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ON SOME MODIFICATION OF THE LEVINSON OPERATOR AND ITS APPLICATION TO A THREE-POINT BOUNDARY VALUE PROBLEM

JAN ANDRES

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INTRODUCTION

The concept of the translation operator and its application to the periodic boundary value problem is well-known since the time of Poincaré [1]. However, Levinson [2] was the first who studied its properties in detail (whence the name) with respect to the second order differential equations.

Since there exists the whole theory concerning the translation operator at present (see e.g. [3]), we will not repeat here the basic notions, but will be concentrated directly to our goal, consisting of a suitable modification of the Levinson operator, corresponding to the following three-point periodic boundary value problem, namely

$$x'' = f(t,x), \quad f \in C(\langle 0,2a\rangle \times R^1) \tag{1}$$

and $x(0) = x(a) = x(2a), \quad 0 \neq a \in R^1.$ (2)

Remark 1. In order to apply the modified Levinson operator, it is necessary to assume that all solutions of (1) exist and are uniquely determined on $\langle 0,2a\rangle$. However, since we use the a priori estimates technique, the existence requirement can be strictly localized (and hence practically omitted; see bellow). Moreover, it is possible (see [3, p.23]) to approximate uniformly f(t,x) by the Lipschitz functions with respect to x on $\langle 0,2a\rangle$ in arbitrary accuracy and to apply the standard limiting argument (cf. [3, p.24]) to satisfy the uniqueness.

2. MODIFICATION OF THE OPERATOR TO SECOND ORDER SCALAR EQUATIONS

Hence, let us define the modified Levinson operator $T_{\mu,\nu}X(0)$, where $\mu, \nu \in \langle 0,1\rangle$ are parameters and $X(0) = (x(0), x'(0))$ are the Cauchy initial values of solutions x(t) of (1), in the following way

$$T_{\mu,\nu}X(0) = \begin{cases} \left\{\frac{1}{a}[x(a)-x(0)], a^{-2}[x(2a)-2x(a)+x(0)]\right\} & \text{for } \mu = \nu = 1, \\ \left\{\mu a[x(\nu a)-x(0)], [x((\mu+\nu)a)-x(\nu a)]-[x(\mu a)-x(0)]\right\}(\mu\nu)^{-1}a^{-2} & \text{for } \mu,\nu \in (0,1\rangle, \\ [x(\nu a)-x(0), x'(\nu a)-x'(0)](\nu a)^{-1} & \text{for } \mu = 0, \nu \in (0,1\rangle, \\ [X'(0), f(0,x(0))] & \text{for } \mu = \nu = 0. \end{cases}$$

It is clear that

$$T_{1,1}X(0) = 0 \text{ iff } x(0) = x(a) = x(2a)$$

$$\text{and } T_{0,1}X(0) = 0 \text{ iff } x(0) = x(a),\ x'(0) = x'(a). \tag{3}$$

Lemma 1. The problem (1) - (2) is solvable, provided

$$\frac{f(0,x(0))}{|f(0,x(0))|} \neq \frac{f(0,-x(0))}{|f(0,-x(0))|} \quad (f(0,x(0)) \neq 0) \tag{i}$$

holds for $|x(0)| \geq R_0$, where R_0 is a suitable positive constant

and $T_{\mu,1}X(0) \neq 0$, $T_{0,\nu}X(0) \neq 0$ (ii)

for $\|X(0)\| \geq R \geq R_0$ (R - great enough real), independently of $\mu, \nu \in (0,1\rangle$.

Proof. As we have already pointed out, the problem considered is solvable if

$$T_{1,1}X(0) = 0.$$

Since we employ here the topological degree arguments, the fundamental requirement reads [4] (see also (3))

$$T_{1,1}X(0) \neq 0, \quad (T_{0,1}X(0) \neq 0) \tag{4}$$

on the sphere $\|X(0)\| = R$. But assuming (ii), condition (4) can be replaced by

$$T_{0,0}X(0) \neq 0 \quad \text{for} \quad \|X(0)\| = R \tag{5}$$

by virtue of the well-known [4] invariance under the homotopy. Furthermore, since the degree of an odd operator is not equal to zero on the sphere according to the classical Borsuk antipodal theorem [4], namely

$$d[T_{0,0}X(0) - T_{0,0}(-X(0)), \|X(0)\| \leq R, 0] \neq 0 \quad \text{for } \|X(0)\| = R,$$

condition (5) can be still replaced by

$$T_{0,0}X(0) - (1-\lambda)T_{0,0}(-X(0)) \neq 0 \quad \text{for all } \lambda \in (0,1\rangle ,$$

which is certainly implied by (i) for $f(0,x(0)) \neq 0$. This completes the proof.

Lemma 2. Condition (ii) of Lemma 1 is fulfilled, if all solutions $x(t)$ of (1), satisfying the following simultaneous boundary conditions

$$x(\nu a) = x(0),\ x'(\nu a) = x'(0) \quad \text{for all } \nu \in (0,1\rangle , \tag{6}$$

$$x((\mu+1)a) = x(\mu a),\ x(a) = x(0) \quad \text{for all } \mu \in (0,1\rangle \tag{7}$$

are uniformly a priori bounded (in case (6) with their derivatives $x'(t)$ as well).

Proof. It can be easily seen (see (ii)) that

$$T_{\mu,1}X(0) \neq 0 \text{ iff } x(a)\neq x(0) \text{ or } x((1+\mu)a)\neq x(\mu a) \quad \text{for all } \mu\in(0,1\rangle \ ,$$
$$T_{0,\nu}X(0) \neq 0 \text{ iff } x(\nu a)\neq x(0) \text{ or } x'(\nu a)\neq x'(0) \quad \text{for all } \nu\in(0,1\rangle \ .$$

Therefore assuming a priori estimates as above (cf. (6), (7)), these inequalities are trivially satisfied for R great enough.

Remark 2. The **criterium of the solvability of (1) - (2) is re**presented by the assumptions of Lemma 2 and (i) (for the problem of Poincaré it is enough to verify besides (i) a priori estimates corresponding to (6), only).

3. A PRIORI ESTIMATES

Now we will proceed to the verification of such a criterium, provided

$$\liminf_{|x|\to\infty} f(t,x)\operatorname{sgn} x > 0, \tag{8}$$

uniformly with respect to $t\in\langle 0,2a\rangle$, by which (i) is guaranteed immediately.

Lemma 3. The uniform a priori boundedness of solutions $x(t)$ of problem (1) - (6) implies the same for their derivatives $x'(t)$.

Proof. According to Hille's version [5] of the Landau inequality, we have

$$\| x'(t)\|^2 \leq 4\| x(t)\| \ \| x''(t)\|$$

for all bounded functions $x(t)\in C^2(R^1)$, where $\| . \| = \sup_{t\in\langle 0,\infty)} | . |$.

If $x(t)$ are uniformly bounded on each subinterval of $\langle 0,2a\rangle$, $|f(t,x)|$ atteins there its maximum, say M_x, with respect to the continuity assumed, and consequently we have

$$\| x'(t) \|^2 \le 4M_x \| x(t) \| ,$$

which was to be proved.

As we can see, only a priori estimates of solutions to (1) - (6) or (1) - (7) are needed.

Lemma 4. There exists such a positive constant S that all solutions x(t) of (1) - (6) or (1) - (7) are under (8) uniformly a priori bounded by it on the appropriate intervals of their existence.

Proof. Relation (8) implies the existence of such an $S > 0$ great enough that

$$x'' \operatorname{sgn} x = f(t,x) \operatorname{sgn} x > 0 \tag{9}$$

for all $t \in \langle 0,2a \rangle$ and $|x| > S$. If $|x(0)| \le S$ and $|x(t_0)| > S$ for some $t_0 \in (0,\nu a)$, then $x(t) > S$ becomes convex (cf. (9)), while $x(t) < -S$ becomes concave there by the same reason, by which x(t) cannot evidently come to $x(\nu a)$ (for $\nu = 1$ to x(a) in (7)). In this respect $|x(\mu a)| \le S$ must be satisfied as well and consequently also $|x(t)| \le S$ for all $t \in \langle 0,(1+\mu)a \rangle$ (for $\mu = 1$ to x(2a)), because of (7). Hence, if $|x(0)| \le S$, then the solutions of boundary value problems (1) - (6) and (1) - (7) are uniformly bounded by the same constant.

If $|x(0)| > S$, then x(t) becomes conve^ν or concave just from the beginning, respectively and either $x(\nu a) \ne x(0)$ again (for $\nu = 1$ $x(a) \ne x(0)$ in (7)) or $x'(\nu a) \ne x'(0)$ in the case corresponding to (6) resp. $x((\mu+1)a) \ne x(\mu a)$ in the case corresponding to (7). Thus, x(t) is bounded in the same way.

4. MAIN STATEMENT AND CONCLUDING REMARKS

Now we can give the principal result.

Theorem. The problem (1) - (2) admits a solution, provided (8).

Proof - follows immediately from Lemmas 1 - 4 with respect to Remark 2.

Remark 3. It would be only a technical matter to generalize the above idea and solve (n+1)-point periodic-like boundary value problem.

Remark 4. If $f(t+a,x) \equiv f(t,x)$, we obtain as a special product of our investigation the existence of an a-periodic solution of (1) (see Remarks 1 - 2). This result is comparable in certain aspects to those obtained earlier by several authors (see e.g. [6], [7] and the references included).

SOUHRN

O JISTÉ MODIFIKACI LEVINSONOVA OPERÁTORU A JEJÍ APLIKACI NA TŘÍBODOVOU OKRAJOVOU ÚLOHU

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Jsou nalezeny postačující podmínky řešení tříbodové okrajové úlohy periodického typu (1) - (2) na bázi adekvátně modifikovaného operátoru Levinsona.

РЕЗЮМЕ

ОБ ОДНОЙ МОДИФИКАЦИИ ОПЕРАТОРА ЛЕВИНСОНА И ЕЕ ПРИЛОЖЕНИИ К ТРЕХТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧЕ

Я. АНДРЕС

Показаны достаточные условия решения трехточечной краевой задачи периодического типа /1/-/2/ на основе соответствующим способом обработанного оператора Левинсона.

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