Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 28 (1989), No. 1, 43--60

Persistent URL: `http://dml.cz/dmlcz/120222`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1989 MATHEMATICA XXVIII VOL. 94

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr. Jindřich Palát, CSc.

ON CERTAIN MULTIPOINT BOUNDARY VALUED PROBLEMS

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(Received April 30, 1988)

In this paper we are concerned with the existence of solutions of the equation

$$u''' = f(t,u',u''), \tag{0.1}$$

satisfying the conditions

$$u(a) = u(t_1),\ u(t_2) = u(t_3),\ u(t_4) = u(b), \tag{0.2}$$

where $-\infty < a < t_1 \leqq t_2 < t_3 \leqq t_4 < b < +\infty$. For similar problems for differential equations of the second order, we refer to [7, 8].

Since the linear operator d^3/dt^3 subjected to the condition (0.2) has the zero eigenvalue, we cannot use existence theorems of the type of C o n t i (see Lemma 4). That is why we prove the proposition which guarantees the existence of solutions of (0.1), (0.2) even for operators having the zero eigenvalue. By means of it we prove Theorems 1 and 2.

Throughout the paper we use the following notations: $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, N is the set of all natural numbers, $D = [a,b] \times R^3$, $D_+ = [a,b] \times R_+^3$;
$c_1 = \max \{|t_2-a|, |t_4-t_2|, |b-t_4|\}$, $c_2 = \max\{|t_3-a|, |b-t_2|\}$;
$C^{(i)}(a,b)$ is the set of all real functions having the continuous i-th derivatives on $[a,b]$;
$AC^{(i)}(a,b)$ is the set of all real functions having the absolutely continuous i-th derivatives on $[a,b]$;
$L^i(a,b)$ is the set of all real functions f such that f^i is Lebesgue integrable on (a,b) ;
$Car_{loc}(D)$ is the set of all real functions satisfying the local Carathéodory conditions on D ;
a.e. = "almost every" ;
we say that some property is satisfied on D, if it is satisfied for a.e. $t \in [a,b]$ and for all $(x,y,z) \in R^3$.

In the following the function f is supposed to be of $Car_{loc}(D)$ and the number $\lambda \in \{-1,1\}$.

By a solution to (0.1), (0.2), we mean a function $u \in AC^2(a,b)$, verifying (0.1) for a.e. $t \in [a,b]$ and satisfying (0.2).

1. The existence results

Theorem 1. Let there exist $r \in (0,+\infty)$ such that on the set D there are satisfied the inequalities

$$\lambda . f(t,x,y,z). \operatorname{sgn} x \geq 0 \quad \text{for } |x| \geq r, \tag{1.1}$$

$$|f(t,x,y,z)| \leq \omega(t,|x|,|y|,|z|) , \tag{1.2}$$

where $\omega \in Car_{loc}(D_+)$ is a non-negative function non-decreasing in its second, third and fourth arguments and

$$\limsup_{\rho \to \infty} \frac{1}{\rho} \int_a^b \omega(t, \rho c_1(b-a), \rho c_1, \rho)dt < 1 . \tag{1.3}$$

Then the problem (0.1), (0.2) has at least one solution.

Corollary. Let there exist $r \in (0,+\infty)$ such that on the set D there are satisfied the inequalities (1.1) and

$$|f(t,x,y,z)| \leq h_1(t)|x| + h_2(t)|y| + h_3(t)|z| + \omega(t,|x|+|y|+|z|), \tag{1.4}$$

where $h_i \in L^1(a,b)$, i=1,2,3, are non-negative functions fulfilling

$$\int_a^b h_3(t)dt + c_1 \int_a^b h_2(t)dt + c_1(b-a)\int_a^b h_1(t)dt < 1 , \tag{1.5}$$

and $\omega \in Car_{loc}([a,b]\times R_+)$ is non-negative, non-decreasing in its second argument and

$$\lim_{\varrho \to \infty} \frac{1}{\varrho} \int_a^b \omega(t,\varrho)dt = 0 . \tag{1.6}$$

Then the problem (0.1), (0.2) is solvable.

Theorem 2. Let there exist $r \in (0,+\infty)$ such that on the set D there are satisfied the inequalities (1.1) and

$$|f(t,x,y,z)| \leq a_1|x| + a_2|y| + a_3|z| + \omega(t,|x|+|y|+|z|) , \tag{1.7}$$

where $a_i \in R_+$, i=1,2,3, are such that

$$a_1(2/\pi)^3 c_2 c_1(b-a) + a_2(2/\pi)^2 c_2 c_1 + a_3(2/\pi)c_2 < 1 \tag{1.8}$$

and $\omega : [a,b] \times R_+ \longrightarrow R_+$ has the properties

$$(1.9)\begin{cases} \omega(\cdot,\varrho) \in L^2(a,b) \quad \text{for any } \varrho \in R_+ \\ \omega(t,\cdot) \in C(R_+) \quad \text{is non-decreasing for a.e. } t \in [a,b] \\ \lim\limits_{\varrho \to \infty} \frac{1}{\varrho} (\int_a^b \omega^2(t,\varrho)dt)^{1/2} = 0 . \end{cases}$$

Then the problem (0.1), (0.2) is solvable.

Theorem 3. Let there exist $a_i \in R_+$ and $h_i \in L^1(a,b)$, $i=1,2,3$, such that

$$(b-a)\left[a_1(2/\pi)^3 c_2 c_1 + a_2(b-a)/2 + a_3\right] \leq 1 , \tag{1.10}$$

$$0 < \lambda h_1(t) \leq a_1,\ |h_2(t)| \leq a_2,\ |h_3(t)| \leq a_3$$
$$\text{for a.e. } t \in [a,b] , \tag{1.11}$$

$$|f(t,x,y,z) - h_1(t)x - h_2(t)y - h_3(t)z| \leq$$
$$\leq \omega(t,|x|+|y|+|z|) \text{ on } D , \tag{1.12}$$

where ω is the function from Corollary.
Then the problem (0.1), (0.2) has at least one solution.

2. Preliminary results

Lemma 1. ([3], Theorem 256, p.219). If $f \in AC(c,d)$, $f' \in L^2(c,d)$ and $f(t_0) = 0$, where $-\infty < c \leq t_0 \leq d < +\infty$, then

$$\int_c^d f^2(t)dt \leq \left[2(d-c)/\pi\right]^2 \int_c^d f'^2(t)dt .$$

Lemma 2. Let us suppose that $a_i \in R_+$ and $h_i \in L^1(a,b)$, $i=1,2,3$, satisfy (1.10) and (1.11). Then the equation

$$u''' = \sum_{i=1}^{3} h_i(t)u^{(i-1)} \tag{2.1}$$

has only the trivial solution fulfilling (0.2).

Proof. Since (1.10),

$$a_2(b-a)^2/2 + a_3(b-a) \leq 1 . \tag{2.2}$$

Therefore, by (1.11), any nontrivial solution of the equation

$$v''' = h_2(t)v' + h_3(t)v''$$

has not more than 2 zeros on [a,b] (see [11], p.157 or [5], p.116). Therefore, in accordance to the Frobenius factorization (see [4], p.87 or [2], p.91-94), the equation (2.1) can be written in the form

$$\frac{1}{p_3(t)}\left(\frac{1}{p_2(t)}\left(\frac{u'}{p_1(t)}\right)'\right)' = h_1(t)u , \qquad (2.3)$$

where $p_1 \in AC^1(a,b)$, $p_2, p_3 \in AC(a,b)$ and $p_i(t) \neq 0$ for $a \leq t \leq b$, $i=1,2,3$.

Now, admitting to the contrary that u is a nontrivial solution of (2.1), (0.2), we get points α_1, α_2, α_3 such that

$$u'(\alpha_i) = 0,\ i=1,2,3,\quad \alpha_1 \in (a,t_1),\quad \alpha_2 \in (t_2,t_3),\quad \alpha_3 \in (t_4,b) \qquad (2.4)$$

and points β_1, β_2 such that

$$u''(\beta_i) = 0,\ i=1,2,\quad \beta_1 \in (\alpha_1, \alpha_2),\quad \beta_2 \in (\alpha_2, \alpha_3). \qquad (2.5)$$

Let us suppose that $u(t) \neq 0$ for $a \leq t \leq b$. In accordance to (2.5) the function $\varphi(t) = (1/p_2(t))(u'/p_1(t))'$ has two zeros on (a,b) and on the other hand from (2.3) it follows that φ is strictly monotonous on (a,b). This contradiction implies the existence of $\tilde{t} \in (a,b)$ such that

$$u(\tilde{t}) = 0 . \qquad (2.6)$$

Put $\rho_0 = (\int_a^b u'''^2(t)dt)^{1/2}$. Since $a < \beta_1 < \alpha_2 < t_3$ and $t_2 < \alpha_2 < \beta_2 < b$, we get by Lemma 1

$$\int_a^{\alpha_2} u''^2(t)dt \leq \left[2(\alpha_2-a)/\pi\right]^2 \int_a^{\alpha_2} u'''^2(t)dt \quad \text{and}$$

$$\int_{\alpha_2}^{b} u''^2(t)dt \leq \left[2(b-\alpha_2)/\pi\right]^2 \int_{\alpha_2}^{b} u'''^2(t)dt. \quad \text{Therefore}$$

$$\left(\int_a^b u''^2(t)dt\right)^{1/2} \leq (2/\pi) c_2 \rho_0 . \qquad (2.7)$$

Similarly, by (2.4) and Lemma 1,

$$\int_a^{t_2} u'^2(t)dt \leq \left[2(t_2-a)/\pi\right]^2 \int_a^{t_2} u''^2(t)dt ,$$

$$\int_{t_2}^{t_4} u'^2(t)dt \leq \left[2(t_4-t_2)/\pi\right]^2 \int_{t_2}^{t_4} u''^2(t)dt ,$$

$$\int_{t_4}^{b} u'^2(t)dt \leq \left[2(b-t_4)/\pi\right]^2 \int_{t_4}^{b} u''^2(t)dt . \quad \text{Hence}$$

$$\left(\int_a^b u'^2(t)dt\right)^{1/2} \leq (2/\pi)^2 c_2 c_1 \rho_0 . \qquad (2.8)$$

Finally, by (2.6) and Lemma 1,

$$\left(\int_a^b u^2(t)dt\right)^{1/2} \leq (2/\pi)^3 . c_2 c_1 (b-a) \rho_0 . \qquad (2.9)$$

Thus we can find from (2.1), (1.10), (2.7), (2.8), (2.9),

$$\rho_0 \leq \rho_0 \left[a_1 (2/\pi)^3 c_2 c_1 (b-a) + a_2 (2/\pi)^2 c_2 c_1 + a_3 (2/\pi) c_2\right] .$$

Consequently, by (1.8), $\rho_0 = 0$. According to (2.9) we obtain

$$u(t) = 0 \text{ for } a \leq t \leq b .$$

<u>Lemma 3</u> ([5], Lemma 2.2, p.12). Let g_i, $k_i \in L^1(a,b)$, $i=1,2,3$, and for any $h_i \in L^1(a,b)$, $i=1,2,3$, satisfying

$$g_i(t) \leq h_i(t) \leq k_i(t), \quad i=1,2,3, \qquad (2.10)$$

for a.e. $t \in [a,b]$, the problem (2.1), (0.2) have only the trivial solution.

Then there exists such $\gamma \in (0,+\infty)$, that for any $h_i \in L^1(a,b)$ satisfying (2.10), it holds

$$\sum_{i=1}^{3} \left| \frac{\partial^{i-1} G(t,s)}{\partial t^{i-1}} \right| \leq \gamma \qquad \text{for } a \leq t \leq b \qquad (2.11)$$

where G is the Green's function for the problem (2.1), (0.2).

Lemma 4 (C o n t i). Let $h_i \in L^1(a,b)$, i=1,2,3, $g \in Car_{loc}(D)$ and let the problem (2.1), (0.2) have only the trivial solution. If there exists $g^* \in L^1(a,b)$ such that

$$|g(t,x,y,z)| \leq g^*(t) \quad \text{on } D ,$$

then the equation

$$u''' = \sum_{i=1}^{3} h_i(t)u^{(i-1)} + g(t,u,u',u'')$$

has a solution satisfying (0.2).

Proof. See for example [5], Theorem 2.4, p.25.

3. Existence proposition

Let there exist $r \in (0,+\infty)$ and a function $h \in L^1(a,b)$ such that on the set D there are satisfied the conditions (1.1) and

$$|f(t,x,y,z)| \leq h(t) . \qquad (3.1)$$

Then the problem (0.1), (0.2) has at least one solution u such that

$$\min\{|u(t)| : a \leq t \leq b\} \leq r . \qquad (3.2)$$

Proof. Let us choose $m_o \in N$ such that

$$m_o \geq (2/\pi)^3 c_2 c_1 (b-a) \qquad (3.3)$$

and consider the equations

$$u''' = (\lambda/m)u \tag{3.4}$$

and

$$u''' = (\lambda/m)u + f(t,u,u',u'') , \tag{3.5}$$

where $m \in N$, $m \geq m_0$.
Using Lemma 2 for $h_1 = \lambda/m$, $h_2 = h_3 = 0$, we can conclude that the problem (3.4), (0.2) has only the trivial solution. Therefore, by Lemma 3, there exists $\gamma = \gamma(m) \in (0,+\infty)$ such that for the Green's function G_m of the problem (3.4), (0.2) the inequality (2.11) is valid.

Let us denote by $\mathcal{B}$ the Banach space of all functions of $C^2(a,b)$ with the norm

$$\|z\| = \max\left\{\sum_{i=1}^{3}|z^{(i-1)}(t)| \ : \ a \leq t \leq b\right\}, \qquad z \in C^2(a,b)$$

and for the fixed number m define a continuous operator $H : \mathcal{B} \longrightarrow \mathcal{B}$ by

$$H(z(t)) = \int_a^b G_m(t,s)f(s,z(s),z'(s),z''(s))ds \ .$$

From (2.10) and (3.1) it follows that H maps $\mathcal{B}$ into its compact subset. (Really, the functions of $H(\mathcal{B})$ are uniformly bounded with their first and second derivatives on $[a,b]$ by the constant $\gamma(m)\int_a^b h(s)ds$. Since $\dfrac{\delta^3 G_m(t,s)}{\delta t^3} = \dfrac{\lambda}{m} G_m(t,s)$ for a.e. $t,s \in [a,b]$, the functions of $H(\mathcal{B})$ are also equi-continuous with their first and second derivatives on $[a,b]$.) Consequently, by Schauder fixed-point theorem, there exists $u_m \in \mathcal{B}$ such that $H(u_m) = u_m$, i.e. u_m is a solution of the problem (3.5), (0.2).

According to (0.2), in the same way as in the proof of

Lemma 2, we get the points $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ satisfying (2.4), (2.5). Now, suppose that $|u_m(t)| \geqq r$ on $[a,b]$. Then, in view of (1.1), $\lambda u_m''' \operatorname{sgn} u_m = \lambda \frac{1}{m} u_m \operatorname{sgn} u_m + \lambda f(t,u_m,u_m',u_m'') \operatorname{sgn} u_m > 0$ for a.e. $t \in [a,b]$, which contradicts to the fact, that u_m'' has two zeros $\beta_1, \beta_2 \in (a,b)$. Thus there exists $t_m \in (a,b)$ such that

$$|u_m(t_m)| < r . \qquad (3.6)$$

By (3.1), (3.3), (3.5),

$$|u_m'''(t)| \leqq h(t) + a_1 |u_m| \quad \text{for a.e. } t \in [a,b], \qquad (3.7)$$

where

$$a_1 \in (0, \tfrac{1}{2}(b-a)^{-3}) . \qquad (3.8)$$

Put $\varrho = \max\{|u_m(t)| : a \leqq t \leqq b\}$ and $h_o = \int_a^b h(s)ds$. Then, by integration (3.7) from β_1 to t, we have

$$|u_m''(t)| \leqq h_o + a_1(b-a)\varrho , \qquad (3.9)$$

integrating (3.9) from α_1 to t, we get

$$|u_m'(t)| \leqq h_o(b-a) + a_1(b-a)^2 \varrho , \qquad (3.10)$$

and integrating (3.10) from t_m to t, we obtain

$$\varrho \leqq r + h_o(b-a)^2 + a_1(b-a)^3 \varrho . \qquad (3.11)$$

From (3.8), (3.11) it follows

$$\varrho \leqq 2(r + h_o(b-a)^2) ,$$

and hence, by (3.9), (3.10),

$$\sum_{i=1}^{3} |u^{(i-1)}(t)| \leqq \varrho^* \quad \text{for } a \leqq t \leqq b , \qquad (3.12)$$

where $\varrho^* = h_o(1+b-a) + (r+h_o(b-a)^2)(2+(b-a)^{-2} + (b-a)^{-1})$.

Since (3.12), the functions $(u_m)_{m=m_o}^{\infty}$ are uniformly bounded and equi-continuous with their first and second derivatives on $[a,b]$, we can suppose without loss of generality, by the Arzelà-Ascoli lemma, that the sequences $(u_m)_{m=m_o}^{\infty}$, $(u'_m)_{m=m_o}^{\infty}$ and $(u''_m)_{m=m_o}^{\infty}$ are uniformly converging on $[a,b]$ and a function

$$u = \lim_{m\to\infty} u_m \quad \text{on} \quad [a,b]$$

is a solution of the problem (0.1), (0.2).

Finally, we prove that u satisfies (3.2). Let us suppose the contrary and put $v_o = \min\{|u(t)| : a \le t \le b\}$ and $\delta = v_o - r > 0$. Since for any $m \in N$, $m \ge m_o$ there exists $t_m \in (a,b)$ fulfilling (3.6), it holds $|u(t_m)-u_m(t_m)| \ge |u(t_m)| - |u_m(t_m)| > v_o - r = \delta$, which contradicts to the uniform convergence of $(u_m)_{m=m_o}^{\infty}$ on $[a,b]$. Thus u satisfies (3.2). This completes the proof.

4. A priori estimates

Lemma 5. Let $\omega \in Car_{loc}(D_+)$ satisfy the conditions of Theorem 1 and $r \in (0,+\infty)$.

Then there exists $r^* \in (r,+\infty)$ such that for any function $u \in AC^2(a,b)$ from the conditions (0.2), (3.2) and

$$|u'''(t)| \le \omega(t,|u|,|u'|,|u''|) \quad \text{for a.e.} \quad t \in [a,b] \qquad (4.1)$$

it follows the estimate

$$\sum_{i=1}^{3} |u^{(i-1)}(t)| \le r^* \quad \text{for} \quad a \le t \le b \quad . \qquad (4.2)$$

Proof. Let $u \in AC^2(a,b)$ satisfy (0.2), (3.2) and (4.1). According to (0.2), in the same way as in the proof of Lemma 2, we get the points $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ satisfying (2.4), (2.5). In view of (3.2) there exists $\tilde{t} \in (a,b)$ such that $|u(\tilde{t})| \le r$.

Let us put $\rho_0 = \max\{|u''(t)| : a \leq t \leq b\}$ and integrate the inequality $|u''(t)| \leq \rho_0$ by sequel from t to α_i, i=1,2,3. We get $|u'(t)| \leq \rho_0 \cdot c_1$. Integrating the latter from t to $\tilde{t}$, we have $|u(t)| \leq r + \rho_0 c_1(b-a)$.

Now, let $t^* \in [a,b]$ be such that $|u''(t^*)| = \rho_0$. Then, integrating (4.1) from t^* to β_1, we obtain

$$\rho_0 \leq \int_a^b \omega(t, r+\rho_0 c_1(b-a), \rho_0 c_1, \rho_0)dt . \qquad (4.3)$$

According to (1.3), there exists $\delta > 0$ such that $(1+\delta) \limsup_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \omega(t, \rho c_1(b-a), \rho c_1, \rho)dt < 1$. Hence there exists $\rho^* > 0$ such that for any $\rho > \rho^*$ it holds $(1+\delta)\rho c_1(b-a) \geq r + \rho c_1(b-a)$ and

$$\frac{1}{\rho} \int_a^b \omega(t, (1+\delta)\rho c_1(b-a), (1+\delta)\rho c_1, (1+\delta)\rho)dt < 1 .$$

Therefore

$$\int_a^b \omega(t, r+\rho c_1(b-a), \rho c_1, \rho)dt < \rho . \qquad (4.4)$$

From (4.3) and (4.4) it follows that $\rho_0 \leq \rho^*$. Putting

$$r^* = r + \rho^*(1+c_1+c_1(b-a)) ,$$

we get the estimate (4.2).

<u>Lemma 6</u>. Let $\omega : [a,b] \times R_+ \to R_+$ has the properties (1.9), $a_i \in R_+$, i=1,2,3, satisfy (1.8) and $r \in (0,+\infty)$.

Then there exists $r^* \in (r,+\infty)$ such that for any function $u \in AC^2(a,b)$ the conditions (0.2), (3.2) and

$$|u'''(t)| \leq \sum_{i=1}^{3} a_i |u^{(i-1)}| + \omega(t, \sum_{i=1}^{3} |u^{(i-1)}|), \text{ for a.e. } t \in [a,b] \quad (4.5)$$

imply the estimate (4.2).

Proof. Let $u \in AC^2(a,b)$ satisfy (0.2), (3.2) and (4.5). According to (0.2), in the same way as in the proof of Lemma 2, we get the points $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ satisfying (2.4), (2.5). In view of (3.2) there exists $\tilde{t} \in (a,b)$ such that $|u(\tilde{t})| \leq r$. Let us put

$$\left(\int_a^b u'''^2(t)dt\right)^{1/2} = \rho_0 \ . \quad (4.6)$$

Then, analogously as in the proof of Lemma 2, we obtain

$$\left(\int_a^b u''^2(t)dt\right)^{1/2} \leq (2/\pi)c_2\rho_0 \ , \ \left(\int_a^b u'^2(t)dt\right)^{1/2} \leq (2/\pi)^2 c_2 c_1 \rho_0 .$$

According to Lemma 1, we get from the latter inequality

$\left(\int_a^b [u(t)-u(\tilde{t})]^2 dt\right)^{1/2} \leq (2/\pi)^3 c_2 c_1 (b-a)\rho_0$. Now, substituting the estimate obtained above into (4.5), we have

$$\rho_0 \leq \rho_0 [a_3(2/\pi)c_2 + a_2(2/\pi)^2 c_2 c_1 + a_1(2/\pi)^3 c_2 c_1 (b-a)] + + r\sqrt{b-a}\, a_1 + \left(\int_a^b \omega^2(t, \sum_{i=1}^{3} |u^{(i-1)}|)dt\right)^{1/2} . \quad (4.7)$$

Let $\rho_i = \max\{|u^{(i-1)}(t)| : a \leq t \leq b$ and $\tau_i \in [a,b]$ be such that $|u^{(i-1)}(\tau_i)| = \rho_i$, $i=1,2,3$. Then, by (4.6) and the Schwarz inequality, we get

$$\rho_3 = \left|\int_{\beta_1}^{\tau_3} u'''(t)dt\right| \leq \int_a^b |u'''(t)|dt \leq \rho_0 \sqrt{b-a} \ ,$$

$$\rho_2 = \left|\int_{\alpha_1}^{\tau_2} u''(t)dt\right| \leq \int_a^b |u''(t)|dt \leq (2/\pi)c_2\,\rho_0\sqrt{b-a}\,,$$

$$\rho_1 \leq \left|\int_{\tilde{t}}^{\tau_1} u'(t)dt\right| \leq \int_a^b |u'(t)|dt + r \leq (2/\pi)^2 c_2 c_1 \rho_0 \sqrt{b-a} + r\,.$$

Therefore

$$\sum_{i=1}^{3} \rho_i \leq l\rho_0, \text{ where } l=\sqrt{b-a}(1+(2/\pi)c_2+(2/\pi)^2 c_2 c_1)+r. \tag{4.8}$$

Inserting (4.8) into (4.7), we have

$$1 \leq \left[a_3(2/\pi)c_2+a_2(2/\pi)^2 c_2 c_1+a_1(2/\pi)^3 c_2 c_1 (b-a)\right] + \\ + \frac{1}{\rho_0} r\sqrt{b-a}\, a_1 + \frac{1}{\rho_0}\left(\int_a^b \omega^2(t, l\rho_0)dt\right)^{1/2}. \tag{4.9}$$

Since (1.8), (1.9), there exists $\rho^* > 0$ such that for any $\rho > \rho^*$ the inequality

$$1 > \left[a_3(2/\pi)c_2+a_2(2/\pi)^2 c_2 c_1+a_1(2/\pi)^3 c_2 c_1 (b-a)\right] + \\ + \frac{1}{\rho} r\sqrt{b-a}\, a_1 + \frac{1}{\rho}\left(\int_a^b \omega^2(t, l\rho)dt\right)^{1/2}. \tag{4.10}$$

Therefore, by (4.9), (4.10), $\rho_0 \leq \rho^*$. Thus the estimate (4.2) is valid for $r^* = l\,\rho^*$, where l is determined by (4.8).

<u>Lemma 7</u>. Let us suppose that $a_i \in R_+$ and $h_i \in L^1(a,b)$, $i=1, 2,3$, satisfy (1.10) and (1.11) and ω be the function from Corollary.

Then there exists $r^* \in (r,+\infty)$ such that for any function $u \in AC^2(a,b)$ the conditions (0.2) and

$$\left| u''' - \sum_{i=1}^{3} h_i(t)u^{(i-1)} \right| \leq \omega(t, \sum_{i=1}^{3} |u^{(i-1)}|) \qquad (4.11)$$

for a.e. $t \in [a,b]$

imply the estimate (4.2).

Proof. Let us put $h_0(t) = u'''(t) - \sum_{i=1}^{3} h_i(t)u^{(i-1)}(t)$ and consider the equation

$$u'''(t) = \sum_{i=1}^{3} h_i(t)u^{(i-1)}(t) + h_0(t) \quad \text{on } [a,b]. \qquad (4.12)$$

Since h_i, a_i, i=1,2,3, satisfy the conditions of Lemma 2, the problem (2.1), (0.2) has only the trivial solution. Consequently, by Lemma 3, there exists $\gamma \in (0,+\infty)$ such that for the Green's function G for the problem (2.1), (0.2) the estimate (2.11) is valid. Therefore the solution

$$u(t) = \int_a^b G(t,s)h_0(s)ds$$

of the problem (4.12), (0.2) satisfies

$\sum_{i=1}^{3} |u^{(i-1)}(t)| \leq \gamma \int_a^b \omega(s, \sum_{i=1}^{3} |u^{(i-1)}|)ds$ for $a \leq t \leq b$. Putting

$\rho_0 = \max \left\{ \sum_{i=1}^{3} |u^{(i-1)}(t)| : a \leq t \leq b \right\}$, we get from the last

inequality $\rho_0 \leq \gamma \int_a^b \omega(s,\rho_0)ds$. From (1.6) it follows that

there exists $r^* > 0$ such that for any $\rho > r^*$, $\gamma \int_a^b \omega(s,\rho)ds < \rho$.

Therefore $\rho_0 \leq r^*$ which proves Lemma 7.

5. Proofs of Theorems

Proof of Theorem 1. Let $r^* \in (r, +\infty)$ be the constant constructed by Lemma 5. Let us put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for } 0 \leq s \leq r^* \\ 2 - s/r^* & \text{for } r^* < s < 2r^* \\ 0 & \text{for } s \geq 2r^* , \end{cases} \quad (5.1)$$

$$g(t,x,y,z) = \chi(r^*, |x|+|y|+|z|)\, f(t,x,y,z) \text{ on } D, \quad (5.2)$$

and consider the equation

$$u''' = g(t,u,u',u'') . \quad (5.3)$$

Then $|g(t,x,y,z)| \leq h(t)$ on D, where

$$h(t) = \sup\{|f(t,x,y,z)| : |x|+|y|+|z| \leq 2r^*\} \in L^1(a,b) .$$

Since g satisfies (1.1), by the Existence proposition, the problem (5.3), (0.2) has a solution u with the property (3.2). Further, by (1.2), (5.2), we get $|u'''(t)| \leq |g(t,u,u',u'')| \leq$ $\leq |f(t,u,u',u'')| \leq \omega(t,|u|,|u'|,|u''|)$ for a.e. $t \in [a,b]$ and so we can conclude, by Lemma 5, that the estimate (4.2) is valid. Thus, in view of (5.1) - (5.3), u is a solution of the problem (0.1), (0.2). Theorem is proved.

Proof of Corollary. Let us put

$$\omega_0(t,|x|,|y|,|z|) = h_1(t)|x| + h_2(t)|y| + h_3(t)|z| + \\ + \omega(t,|x| + |y| + |z|) .$$

Then $|f(t,x,y,z)| \leq \omega_0(t,|x|,|y|,|z|)$ on D and

$$\limsup_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \omega_0(t, \rho c_1(b-a), \rho c_1, \rho)dt < 1 +$$

$$+ \lim_{\rho \to \infty} \frac{1}{\rho} \int_a^b \omega(t, \rho[1+c_1+c_1(b-a)]\, dt < 1.$$ Consequently,

f satisfies all conditions from Theorem 1.

Proof of Theorem 2. Theorem 2 can be proved in the same way as Theorem 1, only we use Lemma 6 instead od Lemma 5.

Proof of Theorem 3. Let r^* be the constant found by Lemma 7, and χ be the function defined by (5.1). Let us put

$$g(t,x,y,z) = \chi(r^*, |x|+|y|+|z|)(f(t,x,y,z) - - h_1(t)x - h_2(t)y - h_3(t)z) \tag{5.4}$$

and consider the equation

$$u''' = \sum_{i=1}^{3} h_i(t)u^{(i-1)} + g(t,u,u',u'') . \tag{5.5}$$

Since h_i, a_i, $i=1,2,3$, satisfy the conditions of Lemma 2, the problem (2.1), (0.2) has only the trivial solution. Further $|g(t,x,y,z)| \leq g^*(t)$ on D, where $g^*(t) = \sup\{|f(t,x,y,z) - - h_1(t)x - h_2(t)y - h_3(t)z| : |x|+|y|+|z| \leq 2r^*\} \in L^1(a,b)$. Therefore, by Lemma 4, the problem (5.5), (0.2) has a solution u. According to (1.12), (5.4) and (5.5), it holds

$$\left|u''' - \sum_{i=1}^{3} h_i(t)u^{(i-1)}\right| \leq |g(t,u,u',u'')| \leq \left|f(t,u,u',u'') - \sum_{i=1}^{3} h_i(t)u^{(i-1)}\right| \leq \omega\left(t, \sum_{i=1}^{3} |u^{(i-1)}|\right) \quad \text{for a.e.}$$

$t \in [a,b]$ and so, by Lemma 7, u satisfies the estimate (4.2). Consequently, in view of (5.4), (5.5), u is also a solution of the problem (0.1), (0.2). This completes the proof.

Summary

The paper deals with the question of existence of solutions of the equation

$$u''' = f(t,u,u',u'')$$

satisfying the conditions

$$u(a) = u(t_1), \quad u(t_2) = u(t_3), \quad u(t_4) = \mu(b),$$

where $-\infty < a < t_1 \leq t_2 < t_3 \leq t_4 < b < +\infty$.

Souhrn

O JISTÝCH VÍCEBODOVÝCH OKRAJOVÝCH PROBLÉMECH

V práci je řešena otázka existence řešení rovnice

$$u''' = f(t, u, u', u''),$$

které splňuje podmínky

$$u(a) = u(t_1),\ u(t_2) = u(t_3),\ u(t_4) = u(b),$$

kde $-\infty < a < t_1 \leqq t_2 < t_3 \leqq t_4 < b < +\infty$.

Резюме

О МНОГОТОЧЕЧНЫХ КРАЕВЫХ ЗАДАЧАХ

В работе решается задача об отыскании решения уравнения

$$u''' = f(t, u, u', u'')$$

удовлетворяющего условиям

$$u(a) = u(t_1),\ u(t_2) = u(t_3),\ u(t_4) = u(b),$$

где

$$-\infty < a < t_1 \leqq t_2 < t_3 \leqq t_4 < b < +\infty .$$

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AUPO, Fac.rer.nat. 94, Mathematika XXVIII (1989), 43-60.