Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Jitka Laitochová

A remark to the Floquet theorem for systems of linear differential equations

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 30 (1991), No. 1, 121--124

Persistent URL: http://dml.cz/dmlcz/120251

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1991 MATHEMATICA XXX VOL. 100

Katedra matematiky
pedagogické fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: RNDr.František Matyášek, CSc.

A REMARK TO THE FLOQUET THEOREM FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

JITKA LAITOCHOVÁ

(Received February 27, 1990)

Abstract: The Floquet theorem on the connection between a differential equation $y' = A(t)y$ and a linear differential equation with constant coefficients without the assumption of $A(t)$ periodic is given in this paper.

Key words: transformation, fundamental system.

MS Classification: 34C20, 34C25.

We consider a linear differential equation

$$y' = A(t)y \;, \tag{1}$$

where $A(t)$ is an nxn matrix of continuous functions such that $A[\varphi(t)]\,\varphi'(t) = A(t)$, $t \in (-\infty, \infty)$. We suppose that the function $\varphi(t)$ is increasing from $-\infty$ to ∞ on the interval $(-\infty, \infty)$, $\varphi'(t) \neq 0$ and $\varphi(t) > t$ for every $t \in (-\infty, \infty)$.

Lemma 1. Let $Y(t)$ be a fundamental matrix for the differential equation (1). Then a composite function $Y[\varphi(t)]$ is also a fundamental matrix for (1).

P r o o f . Setting $Z(t) = Y[\varphi(t)]$ we obtain

$$Z'(t) = Y'[\varphi(t)]\varphi'(t) = A[\varphi(t)]Y[\varphi(t)]\varphi'(t) =$$
$$= A[\varphi(t)]\varphi'(t)Y[\varphi(t)] = A(t)Z(t) .$$

Thus $Z(t) = Y[\varphi(t)]$ is a fundamental matrix for (1).

Lemma 2. To fundamental matrices $Y(t)$, $Y[\varphi(t)]$ there exists a nonsingular constant matrix H such that

$$Y[\varphi(t)] = Y(t)H(t), \quad t \in (-\infty, \infty). \tag{2}$$

P r o o f . It is obvious, it is a property of a linear space of fundamental matrices for (1).

Lemma 3. All constant matrices H satisfying (2) are similar.

P r o o f . If $Y(t)$, $Y_1(t)$ are two fundamental matrices for (1) then there exist nonsingular constant matrices H, H_1 such that

$$Y[\varphi(t)] \equiv Y(t)H,$$
$$Y_1[\varphi(t)] \equiv Y_1(t)H_1 .$$

Since there is a constant matrix C such that

$$Y_1(t) = Y(t)C$$

it follows that

$$Y_1[\varphi(t)] \equiv Y[\varphi(t)]C \equiv Y(t)HC .$$

Since $Y_1[\varphi(t)] \equiv Y_1(t)H_1 \equiv Y(t)CH_1$ we get

$$Y(t)HC \equiv Y(t)CH_1$$

or

$$HC \equiv CH_1$$

hence

$$H_1 \equiv C^{-1}HC .$$

Conversely, if $Y(t)$ is a fundamental matrix for (1) satisfying (2) and $H_1 = C^{-1}HC$ then since $Y_1(t) \equiv Y(t)C$ is a fundamental matrix for (1) the following identity is hold

$$Y_1[\varphi(t)] \equiv Y[\varphi(t)]C \equiv Y(t)H(t)C \equiv Y(t)CH_1 \equiv Y_1(t)H_1 .$$

Theorem. Any fundamental matrix $Y(t)$ for the equation (1) may be written as

$$Y(t) = P(t)\exp\{F(t)S\}, \tag{3}$$

where $P(t)$ is a nonsingular nxn matrix such that $P[\varphi(t)] = P(t)$, $t \in (-\infty, \infty)$, and S is a constant matrix and $F(t)$ is an increasing solution of the Abel functional equation $F[\varphi(t)] - F(t) = 1$.

Conversely, if $P(t)$ and S satisfy (3) with a fundamental matrix $Y(t)$ of (1) and with an increasing solution $F(t)$ of the Abel functional equation, then

$$P(t) + PSF'(t) - A(t)P(t) = 0 \text{ for } t \in (-\infty, \infty),$$

and under the transformation

$$y(t) = P(t)w(t), \quad t \in (-\infty, \infty) \tag{4}$$

the differential equation (1) reduces to

$$w' = SF'(t)w, \quad t \in (-\infty, \infty). \tag{5}$$

P r o o f . a) Let $Y(t)$ be a fundamental matrix for (1), and H a constant matrix satisfying $Y[\varphi(t)] = Y(t)H(t)$. We know [1], [2] that there exists a matrix S such that $H = \exp S$. Thus if we set

$$P(t) = Y(t)\exp\{-F(t)S\}$$

we obtain

$$P[\varphi(t)] = Y[\varphi(t)]\exp\{-F[\varphi(t)]S\} = Y(t)H\exp\{(-F(t)-1)S\} =$$
$$= Y(t)\exp S\exp(-F(t)S - S) = Y(t)\exp\{-F(t)S\} = P(t).$$

Thus

$$P[\varphi(t)] = P(t)$$

and we have

$$Y(t) = P(t)\exp\{F(t)S\}.$$

b) Let $P(t) = Y(t)\exp\{-F(t)S\}$. Since $Y' = A(t)Y$ and $(\exp\{-F(t)S\})' = \exp\{-F(t)S\}F'(t)S$ we have

$$P'(t) = Y'(t)\exp\{-F(t)S\} - Y(t)\exp\{-F(t)S\}F'(t)S$$

or

$$P'(t) - A(t)Y(t)\exp\{-F(t)S\} + Y(t)\exp\{-F(t)S\}F'(t)S = 0.$$

After arrangements we obtain

$$P'(t) - A(t)P(t) + P(t)F'(t)S = 0 .$$

Hence

$$F'(t)S = P^{-1}(t)(A(t)P(t) - P'(t)) \text{ for } t \in (-\infty, \infty) . \qquad (6)$$

With respect to (1) and the transformation

$$y = Pw \qquad (7)$$

we get

$$(y' \equiv) P'w + Pw' = APw$$

or

$$AP - P' = Pw'w^{-1}$$

and inserting into (7) we get

$$w' = F'(t)Sw, \quad t \in (-\infty, \infty) . \qquad (8)$$

REFERENCES

[1] K u r z w e i l, J.: Ordinary Differential Equations, Elsevier, Amsterodam, 1986.

[2] L a i t o ch, M.: Rasširenije metoda Floke, Czech.Math.J. 5(80), (1955), 164-174.

[3] R e i d, W.T.: Ordinary Differential Equations, John Wiley and sons, inc., New York, 1971.

Department of Mathematics
Palacký University
Žerotínovo nám.2, 771 40 Olomouc
Czechoslovakia