x=1}^{c}\binom{n-1}{x-1}\left(\tau\frac{q(q-1)-2}{q(q-1)}\right)^{x-1}\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)^{n-x} =$$

$$= \frac{n-1}{n}\tau^2\sum_{x=0}^{c-2}\binom{n-2}{x}\left(\tau\frac{q(q-1)-2}{q(q-1)}\right)^{x}\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)^{n-2-x} +$$

$$+ \frac{\tau}{n}\frac{q(q-1)}{q(q-1)-2}\sum_{x=0}^{c-1}\binom{n-1}{x-1}\left(\tau\frac{q(q-1)-2}{q(q-1)}\right)^{x}\left(1-\tau\frac{q(q-1)-2}{q(q-1)}\right)^{n-1-x} =$$

$$= \frac{n-1}{n}\tau^2 B\left(c-2 \mid n-2, \tau\frac{q(q-1)-2}{q(q-1)}\right) +$$

$$+ \frac{1}{n}\tau\frac{q(q-1)}{q(q-1)-2} B\left(c-1 \mid n-1, \tau\frac{q(q-1)-2}{q(q-1)}\right)$$

This leads us to express the variance of the estimator as follows

(12)

$$D(\hat{\tau}\mid\tau) = \frac{n-1}{n}\tau^2 B\left(c-2 \mid n-2, \tau\frac{q(q-1)-2}{q(q-1)}\right) +$$

$$+ \frac{1}{n}\tau\frac{q(q-1)}{q(q-1)-2} B\left(c-1 \mid n-1, \tau\frac{q(q-1)-2}{q(q-1)}\right) +$$

$$+ 1 - B\left(c \mid n, \tau\frac{q(q-1)-2}{q(q-1)}\right) -$$

$$- \left\{\tau B\left(c-1 \mid n-1, \tau\frac{q(q-1)-2}{q(q-1)}\right) + 1 - B\left(c \mid n, \tau\frac{q(q-1)-2}{q(q-1)}\right)\right\}^2$$

Calculation of the above relation suggest that the extent of variance of point estimator $\hat{\tau}$ decreases in dependence on the extent of the test n and on the increasing number of offered alternatives q. The test with a relatively low number of alternatives may achieve better effectiveness of the estimator of parameter τ by increasing n.

Interval estimation of parameter τ

In the method of construction of the confidence interval of the part of real unfamiliarity in the whole topic tested we define a certain part of numerical axis $(\underline{\tau},\bar{\tau})$, which comprises parameter τ in dependence on the concrete results of the test with the confidence level $1-\alpha$. The upper or lower limits of confidence interval will be respected only if it is ranges within the interval $\langle 0,1\rangle$.

With regard to the date mentioned in [1], the confidence interval of parameter τ is relative again to the construction of the confidence interval for the parameter of the alternative distribution mentioned in [4].

Confidence interval $(\underline{p},\bar{p})$ for parameter $p = 1-p_2 = p_0+p_1$ with confidence level $1-\alpha$ including a concrete value x_0 of the number of questions to which both correct responses were not given can be found by the solution of the following equation

$$\sum_{x=0}^{x_0} \binom{n}{x} \bar{p}^{x} (1-\bar{p})^{n-x} = \alpha_1$$

expected relation

$$P \{ X \le x_0 | \bar{p} \} = \alpha_1$$

and by the following equation

$$\sum_{x=x_0}^{n} \binom{n}{x} \underline{p}^{x} (1-\underline{p})^{n-x} = \alpha_2$$

expected relation

$$P \{ X \ge x_0 | \underline{p} \} = \alpha_2$$

for given α_1, α_2, where $(\alpha_1+\alpha_2)=\alpha$, $\alpha_1\in\langle 0,1\rangle$, $\alpha_2\in\langle 0,1\rangle$, $\alpha\in\langle 0,1\rangle$

Construction of a one-sided confidence interval is based on the equation

$$\sum_{x=0}^{x_0} \binom{n}{x} \bar{p}^{x} (1-\bar{p})^{n-x} = \alpha$$

expected relation

$$P\{ X \le x_0 | \bar{p} \} = \alpha$$

and the following equation

$$\sum_{x=x_0}^{n} \binom{n}{x} \underline{p}^{x} (1-\underline{p})^{n-x} = \alpha$$

expected relation

$$P\{ X \ge x_0 | \underline{p} \} = \alpha$$

With regard to the fact that parameter p dependens directly on parameter τ expressed by $p = \tau(q(q-1)-2)/q(q-1)$ where $(q(q-1)-2)/q(q-1)$ is a constant >0, the upper limit $\bar{\tau}$ of the one-sided confidence interval for parameter τ can be derived as follows

$$\bar{\tau} = \min_{0\le\tau\le1} \{ \tau: P(X \le x_0 | \tau) \le \alpha \} \qquad (13)$$

and the lower limit $\underline{\tau}$ is defined by

$$\underline{\tau} = \max_{0\le\tau\le1} \{ \tau: P(X \ge x_0 | \tau) \le \alpha \}$$

or

$$\underline{\tau} = \max_{0\le\tau\le1} \{ \tau: P(X \le x_0-1 | \tau) \ge 1-\alpha \} \qquad (14)$$

The distribution of variable X is given by (13) of [1]. Similarly, the confidence interval can also be constructed by means of the distribution of the variable M_2.

With increasing n the binomial distribution of random variable X can be approximated by normal distribution. In this case the upper limit $\bar{\tau}$ of the one-sided confidence interval for parameter τ is established from the following relation

$$\phi\left(\frac{x_0 - n\bar{\tau}\frac{q(q-1)-2}{q(q-1)} + \frac{1}{2}}{\sqrt{n\bar{\tau}\frac{q(q-1)-2}{q(q-1)}\left(1 - \bar{\tau}\frac{q(q-1)-2}{q(q-1)}\right)}} \right) = \alpha \qquad (15)$$

and the lower limit of the one-sided confidence interval is established from the relation

$$\phi\left(\frac{x_0 - n\,\underline{\tau}\,\dfrac{q(q-1)-2}{q(q-1)} - \dfrac{1}{2}}{\sqrt{n\,\underline{\tau}\,\dfrac{q(q-1)-2}{q(q-1)}\left(1 - \underline{\tau}\,\dfrac{q(q-1)-2}{q(q-1)}\right)}}\right) = 1 - \alpha \tag{16}$$

Conditions (15),(16) are equivalent with conditions

$$\left(x_0 + \frac{1}{2} - n\,\bar{\tau}\,\frac{q(q-1)-2}{q(q-1)}\right)^2 = u_\alpha^2\; n\,\bar{\tau}\,\frac{q(q-1)-2}{q(q-1)}\left(1 - \bar{\tau}\,\frac{q(q-1)-2}{q(q-1)}\right)$$

$$\left(x_0 - \frac{1}{2} - n\,\underline{\tau}\,\frac{q(q-1)-2}{q(q-1)}\right)^2 = u_{1-\alpha}^2\; n\,\underline{\tau}\,\frac{q(q-1)-2}{q(q-1)}\left(1 - \underline{\tau}\,\frac{q(q-1)-2}{q(q-1)}\right)$$

where u_α equals α.100%-th quantil of distribution N(0,1). In case of a both-sided confidence interval with confidence level $1-\alpha$ the right-hand sides of the above relations should again comprise α_1, α_2 where $\alpha_1+\alpha_2=\alpha$. Mostly it is considered $\alpha_1=\alpha_2=\alpha/2$. Using the upper and lower limits calculated from conditions of the one-sided interval we get a both-sided confidence interval with confidence level $1-2\alpha$.

Numerical results

The confidence intervals of parameter τ for variant I of the probabilistic model of a school-achievement test with double-choice response are for different q,n and x given in tables on pp. 17-20. The interval is constructed as one-sided with confidence level *0,05*. In the case where the upper limit of confidence interval exceeded the upper limit of real values of parameter τ only figure *1* is printed.

The value of point estimator of parameter τ is for different q,n and x summarized in tables on pp. 14-16. The computation program processed in GWBASIC is given in detail on pp. 21-23.

Point estimator of parameter τ in the school-achievement test with double choice response : VARIANT I

x	*q=4*	*q=5*	*q=6*	*q=7*	*q=8*	*q=9*
0	0,00	0,00	0,00	0,00	0,00	0,00
1	0,12	0,11	0,11	0,11	0,10	0,10
2	0,24	0,22	0,21	0,21	0,21	0,21
3	0,36	0,33	0,32	0,32	0,31	0,31
4	0,48	0,44	0,43	0,42	0,41	0,41
5	0,60	0,56	0,54	0,53	0,52	0,51
6	0,72	0,67	0,64	0,63	0,62	0,62
7	0,84	0,78	0,75	0,74	0,73	0,72
8	0,96	0,89	0,86	0,84	0,83	0,82
9	1	1,00	0,96	0,95	0,93	0,93
10	1	1	1	1	1	1

n=10 number of questions of the test

Point estimator of parameter τ in the school-achievement test with double choice response : VARIANT I

x	*q=4*	*q=5*	*q=6*	*q=7*	*q=8*	*q=9*
0	0,00	0,00	0,00	0,00	0,00	0,00
1	0,06	0,06	0,05	0,05	0,05	0,05
2	0,12	0,11	0,11	0,11	0,10	0,10
3	0,18	0,17	0,16	0,16	0,16	0,15
4	0,24	0,22	0,21	0,21	0,21	0,21
5	0,30	0,28	0,27	0,26	0,26	0,26
6	0,36	0,33	0,32	0,32	0,31	0,31
7	0,42	0,39	0,38	0,37	0,36	0,36
8	0,48	0,44	0,43	0,42	0,41	0,41
9	0,54	0,50	0,48	0,47	0,47	0,46
10	0,60	0,56	0,54	0,53	0,52	0,51
11	0,66	0,61	0,59	0,58	0,57	0,57
12	0,72	0,67	0,64	0,63	0,62	0,62
13	0,78	0,72	0,70	0,68	0,67	0,67
14	0,84	0,78	0,75	0,74	0,73	0.72
15	0,90	0,83	0,80	0,79	0,78	0,77
16	0,96	0,89	0,86	0,84	0,83	0,82
17	1	0,94	0,91	0,89	0,88	0,87
18	1	1,00	0,96	0,95	0,93	0,93
19	1	1	1	1,00	0,99	0,98
20	1	1	1	1	1	1

n=20 number of questions of the test

x **number of questions to which both correct responses were not given**

q **number of offered alternatives**

τ **part of real unfamiliarity**

Point estimator of parameter τ in the school-achievement test with double choice response : VARIANT I

x	*q=4*	*q=5*	*q=6*	*q=7*	*q=8*	*q=9*
0	0,00	0,00	0,00	0,00	0,00	0,00
1	0,04	0,04	0,04	0,04	0,03	0,03
2	0,08	0,07	0,07	0,07	0,07	0,07
3	0,12	0,11	0,11	0,11	0,10	0,10
4	0,16	0,15	0,14	0,14	0,14	0,14
5	0,20	0,19	0,18	0,18	0,17	0,17
6	0,24	0,22	0,21	0,21	0,21	0,21
7	0,28	0,26	0,25	0,25	0,24	0,24
8	0,32	0,30	0,29	0,28	0,28	0,27
9	0,36	0,33	0,32	0,32	0,31	0,31
10	0,40	0,37	0,36	0,35	0,35	0,34
11	0,44	0,41	0,39	0,39	0,38	0,38
12	0,48	0,44	0,43	0,42	0,41	0,41
13	0,52	0,48	0,46	0,46	0,45	0,45
14	0,56	0,52	0,50	0,49	0,48	0,48
15	0,60	0,56	0,54	0,53	0,52	0,51
16	0,64	0,59	0,57	0,56	0,55	0,55
17	0,68	0,63	0,61	0,60	0,59	0,58
18	0,72	0,67	0,64	0,63	0,62	0,62
19	0,76	0,70	0,68	0,67	0,66	0,65
20	0,80	0,74	0,71	0,70	0,69	0,69
21	0,84	0,78	0,75	0,74	0,73	0,72
22	0,88	0,81	0,79	0,77	0,76	0,75
23	0,92	0,85	0,82	0,81	0,80	0,79
24	0,96	0,89	0,86	0,84	0,83	0,82
25	1,00	0,93	0,89	0,88	0,86	0,86
26	1	0,96	0,93	0,91	0,90	0,89
27	1	1,00	0,96	0,95	0,93	0,93
28	1	1	1,00	0,98	0,97	0,96
29	1	1	1	1	1	0,99
30	1	1	1	1	1	1

$n=30$ number of questions of the test

x number of questions to which both correct responses were not given

q number of offered alternatives

τ part of real unfamiliarity

Point estimator of parameter τ in the school-achievement test with double choice response : VARIANT I

x	q=4	q=5	q=6	q=7	q=8	q=9
0	0,00	0,00	0,00	0,00	0,00	0,00
1	0,03	0,03	0,03	0,03	0,03	0,03
2	0,06	0,06	0,05	0,05	0,05	0,05
3	0,09	0,08	0,08	0,08	0,08	0,08
4	0,12	0,11	0,11	0,11	0,10	0,10
5	0,15	0,14	0,13	0,13	0,13	0,13
6	0,18	0,17	0,16	0,16	0,16	0,15
7	0,21	0,19	0,19	0,18	0,18	0,18
8	0,24	0,22	0,21	0,21	0,21	0,21
9	0,27	0,25	0,24	0,24	0,23	0,23
10	0,30	0,28	0,27	0,26	0,26	0,26
11	0,33	0,31	0,29	0,29	0,29	0,28
12	0,36	0,33	0,32	0,32	0,31	0,31
13	0,39	0,36	0,35	0,34	0,34	0,33
14	0,42	0,39	0,38	0,37	0,36	0,36
15	0,45	0,42	0,40	0,39	0,39	0,39
16	0,48	0,44	0,43	0,42	0,41	0,41
17	0,51	0,47	0,46	0,45	0,44	0,44
18	0,54	0,50	0,48	0,47	0,47	0,46
19	0,57	0,53	0,51	0,50	0,49	0,49
20	0,60	0,56	0,54	0,53	0,52	0,51
21	0,63	0,58	0,56	0,55	0,54	0,54
22	0,66	0,61	0,59	0,58	0,57	0,57
23	0,69	0,64	0,62	0,60	0,60	0,59
24	0,72	0,67	0,64	0,63	0,62	0,62
25	0,75	0.69	0,67	0,66	0,65	0,64
26	0,78	0,72	0,70	0,68	0,67	0,67
27	0,81	0,75	0,72	0,71	0,70	0,69
28	0,84	0,78	0,75	0,74	0,73	0,72
29	0,87	0,81	0,78	0,76	0,75	0,75
30	0,90	0,83	0,80	0,79	0,78	0,77
31	0,93	0,86	0,83	0,81	0,80	0,80
32	0,96	0,89	0,86	0,84	0,83	0,82
33	0,99	0,92	0,88	0,87	0,86	0,85
34	1	0,94	0,91	0,89	0,88	0,87
35	1	0,97	0,94	0,92	0,91	0,90
36	1	1,00	0,96	0,95	0,93	0,93
37	1	1	0,99	0,97	0,96	0,95
38	1	1	1	1,00	0,99	0,98
39	1	1	1	1	1	1
40	1	1	1	1	1	1

n=40 number of questions of the test

x number of questions to which both correct responses were not given

q number of offered alternatives

τ part of real unfamiliarity

Confidence interval of parameter τ in the school-achievement test with double choice response : VARIANT I

x	q=4	q=5	q=6	q=7	q=8
0	0,00 0,32	0,00 0,29	0,00 0,28	0,00 0,28	0,00 0,27
1	0,00 0,48	0,00 0,44	0,00 0,43	0,00 0,42	0,00 0,41
2	0,04 0,61	0,04 0,57	0,03 0,55	0,03 0,54	0,03 0,53
3	0,10 0,73	0,09 0,68	0,09 0,65	0,09 0,64	0,09 0,63
4	0,18 0,84	0,16 0,78	0,16 0,75	0,15 0,74	0,15 0,73
5	0,26 0,94	0,24 0,87	0,23 0,84	0,23 0,82	0,23 0,81
6	0,36 1	0,33 0,95	0,32 0,92	0,31 0,90	0,31 0,89
7	0,47 1	0,43 1	0,42 0,98	0,41 0,96	0,40 0,95
8	0,59 1	0,54 1	0,52 1	0,51 1	0,51 1,00
9	0,72 1	0,67 1	0,64 1	0,63 1	0,62 1
10	0,88 1	0,82 1	0,79 1	0,77 1	0,76 1

n=10 number of questions of the test

Confidence interval of parameter τ in the school-achievement test with double choice response : VARIANT I

x	q=4	q=5	q=6	q=7	q=8
0	0,00 0,17	0,00 0,16	0,00 0,15	0,00 0,15	0,00 0,15
1	0,00 0,26	0,00 0,25	0,00 0,24	0,00 0,23	0,00 0,23
2	0,02 0,34	0,02 0,32	0,01 0,31	0,01 0,30	0,01 0,30
3	0,05 0,42	0,04 0,39	0,04 0,37	0,04 0,37	0,04 0,36
4	0,08 0,49	0,07 0,45	0,07 0,43	0,07 0,43	0,07 0,42
5	0,12 0,55	0,11 0,51	0,11 0,49	0,10 0,48	0,10 0,48
6	0,16 0,61	0,15 0,57	0,14 0,55	0,14 0,54	0,14 0,53
7	0,21 0,67	0,19 0,63	0,18 0,60	0,18 0,59	0,18 0,58
8	0,26 0,73	0,24 0,68	0,23 0,65	0,22 0,64	0,22 0,63
9	0,31 0,79	0,28 0,73	0,27 0,70	0,27 0,69	0,26 0,68
10	0,36 0,84	0,33 0,78	0,32 0,75	0,31 0,74	0,31 0,73
11	0,41 0,89	0,38 0,83	0,37 0,80	0,36 0,78	0,35 0,77
12	0,47 0,94	0,43 0,87	0,42 0,84	0,41 0,83	0,40 0,82
13	0,53 0,99	0,49 0,92	0,47 0,89	0,46 0,87	0,45 0,86
14	0,59 1	0,54 0,96	0,52 0,93	0,51 0,91	0,51 0,90
15	0,65 1	0,60 1,00	0,58 0,96	0,57 0,95	0,56 0,93
16	0,71 1	0,66 1	0,64 1,00	0,62 0,98	0,62 0,97
17	0,78 1	0,72 1	0,70 1	0,68 1	0,68 1,00
18	0,86 1	0,79 1	0,76 1	0,75 1	0,74 1
19	0,94 1	0,87 1	0,83 1	0,82 1	0,81 1
20	1,00 1	0,95 1	0,92 1	0,90 1	0,89 1

n=20 **number of questions of the test**

x **number of questions to which both correct responses were not given**

q **number of offered alternatives**

τ **part of real unfamiliarity**

Confidence interval of parameter τ in the school-achievement test with double choice response : VARIANT I

x	*q=4*	*q=5*	*q=6*	*q=7*	*q=8*
0	0,00 0,12	0,00 0,11	0,00 0,11	0,00 0,10	0,00 0,10
1	0,00 0,18	0,00 0,17	0,00 0,16	0,00 0,16	0,00 0,16
2	0,01 0,24	0,01 0,22	0,01 0,21	0,01 0,21	0,01 0,21
3	0,03 0,29	0,03 0,27	0,02 0,26	0,02 0,26	0,02 0,25
4	0,05 0,34	0,05 0,32	0,05 0,30	0,04 0,30	0,04 0,29
5	0,08 0,39	0,07 0,36	0,07 0,35	0,07 0,34	0,07 0,34
6	0,10 0,43	0,10 0,40	0,09 0,39	0,09 0,38	0,09 0,38
7	0,13 0,48	0,12 0,44	0,12 0,43	0,12 0,42	0,11 0,41
8	0,16 0,52	0,15 0,48	0,15 0,47	0,14 0,46	0,14 0,45
9	0,19 0,56	0,18 0,52	0,17 0,50	0,17 0,49	0,17 0,49
10	0,23 0,60	0,21 0,56	0,20 0,54	0,20 0,53	0,20 0,52
11	0,26 0,64	0,24 0,60	0,23 0,58	0,23 0,56	0,22 0,56
12	0,29 0,68	0,27 0,63	0,26 0,61	0,26 0,60	0,25 0,59
13	0,33 0,72	0,30 0,67	0,29 0,65	0,29 0,63	0,28 0,53
14	0,37 0,76	0,34 0,71	0,33 0,68	0,32 0,67	0,31 0,66
15	0,40 0,80	0,37 0,74	0,36 0,71	0,35 0,70	0,35 0,69
16	0,44 0,83	0,41 0,77	0,39 0,75	0,38 0,73	0,38 0,72
17	0,48 0,87	0,44 0,81	0,43 0,78	0,42 0,76	0,41 0,75
18	0,52 0,91	0,48 0,84	0,46 0,81	0,45 0,79	0,45 0,78
19	0,56 0,94	0,51 0,87	0,50 0,84	0,49 0,82	0,48 0,81
20	0,60 0,97	0,55 0,90	0,53 0,87	0,52 0,85	0,51 0,84
21	0,64 1	0,59 0,93	0,57 0,90	0,56 0,88	0,55 0,87
22	0,68 1	0,63 0,96	0,61 0,93	0,59 0,91	0,59 0,90
23	0,72 1	0,67 0,99	0,64 0,95	0,63 0,93	0,62 0,92
24	0,77 1	0,71 1	0,68 0,98	0,67 0,96	0,66 0,95
25	0,81 1	0,75 1	0,72 1,00	0,71 0,98	0,70 0,97
26	0,86 1	0,80 1	0,77 1	0,75 1,00	0,74 0,99
27	0,91 1	0,84 1	0,81 1	0,79 1,00	0,78 1,00
28	0,96 1	0,89 1	0,86 1	0,84 1,00	0,83 1,00
29	1,00 1	0,94 1	0,91 1	0,89 1,00	0,88 1
30	1,00 1	1,00 1	0 96 1	0,95 1,00	0,93 1

n=30 number of questions of the test

x number of questions to which both correct responses were not given

q number of offered alternatives

τ part of real unfamiliarity

Confidence interval of parameter τ in the school-achievement test with double choice response : VARIANT I

x	*q=4*		*q=5*		*q=6*		*q=7*		*q=8*	
0	0,00	0,09	0,00	0,09	0,00	0,08	0,00	0,08	0,00	0,08
1	0,00	0,14	0,00	0,13	0,00	0,13	0,00	0,12	0,00	0,12
2	0,01	0,18	0,00	0,17	0,00	0,16	0,00	0,16	0,00	0,16
3	0,02	0,22	0,02	0,21	0,02	0,20	0,02	0,20	0,02	0,19
4	0,04	0,26	0,03	0,24	0,03	0,23	0,03	0,23	0,03	0,23
5	0,06	0,30	0,05	0,28	0,05	0,27	0,05	0,26	0,05	0,26
6	0,08	0,33	0,07	0,31	0,07	0,30	0,07	0,29	0,06	0,29
7	0,10	0,37	0,09	0,34	0,09	0,33	0,08	0,32	0,08	0,32
8	0,12	0,40	0,11	0,37	0,11	0,36	0,10	0,35	0,10	0,35
9	0,14	0,44	0,13	0,40	0,13	0,39	0,12	0,38	0,12	0,38
10	0,17	0,47	0,15	0,44	0,15	0,42	0,14	0,41	0,14	0,41
11	0,19	0,50	0,18	0,46	0,17	0,45	0,17	0,44	0,16	0,43
12	0,21	0,53	0,20	0,49	0,19	0,48	0,19	0,47	0,18	0,46
13	0,24	0,56	0,22	0,52	0,21	0,50	0,21	0,49	0,21	0,49
14	0,27	0,60	0,25	0,55	0,24	0,53	0,23	0,52	0,23	0,52
15	0,29	0,63	0,27	0,58	0,26	0,56	0,25	0,55	0,25	0,54
16	0,32	0,66	0,29	0,61	0,28	0,59	0,28	0,57	0,27	0,57
17	0,35	0,69	0,32	0,63	0,31	0,61	0,30	0,60	0,30	0,59
18	0,37	0,71	0,34	0,66	0 33	0,64	0,33	0,63	0,32	0,62
19	0,40	0,74	0,37	0,69	0,36	0,66	0,35	0,65	0,35	0,64
20	0,43	0,77	0,40	0,71	0,38	0,69	0,37	0,68	0,37	0,67
21	0,46	0,80	0,42	0,74	0,41	0,71	0,40	0,70	0,39	0,69
22	0,49	0,83	0,45	0,77	0,43	0,74	0,42	0,72	0,42	0,72
23	0,51	0,85	0,48	0,79	0,46	0,76	0,45	0,75	0,44	0,74
24	0,54	0,88	0,50	0,82	0,49	0,79	0,48	0,77	0,47	0,76
25	0,57	0,91	0,53	0,84	0,51	0,81	0,50	0,80	0,50	0,79
26	0,60	0,93	0,56	0,87	0,54	0,83	0,53	0,82	0,52	0,81
27	0,64	0,96	0 59	0,89	0,57	0,86	0,56	0,84	0,55	0,83
28	0,67	0,99	0,62	0,91	0,59	0,88	0,58	0,86	0,58	0,85
29	0,71	1	0,65	0,94	0,62	0,90	0,61	0,88	0,60	0,87
30	0,73	1	0,68	0,96	0,65	0,92	0,64	0,91	0,63	0,89
31	0,76	1	0,71	0,98	0,68	0,94	0,67	0,93	0,66	0,91
32	0,80	1	0,74	1,00	0,71	0,96	0,70	0,94	0,69	0,93
33	0,83	1	0,77	1,00	0,74	0,98	0,73	0,94	0,72	0,94
34	0,87	1	0,80	1,00	-0,77	0,99	0,76	0,96	0,75	0,95
35	0,90	1	0,83	1	0,80	0,99	0,79	0,96	0,78	0,96
36	0,94	1	0,87	1	0,84	1,00	0,82	0,98	0,81	0,97
37	0,98	1	0,90	1	0,87	1,00	0,85	0,99	0,84	0,98
38	1,00	1	0,94	1	0,91	1,00	0,89	0,99	0,88	0,99
39	1,00	1	0,98	1	0,95	1	0,93	1,00	0,91	1,00
40	1,00	1	0,99	1	0,95	1	0,93	1	0,92	1

n=40 number of questions of the test

x number of questions to which both correct responses were not given

q number of offered alternatives

τ part of real unfamiliarity

Computation program

```
100 REM -COMPUTATION PROGRAM FOR ESTIMATION OF PARAMETER τ IN
SCHOOL-ACHIEVEMENT TEST WITH DOUBLE-CHOICE RESPONSE VARIANT I
101 REM             Tesarikova Eva - 1990
103 CLS:LOCATE 4,15:PRINT"Calculation of parameter τ in school
achievement "
104 LOCATE 6,24:PRINT"test with double-choice response"
105 LOCATE 10,10:PRINT"1  Point estimation "
106 LOCATE 12,10:PRINT"2  Confidence interval"
107 LOCATE 14,10:PRINT"3  End"
108 LOCATE 20,10:PRINT"Order a number of questions,confidence
level,number of method"
109 PRINT
110 INPUT "number of questions = ";N
112 INPUT "alfa= ";AL
114 INPUT "number of method = ";V
115 ON V GOTO 120,200,500
120 CLS:LOCATE 10,15 :PRINT "Point estimator is calculated"
125 FOR Q=4 to 10
130 P(Q)=(Q*(Q-1)-2)/(Q*(Q-1))
140 NEXT Q
142 LPRINT :LPRINT"          Point estimator of parameter τ in
the school-achievement test"
143 LPRINT "                     with double choice response"
144 lPRINT:LPRINT"      x   ";
146 FOR Q=4 TO 9
147 LPRINT"    q=";Q
148 NEXT Q
149 LPRINT:LPRINT
150 FOR X=0 TO N
155 FOR Q=4 TO 9
160 TAU(Q)=X/(N*P(Q))
172 NEXT Q
173 LPRINT USING "##########";X;
175 FOR Q=4 TO 9
176 IF TAU(Q)>1 THEN LPRINT "     1  ";ELSE 178
177 GOTO 180
178 LPRINT USING "########.##":TAU(Q);
180 NEXT Q
185 LPRINT
190 NEXT X
191 LPRINT:LPRINT
192 LPRINT"    N=";N;"..number of questions in the test"
193 LPRINT"    X........number of questions to which both cor
rect responses were not given"
194 LPRINT"    Q........number of offered alternatives"
195 LPRINT"    τ........part of real unfamiliarity"
198 GOTO 103
200 CLS:LOCATE 10,15:PRINT"Confidence interval is calculated"
203 LPRINT:LPRINT
205 LPRINT"   Confidence interval of parameter  τ in school-
achievemet test "
206 LPRINT"         with double choice response - VARIANT I"
207 LPRINT
210 LPRINT:LPRINT"     x";
```

```
212 FOR Q=4 TO 9
214 LPRINT "     q=";Q
216 NEXT Q
218 LPRINT:LPRINT:LPRINT
225 FOR Q=4 TO 10
230 P(Q)=(Q*(Q-1)-2)/(Q*(Q-1))
240 NEXT Q
245 FOR X=0 TO N
250 FOR Q=4 TO 10
252 TAU=0
255 B1=P(Q)*TAU:B2=1-B1
260 J=0
265 C2=B2^N
267 S=C2
270 IF J>=X THEN 330
275 J=1:C3=N:C1=B1:C2=C2/B2
280 S=S+C3*C1*C2
285 IF J>=X THEN 330
290 C3=C3*(N-J)/(J+1)
295 C2=C2/B2
300 C1=C1*B1
305 S=S+C1*C2*C3
310 J=J+1
315 IF J>=X THEN 330
320 GOTO 290
330 IF S<=AL THEN 350
335 TAU=TAU+.01
340 GOTO 255
350 HM(Q)=TAU
352 TAU=1
355 B1=P(Q)*TAU:B2=1-B1
360 J=0
365 C2=B2^N
367 S=C2
370 IF J>=X-1 THEN 430
375 J=1:C3=N:C1=B1:C2=C2/B2
380 S=S+C3*C1*C2
385 IF J>=X-1 THEN 430
390 C3=C3*(N-J)/(J+1)
395 C2=C2/B2
400 C1=C1*B1
405 S=S+C1*C2*C3
410 J=J+1
415 IF J>=X-1 THEN 430
420 GOTO 390
430 IF S>=1-AL THEN 450
435 TAU=TAU-0.01
440 GOTO 355
450 DM(Q)=TAU
455 NEXT Q
460 LPRINT USING "#######":X
465 LPRINT "";
470 FOR Q=4 TO 9
475 LPRINT USING "####.##":DM(Q)
476 IF HM(Q)>1 THEN LPRINT " 1  ";ELSE 480
478 GOTO 485
```

```
480 LPRINT USING "##.##";RM(Q);
485 NEXT Q
488 LPRINT
490 NEXT X
491 LPRINT:LPRINT
492 LPRINT "    n=";N;". number of questions in the test"
493 LPRINT "    x ...... number of questions to which both cor
rect responses were not given"
494 LPRINT "    q....... number of offered alternatives"
495 LPRINT "    τ....... part of real unfamiliarity"
496 LPRINT "    α=";1-AL;".confidence level"
499 GOTO 103
500 END
```


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