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Jitka Laitochová

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ON A CANONICAL TWO-DIMENSIONAL SPACE OF CONTINUOUS FUNCTIONS

JITKA LAITOCHOVÁ

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This paper is devoted to a study of the global transformation of two-dimensional regular and strongly regular spaces of continuous functions from a geometrical point of view. Here the so-called canonical space of continuous functions is of great importance, since it enables us to characterize the spaces of continuous functions under consideration.

1. The set of real numbers will be denoted by R. If j denotes an open interval (a,b), $a,b \in R$, where a may be $-\infty$ and b may be $+\infty$, then the symbol $C^{(0)}(j)$ will stand for a set of continous real functions of the real variable t in the interval j, while the symbol $C^{(n)}(j)$, where n is a natural number, will stand for a set of real functions of the real variable t in the interval j, having continuous derivatives up to and including the order n.

Definition 1.1. Let $y_1, y_2 \in C^{(0)}(j)$. We say, the functions are dependent on the interval j if there exist such numbers $k_1, k_2 \in R$, $k_1^2 + k_2^2 > 0$ that the identity

$$k_1\, y_1(t) + k_2\, y_2(t) \equiv 0$$

is valid in the interval j.

If for every two numbers $k_1, k_2 \in R$, $k_1^2 + k_2^2 > 0$ and for every interval j_1, $j_1 \subset j$

$$k_1\, y_1(t) + k_2\, y_2(t) \not\equiv 0 \quad \text{on } j_1$$

holds, we say that the functions y_1, y_2 are independent of the interval j.

Definition 1.2. Let $y_1, y_2 \in C^{(0)}(j)$ be independent functions of the interval j, and $k_1, k_2 \in R$ be arbitrary numbers. By a set S of all functions in the form

$$k_1\, y_1 + k_2\, y_2$$

we mean a two-dimensional space of continuous functions in the interval j or also a space generated by the functions y_1, y_2 with a definition interval j.

Any ordered pair (z_1, z_2) of independent functions $z_1, z_2 \in S$ will be called the basis of the space S.

Definition 1.3. Let $t_0 \in j$, $z \in S$. The point t_0 will be called the zero of the function z if $z(t_0) = 0$.

If t_0 is a zero of all functions of the space S, then it is called the singular point of the definition interval j of the space S. In the contrary case the point t_0 will be named the regular point of the definition interval j of the space S.

The space S is called regular if the definition interval j possesses regular points, only.

The above definitions are used by K. S t a ch in [5].

Definition 1.4. Let j,J be open intervals in R. Further let S_1 and S_2 be spaces of continuous functions generated by the functions y_1, y_2 and Y_1, Y_2 with the definition intervals j and J, respectively. Say, the space S_2 is globally transformed onto the space S_1 if there exist

a) a bijection $h : j \to J$, $h \in C^{(0)}(j)$,

b) a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$,

c) a matrix $A = \|a_{ik}\|$, $i,k = 1,2$, $a_{ik} \in R$, $\det A \neq 0$

to the vectors $\underline{y} = (y_1, y_2)^T$, $\underline{Y} = (Y_1, Y_2)^T$, so that for every $t \in j$ the equality

$$\underline{y}(t) = Af(t)\, \underline{Y}[h(t)] \tag{1.1}$$

holds, where $(.,.)^T$ denotes the transposed vector to the vector $(.,.)$. The mapping (1.1) will be called the global transformation and will be written as $\mathcal{T} = \langle Af, h \rangle$.

The above definition of global transformation corresponds to that used by F. N e u m a n in [4] for spaces of linear n-th order differential equation solutions.

The equivalence of definitions of the global transformation used by O. B o r ů v k a in [1], F. N e u m a n in [4] and K. S t a ch in [6] is discussed in [2].

Definition 1.5. Let $y_1, y_2 \in C^{(0)}(j)$ be independent functions of the interval j. We say that the quotient $y_2(t)/y_1(t)$ is by parts increasing resp. decreasing in the interval $j=(a,b)$ provided the following two conditions A and B are satisfied.

Condition A. The quotient $y_2(t)/y_1(t)$ is an increasing resp. decreasing continuous function in the intervals described by some of the following situations a) - d):

a) in the interval j if $y_1(t) \neq 0$ for $t \in j$;

b) in every interval (t_i, t_{i+1}), where t_i, t_{i+1} for $i = 0, \pm 1, \pm 2, \ldots$ are the neighbouring zeros of the function $y_1(t)$ in j and the endpoints a,b of the interval j are the only cluster points of zeros of the function $y_1(t)$;

c) in every interval (t_i, t_{i+1}), where t_i, t_{i+1} for $i = 1,2,3,\ldots$ are the neighbouring zeros of the function $y_1(t)$ in j and in the interval (a, t_1), if b is the only cluster point of zeros of the function $y_1(t)$, or
in every interval (t_{-i}, t_{-i+1}), where t_{-i}, t_{-i+1} for $i = 1,2,3,\ldots$ are the neighbouring zeros of the function $y_1(t)$ in j and in the interval (t_o, b) if a is the only cluster point of zeros of the function $y_1(t)$;

d) in every interval (t_i, t_{i+1}), where t_i, t_{i+1} for $i = 1,2,\ldots,n-1$ are the neighbouring zeros of the function $y_1(t)$ in j and in the intervals (a, t_1), (t_n, b) in case the function $y_1(t)$ in the interval j has precisely n zeros t_i; for $n=1$ the interval (t_1, t_2) is an empty set.

Condition B. At the points t_i there exist the following limits

$$\lim_{t \to t_i +} \frac{y_2(t)}{y_1(t)} = -\infty , \qquad \lim_{t \to t_i -} \frac{y_2(t)}{y_1(t)} = +\infty ,$$

resp.

$$\lim_{t \to t_i +} \frac{y_2(t)}{y_1(t)} = +\infty , \qquad \lim_{t \to t_i -} \frac{y_2(t)}{y_1(t)} = -\infty .$$

<u>Definition 1.6</u>. The regular space S of continuous functions generated by the functions y_1, y_2 with the definition interval j is called strongly regular exactly if the quotient $y_2(t)/y_1(t)$ is by parts increasing or decreasing on the interval j.

Lemma 1.1. Let S be the space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Let $u, v \in S$ be arbitrary functions, whereby

$$u = c_{11} y_1 + c_{12} y_2 ,$$
$$v = c_{21} y_1 + c_{22} y_2 , \tag{1.2}$$

where $c_{ik} \in R$ are convenient numbers, $i,k = 1,2$.
Then it holds: The functions u,v are dependent (independent) on j if and only if

$$c_{11} c_{22} - c_{12} c_{21} = 0 \quad (c_{11} c_{22} - c_{12} c_{21} \neq 0).$$

P r o o f. Let $u = c_{11} y_1 + c_{21} y_2$,
$v = c_{21} y_1 + c_{22} y_2$,

where the numbers $c_{ik} \in R$, $i,k = 1,2$. Two possibilities can occur:

a) the determinant $|c_{ik}| = 0$ or
b) the determinant $|c_{ik}| \neq 0$.

Ad a) It holds

$$c_{11} c_{22} - c_{12} c_{21} = 0. \tag{1.3}$$

Now there can two cases arise: Either all numbers c_{ik}, $i,k = 1,2$ are equal to zero or at least one of the numbers c_{ik} is different from zero. We will show that in both cases the functions u,v are dependent on the interval j.

1) Let $c_{11} = c_{12} = c_{21} = c_{22} = 0$. Then $u \equiv 0$, $v \equiv 0$. Thus u,v are dependent on j.

2) Let f.i. $c_{11} \neq 0$. It then follows from (1.2) on taking account of (1.3) that

$$c_{11}v = c_{11}(c_{21}y_1 + c_{22}y_2) = c_{11}c_{21}y_1 + c_{11}c_{22}y_2 =$$

$$= c_{11}c_{21}y_1 + c_{12}c_{21}y_2 = c_{21}(c_{11}y_1 + c_{12}y_2) = c_{21}u;$$

whence it follows that the functions u,v are dependent on j, since the identity

$$c_{21}\ u - c_{11}\ v \equiv 0$$

on condition $c_{21}^2 + (-c_{11})^2 > 0$ is satisfied.

Likewise we may proceed for $c_{12} \neq 0$ or $c_{21} \neq 0$ or $c_{22} \neq 0$.

Ad b) Let $c_{ik} \neq 0$. We will show that in this case the functions u,v are independent of the interval j We argue by contradiction. If there were for every two numbers k_1, k_2, $k_1^2 + k_2^2 > 0$,

$$k_1 u(t) + k_2 v(t) \equiv 0$$

in the interval j_1, $j_1 \subset j$, then it would also be

$$k_1(c_{11}y_1 + c_{12}y_2) + k_2(c_{21}y_1 + c_{22}y_2) \equiv 0$$

or

$$(k_1c_{11} + k_2c_{21})y_1 + (k_1c_{12} + k_2c_{22})y_2 \equiv 0$$

whereby $(k_1c_{11} + k_2c_{21})^2 + (k_1c_{12} + k_2c_{22})^2 > 0$, since there cannot simultaneously be

$$\begin{aligned} k_1c_{11} + k_2c_{21} &= 0\ , \\ k_1c_{12} + k_2c_{22} &= 0\ , \end{aligned} \qquad (1.4)$$

because of the fact that $|c_{ik}| \neq 0$ and the system of equations (1.4) would possess a trivial solution $k_1 = k_2 = 0$, only. This implies that the functions y_1, y_2 would be dependent on j, contrary to the assumption. Hence

$$k_1 u + k_2 v \not\equiv 0$$

in every interval j_1, $j_1 \subset j$ and the functions u,v are independent of j.

Lemma 1.2. Let S_1 and S_2 be spaces of continuous functions generated by the functions y_1, y_2 and Y_1, Y_2 with the definition intervals j and J, respectively.

Let S_2 be globally transformed onto S_1. If S_2 is a regular space, then S_1 is also regular. If S_2 is a strongly regular space, then S_1 is also strongly regular.

P r o o f. We argue by contradiction. Let S_2 be a regular space. If S_1 were not a regular space, there would exist a singular point $t_o \in j$ and it would hold

$$k_1\, y_1(t_o) + k_2\, y_2(t_o) = 0$$

for every $k_1, k_2 \in R$.

Consequently, it would also hold

$$f(t_o)\Big[k_1(a_{11}Y_1(h_o) + a_{12}Y_2(h_o)) + k_2(a_{21}Y_1(h_o) + a_{22}Y_2(h_o))\Big] = 0 \qquad (1.5)$$

where $h_o = h(t_o)$. Since in consequence of Lemma 1.1 the functions $\tilde{Y}_1, \tilde{Y}_2$, where

$$\tilde{Y}_1 = a_{11}Y_1 + a_{12}Y_2 ,$$

$$\tilde{Y}_2 = a_{21}Y_1 + a_{22}Y_2 ,$$

are independent, then by condition (1.5) the point h_o would be a singular point of the space S_2, because $f(t_o) \neq 0$, which contradicts our assumption.

Let S_2 be a strongly regular space. From equation (1.1) we have

$$\frac{y_2(t)}{y_1(t)} = \frac{f(t)(a_{21}Y_1[h(t)] + a_{22}Y_2[h(t)])}{f(t)(a_{11}Y_1[h(t)] + a_{12}Y_2[h(t)])} = \frac{\tilde{Y}_2[h(t)]}{\tilde{Y}_1[h(t)]},$$

where $\tilde{Y}_1 = a_{11}Y_1 + a_{12}Y_2$, $\tilde{Y}_2 = a_{21}Y_1 + a_{22}Y_2$ are independent functions of S_2. Since S_2 is strongly regular, the quotient $\tilde{Y}_2/\tilde{Y}_1$ is by parts increasing or decreasing. It follows from the conditions for the bijection $h = h(t)$ that the function h either increases or decreases in j, consequently the composite function $\tilde{Y}_2(h)/\tilde{Y}_1(h)$ is by parts increasing or decreasing in j. Hence, the quotient y_2/y_1 is by parts increasing or decreasing in j. So the space S_1 is strongly regular.

2. We will apply the geometrical methods presented by F. Neuman, f.i. in [3] and [4] to construct now a canonical form of a strongly regular space of continuous functions.

Lemma 2.1. Let S be a regular space of continuous functions generated by the functions $y_1 = y_1(t)$, $y_2 = y_2(t)$ with the definition interval j. Let next

$$\begin{aligned} \xi_1 &= y_1(t)\,, \\ \xi_2 &= y_2(t)\,, \end{aligned} \tag{2.1}$$

where $t \in j$, is a parametrically defined curve $\mathcal{K}$ in a rectangular coordinate system $O\xi_1\xi_2$. Then the curve $\mathcal{K}$ is not going through the origin of the coordinates.

P r o o f. By the assumption the functions y_1, y_2 are understood to be independent in the interval j and every function $y \in S$ is of the form $y = k_1y_1(t) + k_2y_2(t)$, where $k_1, k_2 \in R$ are convenient numbers. If the curve $\mathcal{K}$ were going through the origin of the coordinates, then there would exist a number $t_0 \in j$ such that $y_1(t_0) = y_2(t_0) = 0$. However, in such a case there would hold $y(t_0) = k_1y_1(t_0) + k_2y_2(t_0) = 0$ for every pair $k_1, k_2 \in R$. The point t_0 would thus be a singular point of the space S, which contradicts our assumption on the regularity of the space S.

Lemma 2.2. Let S be a regular space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Let for $y_3 \in S$ be $y_3 = k_1 y_1 + k_2 y_2$, where $k_1, k_2 \in R$ are convenient numbers. The point t_i is a zero of the function y_3 exactly if the point $P_i = [y_1(t_i), y_2(t_i)]$ is the intersection of the curve $\mathcal{K}$ with the following straight line

$$k_1 \xi_1 + k_2 \xi_2 = 0 .$$

P r o o f. If the point t_i is a zero of the function $y_3(t)$, then $k_1 y_1(t_i) + k_2 y_2(t_i) = 0$.

Thus, the point P_i with the coordinates $[y_1(t_i), y_2(t_i)]$ is the intersection of the straight line $k_1\xi_1 + k_2\xi_2 = 0$ going through the origin of the coordinates with the curve $\mathcal{K}$ defined by the equations (2.1).

If the point $P_i = [y_1(t_i), y_2(t_i)]$ is the intersection of the straight line $k_1\xi_1 + k_2\xi_2 = 0$ going through the origin of the coordinates with the curve $\mathcal{K}$ defined by the equations (2.1), then $k_1 y_1(t_i) + k_2 y_2(t_i) = 0$ or $y_3(t_i) = 0$. Hence, the point t_i is a zero of the function $y_3(t)$.

Expressing the curve $\mathcal{K}$ defined by the equations of (2.1) in polar coordinates ρ, φ gives

$$\rho = \sqrt{y_1^2(t) + y_2^2(t)} \quad (2.2)$$

$$\mathrm{tg}\,\varphi = y_2(t)/y_1(t) \quad (2.3)$$

for $t \in j$.

Theorem 2.1. Let S be a regular space of continuous functions generated by the functions y_1, y_2 with the definition interval $j = (a,b)$.

The function $\rho = \rho(t)$ defined by (2.2) is continuous and positive for $t \in j$.

The function $\varphi = \varphi(t)$ defined by (2.3) is an increasing (decreasing) continuous function in j exactly if S is a strongly regular space.

P r o o f. Because of the fact that $y_1, y_2 \in C^{(0)}(j)$ we have $\rho \in C^{(0)}(j)$. The positivity of the function $\rho = \rho(t)$ in j follows from the fact that S is a regular space. Hence, there cannot exist any $t_0 \in j$ such that $y_1(t_0) = y_2(t_0) = 0$ would hold.

Suppose now that $\varphi = \varphi(t)$ is an increasing (decreasing) continuous function in j. Let us denote by J a set of functional values of the function $\varphi = \varphi(t)$ for $t \in j$. Now there may occur the following cases:

a) J does not contain any of the numbers $-\frac{\pi}{2} + i\pi$, i being an integer,

b) J contains all numbers $-\frac{\pi}{2} + i\pi$, i being an integer,

c) there exists such an integer k that J contains all numbers $-\frac{\pi}{2} + k\pi + i\pi$, $i=1,2,\dots$, or it contains all numbers $-\frac{\pi}{2} + k\pi - i\pi$ for $i=1,2,\dots$,

d) there exists such an integer k that J contains all numbers $-\frac{\pi}{2} + k\pi + i\pi$, $i=1,\dots,n$, where n is a natural number.

In then follows from the equality $\operatorname{tg}\varphi(t) = y_2(t)/y_1(t)$ for $t \in j$ that the quotient $y_2(t)/y_1(t)$ is in case of a) an increasing (decreasing) continuous function in j and therefore $y_1(t) \neq 0$ in j.

Denoting in case b) $t_i = \varphi^{-1}(-\frac{\pi}{2} + i\pi)$, where i is an integer, φ^{-1} is the inverse function to φ, then the quotient $y_2(t)/y_1(t)$ is an increasing (decreasing) function in every interval (t_i, t_{i+1}), $i=0,\pm 1,\pm 2,\dots$.

Denoting in case c) $t_i = \varphi^{-1}(-\frac{\pi}{2} + k\pi + i\pi)$,

$i=1,2,\ldots$, then the quotient $y_2(t)/y_1(t)$ is an increasing (decreasing) continuous function in every interval (t_i,t_{i+1}), $i=1,2,\ldots$ and in the interval (a,t_1), the point b is a cluster point of the points t_i, or if we denote $t_i = \varphi^{-1}(-\frac{\pi}{2} + k\pi - i\pi)$, $i=1,2,\ldots$, then the quotient $y_2(t)/y_1(t)$ is an increasing (decreasing) continuous function in every interval (t_{-i},t_{-i+1}), $i = 1,2,\ldots$, and in the interval (t_o,b), the point a is a cluster point of the points t_i.

Denoting in case d) $t_i = \varphi^{-1}(-\frac{\pi}{2} + k\pi + i\pi)$ for $i=1,\ldots,n$, then the quotient $y_2(t)/y_1(t)$ is an increasing (decreasing) function in every interval (t_i,t_{i+1}), $i=1,2,\ldots,n-1$ and in the intervals (a,t_1), (t_n,b), $n \geqq 1$.

Because of the fact that y_1,y_2 are independent functions having thus no zeros in common, it follows from the equality $\operatorname{tg}\varphi(t) = y_2(t)/y_1(t)$ for $t\in j$ that

$$\lim_{t\to t_i+} \frac{y_2(t)}{y_1(t)} = (-\delta).\infty\ , \qquad \lim_{t\to t_i-} \frac{y_2(t)}{y_1(t)} = \delta.\infty \qquad (2.4)$$

where $\delta = 1$, or $\delta = -1$ according as φ is increasing or decreasing in j.

On the contrary. Ad a) Let the quotient $y_2(t)/y_1(t)$ is an increasing (decreasing) continuous function in j. Then the function

$$\varphi(t) = \operatorname{arctg}\frac{y_2(t)}{y_1(t)}$$

is in j an increasing (decreasing) continuous function.

Ad b) Let the quotient $y_2(t)/y_1(t)$ be a function by parts increasing (decreasing) in j and let (2.4) hold. Then the function

$$\varphi(t) = \begin{cases} \operatorname{arctg}\dfrac{y_2(t)}{y_1(t)} + \delta i\pi & \text{for } t\in(t_i,t_{i+1}) \\ & ,\ i=0,\pm1,\pm2,\ldots, \\ \delta(-\frac{1}{2}\pi + i\pi) & \text{for } t = t_i \end{cases}$$

is an increasing (decreasing) continuous function in j, where $\delta = 1$, or $\delta = -1$ according as y_2/y_1 by parts increases or decreases in j.

Ad c) Let the quotient $y_2(t)/y_1(t)$ be a function by parts increasing (decreasing) in j and let (2.4) hold. Then the function

$$\varphi(t) = \begin{cases} \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} + \delta i \pi & \text{for } t \in (t_i, t_{i+1}) \\ & , \ i=1,2,\dots, \\ \delta(-\frac{1}{2}\pi + i\pi) & \text{for } t = t_i \\ \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} & \text{for } t \in (a, t_1) \end{cases}$$

or

$$\varphi(t) = \begin{cases} \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} + \delta i \pi & \text{for } t \in (t_{-i}, t_{-i+1}) \\ & , \ i=1,2,\dots, \\ \delta(-\frac{1}{2}\pi + i\pi) & \text{for } t = t_i \\ \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} & \text{for } t \in (t_o, b) \end{cases}$$

is an increasing (decreasing) continuous function in j, where $\delta = 1$, or $\delta = -1$ according as y_2/y_1 by parts increases or decreases in j.

Ad d) Let the quotient $y_2(t)/y_1(t)$ be a function by parts increasing (decreasing) in j and let (2.4) hold. Then the function

$$\varphi(t) = \begin{cases} \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} + \delta i \pi & \text{for } t \in (t_i, t_{i+1}), \\ & , \ i=1,2,\dots,n-1, \\ \delta(-\frac{1}{2}\pi + i\pi) & \text{for } t = t_i, \quad i=1,2,\dots,n, \\ \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} & \text{for } t \in (a, t_1) \\ \operatorname{arctg} \dfrac{y_2(t)}{y_1(t)} + \delta n \pi & \text{for } t \in (t_n, b) \end{cases}$$

is an increasing (decreasing) continuous function in j, where $\delta = 1$, or $\delta = -1$ according as the y_2/y_1 in parts increases (decreases) in j.

Theorem 2.2. Let S be a strongly regular space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Further let $\mathcal{K}^*$ be the projection of the curve $\mathcal{K}$ defined by equations (2.1) on a unit circle from the origin O defined as follows: For every $t \in j$ there is assigned to the point $P_t \in \mathcal{K}$ with the coordinates $[y_1(t), y_2(t)]$ a point $P_t^* \in \mathcal{K}^*$ with the coordinates $[u_1(t), u_2(t)]$ lying on the half-line $\overline{OP}_t$. Then the curve $\mathcal{K}^*$ is expressed by the equations

$$\begin{aligned} \xi_1 &= u_1(t), \\ \xi_2 &= u_2(t). \end{aligned} \tag{2.5}$$

Hereby it holds for the functions u_1, u_2

$$u_1^2(t) + u_2^2(t) = 1 \tag{2.6}$$

and

$$\begin{aligned} f(t)\, u_1(t) &= y_1(t)\ , \\ f(t)\, u_2(t) &= y_2(t)\ , \end{aligned} \tag{2.7}$$

where f is given by the formula

$$f(t) = \sqrt{y_1^2(t) + y_2^2(t)}. \tag{2.8}$$

P r o o f. From the definition of the curve $\mathcal{K}^*$ then follows its expression by equations (2.5). Relation (2.6) holds, because for every $t \in j$ there are $u_1(t)$ and $u_2(t)$ the coordinates of the point P_t^* which lies on the unit circle.

Since $\dfrac{y_2(t)}{y_1(t)} = \dfrac{u_2(t)}{u_1(t)}$, we obtain from this equations (2.7), whereby $f(t) > 0$.

Squaring equations (2.7) and adding them together, we obtain

$$f^2(t)\left[u_1^2(t) + u_2^2(t)\right] = y_1^2(t) + y_2^2(t)$$

from which and with respect to (2.6) we get (2.8).

The image $\mathcal{K}^*$ of the curve $\mathcal{K}$ is thus given by equations

$$\xi_1 = u_1(t)$$

$$\xi_2 = u_2(t)$$

for $t \in j$, expressing with respect to (2.6) the arc of the unit circle.

The arc $\mathcal{K}^*$ of the unit circle with the center at the origin O may be expressed by the following equations

$$\overline{Y}_1 = \cos(s-s_o)\ , \qquad \overline{Y}_2 = \sin(s-s_o)\ ,$$

or (2.9)

$$\overline{Y}_1 = \cos(s-s_o)\ , \qquad \overline{Y}_2 = -\sin(s-s_o)\ ,$$

where $s \in J$, according as the curve revolves the origin in a positive or negative sense, whereby the point $[1,0]$ on the unit circle corresponds to the value of the parameter $s = s_o$.

Theorem 2.3. Let S be a strongly regular space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Let the quotient $y_2(t)/y_1(t)$ be increasing or decreasing in the interval j. Then there exists an increasing or decreasing function $\varphi = \varphi(t)$, $t \in j$ satisfying the equality

$$\operatorname{tg}\varphi(t) = \frac{y_2(t)}{y_1(t)} \quad \text{for every } t \in j \tag{2.10}$$

characterized by the fact that if φ increases there holds

$$u_1\left[\overline{h}\ (s)\right] = \cos s, \quad u_2\left[\overline{h}\ (s)\right] = \sin s\ , \tag{2.11}$$

whereby $\bar{h}(s) = \varphi^{-1}(s)$, φ^{-1} denotes the inverse function to φ, $s \in J$, $J = \varphi(j)$ and if φ decreases there holds

$$u_1[\bar{h}(s)] = \cos s, \quad u_2[\bar{h}(s)] = -\sin s, \tag{2.12}$$

whereby $\bar{h}(s) = \varphi^{-1}(-s)$, $-s \in J$.

The functions u_i are determined by the conditions $y_i(t) = f(t)u_i(t)$, $t \in j$, $i=1,2$, $u_1^2(t) + u_2^2(t) \equiv 1$, $f(t) > 0$.

P r o o f. Let us first remark that there exists the function $\bar{h} = \varphi^{-1}(s)$ and it is a bijection for which

$$\bar{h}: J \to j, \quad \bar{h} \in C^{(0)}(J)$$

holds, which follows on one hand from the monotonicity and on the other hand from the continuity of the function φ.

Because of $\operatorname{tg} \varphi(t) = \dfrac{y_2(t)}{y_1(t)} = \dfrac{f(t)\, u_2(t)}{f(t)\, u_1(t)} = \dfrac{u_2(t)}{u_1(t)}$

for $t \in j$, we obtain from this

$$\sin \varphi(t) = p(t)\, u_2(t),$$

$$\cos \varphi(t) = p(t)\, u_1(t).$$

On squaring and adding together we obtain $p^2(t) \equiv 1$. Consequently there is either

$$u_1(t) = \cos \varphi(t), \qquad u_2(t) = \sin \varphi(t) \tag{2.13}$$

or

$$u_1(t) = -\cos \varphi(t), \qquad u_2(t) = -\sin \varphi(t). \tag{2.14}$$

However, besides the function $\varphi(t)$ there are also the functions $\varphi(t) + k\pi$, k being an integer, satisfying (2.10), it suffices to take the function $\varphi_1(t) = \varphi(t) + \pi$ to get (2.13).

If φ is an increasing or decreasing function, then equations (2.13) define the positively or negatively oriented arc of the unit circle.

Setting in (2.13) in case of the increasing function φ the expression $\bar{h} = \varphi^{-1}(s)$, $s \in J$, instead of t, we obtain

$$u_1\left[\bar{h}(s)\right] = \cos s, \qquad u_2\left[\bar{h}(s)\right] = \sin s ,$$

which is the formula of (2.11).

Setting in (2.13) in case of the decreasing function φ the expression $\bar{h} = \varphi^{-1}(-s)$, $-s \in J$, instead of t, we obtain

$$u_1\left[\bar{h}(s)\right] = \cos(-s) = \cos s, \qquad u_2\left[\bar{h}(s)\right] = \sin(-s) = -\sin s,$$

which is the formula of (2.12).

Theorem 2.4. Let S be a strongly regular space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Let the functions φ, $\bar{h}$, u_i, $i=1,2$ have the meaning stated in the foregoing theorem.
Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Let the functions $\bar{Y}_1$, $\bar{Y}_2$ be defined as follows

$$\begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{pmatrix} = A \begin{pmatrix} u_1\left[\bar{h}(s)\right] \\ u_2\left[\bar{h}(s)\right] \end{pmatrix}. \tag{2.15}$$

$$\text{If} \quad A = \begin{pmatrix} \cos s_o & \sin s_o \\ -\sin s_o & \cos s_o \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \cos s_o & \sin s_o \\ \sin s_o & -\cos s_o \end{pmatrix}, \tag{2.16}$$

then the functions $\bar{Y}_1$, $\bar{Y}_2$ are given by the formulas

$$\bar{Y}_1 = \cos(s-s_o), \qquad \bar{Y}_2 = \sin(s-s_o)$$

or (2.17)

$$\bar{Y}_1 = \cos(s-s_o), \qquad \bar{Y}_2 = -\sin(s-s_o)$$

for $s \in J$.

P r o o f. From (2.15) on taking account of (2.11) or (2.12) and of (2.16) we get

$$\bar{Y}_1 = a_{11}\, u_1\left[\bar{h}(s)\right] + a_{12}\, u_2\left[\bar{h}(s)\right] = \cos s_0 \cos s + \sin s_0 \sin s = \cos(s-s_0),$$

$$\bar{Y}_2 = a_{21}\, u_1\left[\bar{h}(s)\right] + a_{22}\, u_2\left[\bar{h}(s)\right] = -\sin s_0 \cos s + \cos s_0 \sin s = \sin(s-s_0),$$

or

$$\bar{Y}_1 = a_{11}\, u_1\left[\bar{h}(s)\right] + a_{12}\, u_2\left[\bar{h}(s)\right] = \cos s_0 \cos s + \sin s_0 \sin s = \cos(s-s_0),$$

$$\bar{Y}_2 = a_{21}\, u_1\left[\bar{h}(s)\right] + a_{22}\, u_2\left[\bar{h}(s)\right] = \sin s_0 \cos s - \cos s_0 \sin s = -\sin(s-s_0)$$

for $s \in J$, which are the formulas of (2.17).

Theorem 2.5. Let S be a strongly regular space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Let φ be an increasing or decreasing continuous function satisfying the functional equation

$$\operatorname{tg}\varphi(t) = \frac{y_2(t)}{y_1(t)} \qquad \text{for } t \in j.$$

Let $J = \varphi(j)$. Then for

a) the bijection $\varphi : j \to J$, $\varphi \in C^{(0)}(j)$,

b) the function $f = \sqrt{y_1^2 + y_2^2}$, $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$,

c) the matrix $A = \begin{pmatrix} \cos s_0 & \sin s_0 \\ -\sin s_0 & \cos s_0 \end{pmatrix}$ or $A = \begin{pmatrix} \cos s_0 & \sin s_0 \\ \sin s_0 & -\cos s_0 \end{pmatrix}$

$s_0 \in J$, there exists the global transformation $\mathcal{T} = \langle Af, \varphi \rangle$ of the space S^* generated by the functions cos s, sin s with the definition interval J onto the space S given by the relation

$$y(t) = Af(t)\, Y[\varphi(t)], \quad t \in j,$$

where

$$Y = (\cos s, \sin s)^T, \qquad y = (y_1, y_2)^T.$$

P r o o f. Let us put $\tilde{\underline{Y}}(s) = (\tilde{Y}_1(s), \tilde{Y}_2(s))$ and define $\tilde{\underline{Y}}(s)$ by the equation

$$\tilde{\underline{Y}}(s) = A\, \underline{Y}(s).$$

We easily observe that for $A = \begin{pmatrix} \cos s_o & \sin s_o \\ -\sin s_o & \cos s_o \end{pmatrix}$ we have

$\tilde{Y}_1 = \cos(s - s_o)$, $\tilde{Y}_2 = \sin(s - s_o)$ and for

$A = \begin{pmatrix} \cos s_o & \sin s_o \\ \sin s_o & -\cos s_o \end{pmatrix}$ we have $\tilde{Y}_1 = \cos(s - s_o)$ $\tilde{Y}_2 = -\sin(s - s_o)$.

We are looking for $s = h(t)$ such that $\tilde{\underline{Y}}[h(t)] = \underline{u}(t)$, where $\underline{u} = (u_1, u_2)$, whereby the functions u_i, $i=1,2$, have the meaning stated in Theorem 2.3.

Then

$$\underline{u}(t) = \tilde{\underline{Y}}[h(t)] = A\, \underline{Y}[h(t)]$$

and since $y_i = f\, u_i$, $i=1,2$, we have for $\underline{y} = (y_1, y_2)$ that

$$\underline{y}(t) = f(t)\, \tilde{\underline{Y}}[h(t)] = f(t)\, A\, \underline{Y}[h(t)].$$

Since for the function φ

$$\operatorname{tg} \varphi = \frac{y_2(t)}{y_1(t)} = \frac{u_2(t)}{u_1(t)} = \frac{\sin[h(t) - s_o]}{\cos[h(t) - s_o]} = \operatorname{tg}[h(t) - s_o]$$

holds, we have from this

$$\varphi = h(t) - s_o + k\pi, \qquad k \text{ being an integer.}$$

In case of $s_o = 0$ we have $A = E$ and $\tilde{\underline{Y}} = \underline{Y}$ and the transformation equation becomes the form

$$\underline{y}(t) = f(t)\, \underline{Y}[h(t)], \quad \text{where } \varphi = h(t) + k\pi .$$

Definition 2.1. By a space of continuous functions S^* generated by the functions cos s, sin s with the definition interval J we mean the canonical form of a strongly regular space S of continuous functions generated by the functions y_1, y_2 with the definition interval j, more briefly the canonical space of continuous functions.

It follows from Definition 2.1 and from Lemma 1.2 that the canonical space of continuous functions S^* is strongly regular.

Let us remark that the elements of the space S^* are the functions $k_1 \cos s + k_2 \sin s$, $s \in J$, $k_1, k_2 \in R$, where $J = \varphi(j)$, φ is a continuous function satisfying the functional equation

$$\operatorname{tg} \varphi(t) = \frac{y_2(t)}{y_1(t)}, \qquad t \in j .$$

First phase. Conformably to the definition of the first phase of an ordered pair of solutions of a linear second-order differential equation of the Jacobian form, introduced by O. B o r ů v k a in [1], we will express the followin[g]

Definition 2.2. Let (y_1, y_2) be a basis of the strongly regular space S with the definition interval j. Every function $\alpha \in C^{(0)}(j)$, $\alpha : j \to J$ satisfying in j the functional equation

$$\operatorname{tg} \alpha(t) = \frac{y_1(t)}{y_2(t)} \tag{2.18}$$

will be called the first phase, more briefly the phase of the ordered pair of functions $y_1, y_2 \in S$.

Theorem 2.6. If α is the phase of the ordered pair $(y_1, y_2) \in S$, then any other phase α_k, k being an integer, is given by the formula

$$\alpha_k(t) = \alpha(t) + k\pi ,$$

where

$$\alpha_o = \alpha .$$

Thus (2.18) defines the countable system of phase α_k, k being an integer.

P r o o f. This follows immediately from (2.18).

Theorem 2.7. Let $(\tilde{y}_1, \tilde{y}_2)$ be the basis of the strongly regular space S with the definition interval j and $(\tilde{Y}_1, \tilde{Y}_2)$ be the basis of the space S^* with the definition interval J. Next let $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$, $\tilde{\underline{Y}} = (\tilde{Y}_1, \tilde{Y}_2)$. Let the space S^* be globally transformed onto the space S as follows

$$\tilde{\underline{y}}(t) = Af(t)\, \tilde{\underline{Y}}[h(t)] \qquad (2.19)$$

by means of the function f, of the parametrization h and of the matrix A. Let $\mathcal{X}$ be the matrix given by the equation

$$\tilde{\underline{Y}} = \mathcal{X} \underline{Y} ,$$

where $\underline{Y} = (\cos s, \sin s)^T$. Let $\underline{y} = (y_1, y_2)^T$, where $\underline{y} = \mathcal{X}^{-1} A^{-1} \tilde{\underline{y}}$, whereby $\mathcal{X}^{-1}$, A^{-1} denote inverse matrices to the matrix $\mathcal{X}$ or A. Let $\alpha = \alpha(t)$ be a first phase of the basis $(y_1, y_2) \in S$.

Then

$$\alpha_k(t) = h(t) + k\pi , \qquad k \text{ being an integer,}$$

$\alpha_o = \alpha(t)$.

P r o o f. From equation (2.19) we obtain $\tilde{\underline{y}}(t) = A\mathcal{X} f(t)\, \underline{Y}[h(t)]$, whence

$$\underline{y} = \varkappa^{-1} A^{-1} \underline{\tilde{Y}} = f(t)\, \underline{Y}\left[h(t)\right]$$

or

$$y_1 = f(t)\, Y_1\left[h(t)\right] = f(t) \cos h(t) \, ,$$
$$y_2 = f(t)\, Y_2\left[h(t)\right] = f(t) \sin h(t) \, .$$

From this we get

$$\operatorname{tg} h(t) = \frac{y_2(t)}{y_1(t)} \, . \tag{2.20}$$

From the definition equation (2.18) we obtain the expression for the phase α of the basis (y_2, y_1)

$$\operatorname{tg} \alpha(t) = \frac{y_2(t)}{y_1(t)} \, . \tag{2.21}$$

Comparing (2.20) and (2.21) we observe that the parametrization $h(t)$ represents the phase $\alpha(t)$ of the ordered pair (y_2, y_1). Thus

$\alpha_k(t) = h(t) + k\pi$, k being an integer, $\alpha_0 = \alpha(t) = h(t)$.

The following theorem is a certain modification of the Stach theorem [6] and presents a necessary and sufficient condition of the global transformation of the space S_2 onto the space S_1.

<u>Theorem 2.8</u>. Let S_i, i=1,2, be strongly regular spaces of the continuous functions with definition intervals j_i. Let S^* be a canonical space of continuous functions with the definition interval J. Let $\underline{\tilde{Y}} = (\tilde{Y}_1, \tilde{Y}_2)$ be a basis of S^* and $(\tilde{u}_1, \tilde{u}_2) \in S_1$, $(\tilde{U}_1, \tilde{U}_2) \in S_2$ be the space bases. Let the space S^* be globally transformed onto the space S_1 as follows

$\tilde{u}(t) = A_1 f(t)\underline{\tilde{Y}}[h(t)]$, where h is a bijection $h: j_1 \to J$, $h \in C^{(0)}(j_1)$, $\underline{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2)^T$.

Suppose the space S^* is globally transformed onto the space S_2 by the equation

$\underline{\tilde{U}}(T) = A_2 F(T)\underline{\tilde{Y}}[H(T)]$, where H is a bijection, $H: j_2 \to J$, $H \in C^{(0)}(j_2)$, $\underline{\tilde{U}} = (\tilde{U}_1, \tilde{U}_2)^T$.

The necessary and sufficient condition for the existence of a global transformation of the space S_2 onto the space S_1, transforming the basis $(\tilde{u}_1, \tilde{u}_2)$ into the basis $(\tilde{U}_1, \tilde{U}_2)$ by means of

a) the bijection $k: j_1 \to j_2$, $k \in C^{(0)}(j_1)$,

b) the function $g \in C^{(0)}(j_1)$, $g(t) \neq 0$ for $t \in j_1$,

c) the matrix $B = A_1 A_2^{-1}$

by the formula

$$\underline{\tilde{u}}(t) = Bg(t)\underline{\tilde{U}}[k(t)], \quad t \in j_1 \tag{2.22}$$

is the existence of

1) the bijection $T = X(t)$, $X: j_1 \to j_2$, $X \in C^{(0)}(j_1)$,

2) the integer ℓ, for which $h(t) = H[X(t)] + \ell\pi$

for $t \in j_1$.

P r o o f. Let the matrix $\mathcal{X}$ be given by the equation $\underline{\tilde{Y}} = \mathcal{X}\,\underline{Y}$, where $\underline{Y} = (\cos s, \sin s)^T$.

Assume the space S_2 to be globally transformed onto the space S_1. Since $\underline{\tilde{u}}(t) = A_1 f(t)\underline{\tilde{Y}}[h(t)]$, $\underline{\tilde{U}}(T) = A_2 F(T)\underline{\tilde{Y}}[H(T)]$ we obtain from this on introducing the following relations

$$\underline{u} = \mathcal{X}^{-1} A_1^{-1} \underline{\tilde{u}}, \quad \underline{U} = \mathcal{X}^{-1} A_2^{-1} \underline{\tilde{U}}, \tag{2.23}$$

hat

$$A_1 \mathcal{K}\, \underline{u} = A_1 f(t) \mathcal{K}\, \underline{Y}\left[h(t)\right], \quad A_2 \mathcal{K}\, \underline{U} = A_2 F(T) \mathcal{K}\, \underline{Y}\left[H(T)\right]$$

i.e.

$$\underline{u} = f(t) Y\left[h(t)\right], \quad \underline{U} = F(T)\ Y\left[H(T)\right] .$$

From this we have

$$\mathrm{tg}\ h = \frac{u_2(t)}{u_1(t)}, \qquad \mathrm{tg}\ H = \frac{U_2(T)}{U_1(T)} .$$

It thus follows from (2.22) and (2.23) that

$$A_1 \mathcal{K}\, \underline{u} = A_1 A_2^{-1} g(t) \quad A_2 \quad F(T) \mathcal{K}\, \underline{Y}\left[k(t)\right]$$

i.e.

$$\underline{u} = g(t)\ F(T)\ \underline{Y}\left[k(t)\right] = g(t) \underline{U}\left[k(t)\right] .$$

So, we have

$$\mathrm{tg}\ h = \frac{u_2(t)}{u_1(t)} = \frac{U_2(k)}{U_1(k)} = \mathrm{tg}\ H(k) .$$

If we set $X(t) = k(t)$, $k(t) = H^{-1}(h)$, then $X\colon j_1 \to j_2$, $X \in C^{(0)}(j_1)$ and we see that for $t \in j_1$ there exists an integer ℓ such that

$$h(t) = H\left[X(t)\right] + \ell\pi .$$

Suppose conversely the existence of the function $X = X(t)$, $X\colon j_1 \to j_2$, $X \in C^{(0)}(j_1)$ and of the integer ℓ, for which $h(t) = H\left[X(t)\right] + \ell\pi$.

Then

$$\mathrm{tg}\ h(t) = \mathrm{tg}\ H\left[X(t)\right],$$

i.e.

$$(\mathrm{tg}\ h(t) \equiv) \frac{u_2(t)}{u_1(t)} = \frac{U_2\left[X(t)\right]}{U_1\left[X(t)\right]} (\equiv \mathrm{tg}\ H\left[X(t)\right]) .$$

Setting $X(t) = k(t)$ yields

$$\underline{u} = g(t)\, U(k)\ ,$$

where $g \in C^{(0)}(j_1)$, $g(t) \neq 0$ for $t \in j_1$, $k: j_1 \to j_2$, $k \in C^{(0)}(j_1)$.

Since (2.23) holds, we obtain

$$\varkappa^{-1} A_1^{-1} \tilde{\underline{u}} = g(t) \varkappa^{-1} A_2^{-1} \tilde{\underline{U}}(k)$$

i.e.

$$\tilde{\underline{u}} = g(t)\, A_1 A_2^{-1} \tilde{\underline{U}}(k)\ .$$

Hence

$$\tilde{\underline{u}} = Bg(t)\, \tilde{\underline{U}}\ (k)\ ,$$

where $B = A_1 A_2^{-1}$, so that (2.22) holds.

3. As special cases of strongly regular spaces of continuous functions there may be named the spaces of solutions of linear second-order differential equations of the general form

$$\text{(ab)} \qquad y'' + a(t)y' + b(t)y = 0\ ,$$

where $a, b \in C^{(0)}(j)$ and further of the Sturm form

$$\text{(pq)} \qquad (p(t)y')' + q(t)y = 0\ ,$$

where $p, q \in C^{(0)}(j)$, $py' \in C^{(1)}(j)$, $p(t) \neq 0$ in j .

The space of solutions of the differential equation (ab) will be denoted by S_{ab} and the space of solutions of the differential equation (pq) by S_{pq}.

It can be easily seen that the spaces of solutions S_{ab}, S_{pq} are generated by the functions of the basis of the differential equation (ab), or (pq). The elements of the basis, f.i. the solutions y_1, y_2 are linearly independent, so that $k_1y_1(t) + k_2y_2(t) \not\equiv 0$ in j, where $k_1^2 + k_2^2 > 0$. We will now show that y_1, y_2 are also independent functions of the interval j, in other words that

$$k_1 y_1(t) + k_2 y_2(t) \not\equiv 0 \quad \text{in } j_1 , \quad j_1 \subset j .$$

Indeed, there holds for $k_2 \neq 0$ or $k_1 \neq 0$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \frac{1}{k_2} \begin{vmatrix} y_1 & k_1 y_1 + k_2 y_2 \\ y_1' & k_1 y_1' + k_2 y_2' \end{vmatrix} = \frac{1}{k_1} \begin{vmatrix} k_1 y_1 + k_2 y_2 & y_2 \\ k_1 y_1' + k_2 y_2' & y_2' \end{vmatrix} .$$

If the interval $j_1 \subset j$ and the numbers k_1, k_2, $k_1^2 + k_2^2 > 0$ existed such that $k_1 y_1(t) + k_2 y_2(t) \equiv 0$ in j_1, then the Wronskian W would be equal to zero in j_1 and the functions y_1, y_2 would be linearly dependent, contrary to our assumption.

The spaces S_{ab} and S_{pq} are regular spaces of continuous functions with the definition interval j.

Indeed, if $y_1 = y_1(t)$, $y_2 = y_2(t)$ is the basis of S_{ab} or S_{pq}, there cannot simultaneously be $y_1(t_o) = y_2(t_o) = 0$ for any $t_o \in j$, for otherwise the functions y_1, y_2 would be linearly dependent.

The spaces S_{ab} and S_{pq} are strongly regular spaces of continuous functions with the definition interval j.

Indeed, if $y_1 = y_1(t)$, $y_2 = y_2(t)$ is the basis of S_{ab} or S_{pq}, there cannot simultaneously be $y_1(t_o) = y_2(t_o) = 0$ for any $t_o \in j$, for otherwise the functions y_1, y_2 would be linearly dependent.

The spaces S_{ab} and S_{pq} are strongly regular spaces of continuous functions, since the quotients of the independent solutions $y_2(t)/y_1(t)$ are by parts monotonic functions in j.

In fact

$$\left(\frac{y_2(t)}{y_1(t)} \right)' = \frac{y_1(t) y_2'(t) - y_1'(t) y_2(t)}{y_1^2(t)} \gtrless 0 ,$$

because the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is different from zero for independent functions y_1, y_2.

Let us remark that the elements of the space S_{ab} are of the class $C^{(2)}(j)$ and those of the space S_{pq} are of the class $C^{(1)}(j)$ with the property that $py' \in C^{(1)}(j)$.

We will now show how our results on spaces of functions obtained in this paper may be applied to solutions of spaces S_{ab} and S_{pq} especially as regards the concepts of the global transformation and the canonical space. In point of the global transformation of spaces of linear differential equations (ab), (pq) solutions, we will introduce - conformably with the definition of the global transformation of linear n-th order equations in [4] - some simple conditions on the coefficients of differential equations, such that the multiplicator f and the parametrization h may be the functions of the class $C^{(2)}$ and this at the transformation of the equation (ab) or (pq) into the differential equation $y'' = -y$, which will be considered to be canonical.

With the above approach to the definition of global transformation we may show that in the differential equation (pq) it suffices besides $p, q \in C^{(0)}$ to assume f.i. $p \in C^{(1)}$. In case we require in the definition of global transformation $f, h \in C^{(1)}$, it suffices to assume $p, q \in C^{(0)}$.

The space of solutions of the linear differential equation (ab)

We will now express the main results of this paper applied to the space of solutions of the linear differential equation (ab).

A modification of Theorem 2.1.:

Theorem 3.1. Let S_{ab} be a space of solutions of the linear differential equation (ab) with the definition interval j.

Let (y_1, y_2), where $y_1 = y_1(t)$, $y_2 = y_2(t)$, be a basis of the space S_{ab} and $\mathcal{K}$ be a curve defined by the equations $\xi_1 = y_1(t)$, $\xi_2 = y_2(t)$, $t \in j$. For the polar coordinates $\rho = \rho(t)$, $\varphi = \varphi(t)$ of the curve $\mathcal{K}$ we have: $\rho \in C^{(2)}(j)$, $\varphi \in C^{(2)}(j)$, $\varphi'(t) \neq 0$ in j.

P r o o f. Since $y_1, y_2 \in C^{(2)}(j)$, it follows from (2.2) $\rho = \sqrt{y_1^2(t) + y_2^2(t)}$, $t \in j$, the existence of the continuous derivative of the second order of the function ρ, i.e. $\rho \in C^{(2)}(j)$.

From formula (2.3) $\operatorname{tg}\varphi = y_2(t)/y_1(t)$ we get $\sin\varphi = ky_2(t)$, $\cos\varphi = ky_1(t)$, $k = 1/\sqrt{y_1^2 + y_2^2}$. By differentiating this formula we find that $\varphi'/\cos^2\varphi = -(y_1'y_2 - y_1y_2')/y_1^2$ whence by rearrangement

$$\varphi' = -(y_1y_2' - y_1'y_2) / (y_1^2 + y_2^2) \quad ,$$

$t \in j$. We see that $\varphi'(t) \neq 0$ for $t \in j$ and that there exists a continuous φ'' in j, i.e. $\varphi \in C^{(2)}(j)$.

Since the space of solutions of the differential equation (ab) is strongly regular, we can express (conformably with Theorem 2.3 and Theorem 2.4) the following theorems for the space S_{ab}.

Theorem 3.2. Let S_{ab} be the space of solutions of the differential equation (ab) with the definition j. Let (y_1, y_2) be the basis of S_{ab}. Let $s = \varphi(t)$ be the polar coordinate of the curve $\mathcal{K}$: $\xi_1 = y_1(t)$, $\xi_2 = y_2(t)$, $t \in j$, defined by formula (2.3): $\operatorname{tg}\varphi = y_1(t)/y_2(t)$, $t \in j$.

Then we have for the increasing or decreasing function $\varphi = \varphi(t)$

$$u_1[\varphi^{-1}(s)] = \cos s, \quad u_2[\varphi^{-1}(s)] = \sin s, \quad s \in J, \ J = \varphi(j),$$

or

$$u_1\left[\varphi^{-1}(s)\right] = \cos s, \quad u_2\left[\varphi^{-1}(s)\right] = -\sin s, \quad -s\in J,$$
$$J = \varphi(j),$$

where φ^{-1} denotes an inverse function to φ and the functions u_i are determined by the conditions $y_i(t) = f(t)u_i(t)$, $t\in j$, $i=1,2$, $u_1^2(t) + u_2^2(t) \equiv 1$, $f(t) = \sqrt{y_1^2(t) + y_2^2(t)}$.

Theorem 3.3. Suppose the assumptions of Theorem 3.2 are satisfied. Let $A = \|a_{ik}\|$, $i,k = 1,2$, $\overline{Y} = A\,\underline{u}\left[\varphi^{-1}(s)\right]$.

If the matrix A is of the form (2.16), then the functions $\overline{Y}_1$, $\overline{Y}_2$ are given by formulas (2.17).

Theorem 3.4. Let S_{ab} be a space of solutions of the differential equation (ab) with the definition interval j. Let (y_1, y_2) be a basis of the space S_{ab}. Let $\varphi = \varphi(t)$ be an increasing or decreasing function satisfying the functional equation $\operatorname{tg}\varphi(t) = y_2(t)/y_1(t)$ for $t\in j$. Let $J = \varphi(j)$. Then for

a) the bijection $\varphi: j \to J$, $\varphi\in C^{(2)}(j)$,

b) the function $f = \sqrt{y_1^2(t) + y_2^2(t)} \in C^{(2)}(j)$, $f(t) \neq 0$ for $t\in j$,

c) the matrix $A = \begin{pmatrix} \cos s_0 & \sin s_0 \\ -\sin s_0 & \cos s_0 \end{pmatrix}$ or $A = \begin{pmatrix} \cos s_0 & \sin s_0 \\ \sin s_0 & -\cos s_0 \end{pmatrix}$,

$s_0\in J$, $s_0 = \varphi(t_0)$, $t_0\in j$,

there exists a global transformation of the space S^* onto the space S_{ab} given by the relation

$$y(t) = Af(t)\,\underline{Y}\left[\varphi(t)\right],$$

$t\in j$, $\underline{Y} = (\cos s, \sin s)^T$, $\underline{y} = (y_1, y_2)^T$.

P r o o f. The above theorem is a modification of Theorem 2.5. in case of the space S_{ab}.

Definition 3.1. The differential equation

$$Y'' = -Y \qquad (-1)$$

with the definition interval J, $J = \varphi(j)$ will be called the canonical form of the second order of the space S_{ab} of the solution of the differential equation (ab), or briefly the canonical differential equation of the second order of the space S_{ab}.

Let us remark that by the space of solutions of the differential equation (-1) we mean the space S^* with the definition interval J. The elements of the space S^* are the functions of the form

$$\bar{Y} = k_1 \cos s + k_2 \cos s, \quad s \in J, \quad K_1, k_2 \in R$$

Definition 3.2. Let (y_1, y_2) be a basis of the space S_{ab} with the definition interval j. Every function $\alpha \in C^{(2)}(j)$, α : $j \to J$, satisfying in j the functional equation

$$\operatorname{tg}\alpha(t) = \frac{y_1(t)}{y_2(t)}$$

will be named the first phase, briefly the phase of an ordered pair of solutions $y_1, y_2 \in S_{ab}$.

Theorem 3.5. Let $(\tilde{y}_1, \tilde{y}_2)$ be a basis of the space S_{ab} with the definition interval j. Let $(\tilde{Y}_1, \tilde{Y}_2)$ be a basis of the space S^* with the definition interval J. Suppose the space S^* is globally transformed onto the space S_{ab} by the equation

$$\tilde{\underline{y}}(t) = Af(t)\tilde{\underline{Y}}\left[h(t)\right]$$

by means of the function f, the parametrization h and the matrix A. Let $\mathcal{H}$ be the matrix given by the equation $\tilde{\underline{Y}} = \mathcal{H}\, \underline{Y}$, where $\underline{Y} = (\cos s, \sin s)^T$. Let $\underline{y} = (y_1, y_2)^T$, where $\underline{y} = \mathcal{H}^{-1}A^{-1}\underline{y}$, whereby $\mathcal{H}^{-1}$, A^{-1} denote the inverse matrices to the matrix $\mathcal{H}$, or A. Let $\alpha = \alpha(t)$ be a first phase of the basis $(y_2, y_1) \subset S_{ab}$. Then

$$\alpha_k(t) = h(t) + k\pi, \quad k \text{ being an integer},$$

$$\alpha_o = \alpha(t) \quad \text{for} \quad t \in j.$$

P r o o f. The above theorem is a modification of Theorem 2.7 in case of the space S_{ab}.

<u>Theorem 3.6</u>. Let S_{ab} and S_{AB} be the spaces of solutions of the differential equations (ab) and (AB) with the definition intervals j_1 and j_2, respectively. Let S^* be the canonical space with the definition interval J. Let $\underline{\tilde{Y}} = (\tilde{Y}_1, \tilde{Y}_2)$ be a basis of S^*. Let $(\tilde{u}_1, \tilde{u}_2) \in S_{ab}$, $(\tilde{U}_1, \tilde{U}_2) \in S_{AB}$ be the space bases.

Suppose the space S^* is globally transformed onto the space S_{ab} by the equation $\underline{\tilde{u}}(t) = A_1 f(t) \underline{\tilde{Y}}[h(t)]$, where h: $j_1 \to J$, $h \in C^{(2)}(j)$, $\underline{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2)^T$. Suppose the space S^* is globally transformed onto the space S_{AB} by the equation $\underline{\tilde{U}}(T) = A_2 F(T) \underline{\tilde{Y}}[H(T)]$, where H: $j_2 \to J$, $H \in C^{(2)}(j_2)$, $\underline{\tilde{U}} = (\tilde{U}_1, \tilde{U}_2)^{(T)}$.

The necessary and sufficient condition for the existence of the global transformation of the space S_{AB} onto the space S_{ab} transforming the basis $(\tilde{u}_1, \tilde{u}_2)$ into the basis $(\tilde{U}_1, \tilde{U}_2)$ by means of

a) the bijection k: $j_1 \to j_2$, $k \in C^{(2)}(j_1)$.

b) the function $g \in C^{(2)}(j_1)$, $g(t) \neq 0$ for $t \in j_1$,

c) the matrix $B = A_1 A_2^{-1}$

by the formula

$$\underline{\tilde{u}}(t) = B\, g(t) \quad \underline{\tilde{U}}[k(t)], \qquad t \in j_1$$

is the existence of

1. a bijection $T = X(t)$, X: $j_1 \to j_2$, $X \in C^{(2)}(j)$,

2. an integer ℓ, for which $h(t) = H[X(t)] + \ell\pi$ for $t \in j_1$.

P r o o f. The above theorem is a modification of Theorem 2.8 in case of the spaces S_{ab}, S_{AB} .

The space of solutions of the linear differential equation (pq)

We will now express the main results obtained in this paper applied to the space of solutions of the linear differential equation (pq).

A modification to Theorem 2.1.:

Theorem 3.7. Let S_{pq} be a space of solutions of the linear differential equation (pq) with the definition interval j. Let (y_1, y_2), where $y_1 = y_1(t)$, $y_2 = y_2(t)$, be a basis of the space S_{pq} and $\mathcal{K}$ be a curve defined by the equations $\xi_1 = y_1(t)$, $\xi_2 = y_2(t)$, $t \in j$. For the polar coordinates $\varrho = \varrho(t)$, $\varphi = \varphi(t)$ of the curve $\mathcal{K}$ we have $\varrho \in C^{(1)}(j)$, $\varphi \in C^{(1)}(j)$, $\varphi'(t) \neq 0$ in j.

P r o o f. Since $p \in C^{(0)}(j)$, $y_1, y_2, py_1', py_2' \in C^{(1)}(j)$, $p(t) \neq 0$ in j, it follows from formula (2.2) $\varrho = \sqrt{y_1^2(t) + y_2^2(t)}$, $t \in j$, that

$$\varrho' = (y_1^2 + y_2^2)^{-\frac{1}{2}} (y_1 y_1' + y_2 y_2') = \left[y_1(py_1') + y_2(py_2')\right] / $$

$$/ \, p\sqrt{y_1^2 + y_2^2} \ .$$

From this we see that ϱ' is a continuous function, i.e. $\varrho \in C^{(1)}(j)$. From 2.3): $\operatorname{tg} \varphi = y_2(t)/y_1(t)$ we obtain $\sin\varphi = ky_2(t)$, $\cos\varphi = ky_1(t)$, $k = 1/\sqrt{y_1^2 + y_2^2}$. By differentiatin and rearrangement we obtain

$$\varphi' = -(y_1 y_2' - y_1' y_2)/(y_1^2 + y_2^2) = -\left[y_1(py_2') - \right.$$

$$\left. - (py_1')y_2\right] / p(y_1^2 + y_2^2) , \qquad t \in j$$

and we can see that $\varphi'(t) \neq 0$ for $t \in j$ and that φ' is continuous in j, i.e. $\varphi \in C^{(1)}(j)$.

Assuming that the coefficient p of the differential equation (pq) satisfies the condition $p \in C^{(1)}(j)$, then there holds for the polar coordinates of the curve $\mathcal{K}$ from Theorem 3.7. that $\rho \in C^{(2)}(j)$, $\varphi \in C^{(2)}(j)$, $\varphi'(t) \neq 0$ in j.

Since the space S_{pq} is strongly regular we may express (conformably with Theorems 2.3. and 2.4.) the following theorems for the space S_{pq}.

Theorem 3.8. Let S_{pq} be a space of solutions of the differential equation (pq) with the definition interval j. Let (y_1, y_2) be a basis of S_{pq}. Let $s = \varphi(t)$ be a polar coordinate of the curve $\mathcal{K}$: $\xi_1 = y_1(t)$, $\xi_2 = y_2(t)$, $t \in j$, defined by formula (2.3).

Then it holds for the increasing or decreasing function $\varphi = \varphi(t)$

$$u_1\left[\varphi^{-1}(s)\right] = \cos s, \quad u_2\left[\varphi^{-1}(s)\right] = \sin s, \quad s \in J, \quad J = \varphi(j)$$

or

$$u_1\left[\varphi^{-1}(s)\right] = \cos s, \quad u_2\left[\varphi^{-1}(s)\right] = -\sin s, \quad -s \in J, \quad J = \varphi(j) ,$$

where φ^{-1} denotes the inverse function to φ and the functions u_i are determined by the conditions $y_i(t) = f(t)u_i(t)$, $t \in j$, $i=1,2$, $u_1^2(t) + u_2^2(t) \equiv 1$, $f(t) = \sqrt{y_1^2(t) + y_2^2(t)}$.

Theorem 3.9. Suppose the assumptions of Theorem 3.8. are fulfilled. Let $A = \| a_{ik} \|$, $i,k = 1,2$, $\underline{\bar{Y}} = A . \underline{u}\left[\varphi^{-1}(s)\right]$.

If the matrix A is of the form (2.16) then the functions $\bar{Y}_1, \bar{Y}_2$ are given by formulas (2.17).

Theorem 3.10. Let S_{pq} be a space of solutions of the differential equation (pq) with the definition interval j. Let (y_1, y_2) be a basis of the space S_{pq}. Let $\varphi = \varphi(t)$ be an increasing or decreasing function satisfying the functional equation $\operatorname{tg} \varphi(t) = y_2(t)/y_1(t)$ for $t \in j$. Let $J = \varphi(j)$. Then for

a) the bijection $\varphi : j \to J$, $\varphi \in C^{(2)}(j)$,

b) the function $f = \sqrt{y_1^2(t) \text{ a } y_2^2(t)} \in C^{(2)}(j)$, $f(t) \neq 0$ for $t \in j$,

c) the matrix $A = \begin{pmatrix} \cos s_0 & \sin s_0 \\ -\sin s_0 & \cos s_0 \end{pmatrix}$, or $A = \begin{pmatrix} \cos s_0 & \sin s_0 \\ \sin s_0 & -\cos s_0 \end{pmatrix}$,

$s_0 \in J$, $s_0 = \varphi(t_0)$, $t_0 \in j$,

there exists a global transformation S^* onto the space S_{pq} given by the relation

$$\underline{y}(t) = Af(t)\, \underline{Y}\left[h(t)\right],$$

$t \in j$, $\underline{Y} = (\cos s, \sin s)^T$, $\underline{y} = (y_1, y_2)^T$.

P r o o f. The above theorem is a modification of Theorem 2.5. in case of the space S_{pq}.

Definition 3.3. The differential equation

$$Y'' = -Y \qquad (-1)$$

with the definition interval J, $J = \varphi(j)$ will be called the canonical form of the second order of the space S_{pq} of the solution of the differential equation (pq), more briefly the canonical differential equation of the second order of the space S_{pq}.

Let us note that by the space of solutions of the differential equation (-1) we mean the space S^* with the definition interval J. The elements of the space S^* are the functions

$$\bar{Y} = k_1 \cos s + k_2 \sin s , \qquad s \in J, \quad k_1, k_2 \in R .$$

Definition 3.4. Let (y_1, y_2) be a basis of the space S_{pq} with the definition interval j. Every function $\alpha \in C^{(2)}(j)$, $\alpha : j \to J$ satisfying in j the functional equation

$$\operatorname{tg} \alpha(t) = \frac{y_1(t)}{y_2(t)} ,$$

will be called a first phase, more briefly a phase of an ordered pair of solutions $y_1, y_2 \in S_{pq}$.

Theorem 3.11. Let $(\tilde{y}_1, \tilde{y}_2)$ be a basis of the space S_{pq} with the definition interval j. Let $(\tilde{Y}_1, \tilde{Y}_2)$ be a basis of the space S^* with the definition interval J. Suppose the space S^* is globally transformed onto the space S_{pq} by the equation.

$$\underline{\tilde{y}}(t) = A f(t) \, \underline{\tilde{Y}}\left[h(t)\right]$$

by means of the function f, the parametrization h and the matrix A. Let $\mathcal{K}$ be a matrix given by the equation $\underline{\tilde{Y}} = \mathcal{K} \, \underline{Y}$ where $\underline{Y} = (\cos s, \sin s)^T$. Let $\underline{y} = (y_1, y_2)^T$, where $\underline{y} = \mathcal{K}^{-1} A^{-1} \underline{\tilde{y}}$, whereby $\mathcal{K}^{-1}$, A^{-1} denote the inverse matrices to the matrix $\mathcal{K}$, or A. Let $\alpha = \alpha(t)$ a first phase of the basis $(y_2, y_1) \in S_{pq}$. Then

$$\alpha_k(t) = h(t) + k\pi , \qquad k \text{ being an integer,}$$

$\alpha_0 = \alpha(t)$ for $t \in j$.

P r o o f. The above theorem is a modification of Theorem 2.7. in case of the space S_{pq}.

Theorem 3.12. Let S_{pq} be the space of solutions of the differential equation (pq) with the definition interval j_1 and S_{PQ} be the space of solutions of the differential equation (PQ) with the definition interval j_2. Next let S^* be a canonical space with the definition interval J. Let $\tilde{\underline{Y}} = (\tilde{Y}_1, \tilde{Y}_2)$ be a basis of S^*. Let $(\tilde{u}_1, \tilde{u}_2) \in S_{pq}$, $(\tilde{U}_1, \tilde{U}_2) \in S_{PQ}$ be a space basis. Suppose the space S^* is globally transformed onto the space S_{pq} by the equation

$$\tilde{\underline{u}}(t) = A_1 f(t)\, \tilde{\underline{Y}}[h(t)], \quad \text{where } h: j_1 \to J, \quad h \in C^{(2)}(j_1),$$

$\tilde{\underline{u}} = (\tilde{u}_1, \tilde{u}_2)^T$.

Suppose the space S^* is globally transformed onto the space S_{PQ} by the equation

$$\tilde{\underline{U}}(T) = A_2 F(T)\, \tilde{\underline{Y}}[H(T)], \quad \text{where } H: j_2 \to J, \; H \in C^{(2)}(j_2),$$

$\tilde{\underline{U}} = (U_1, U_2)^T$.

The necessary and sufficient condition for the existence of the global transformation of the space S_{PQ} onto the space S_{pq} transforming the basis $(\tilde{u}_1, \tilde{u}_2)$ into the basis $(\tilde{U}_1, \tilde{U}_2)$ by

a) the bijection $k: j_1 \to j_2$, $k \in C^{(2)}(j_1)$,

b) the function $g \in C^{(2)}(j)$, $g(t) \neq 0$ for $t \in j_1$,

c) the matrix $B = A_1 A_2^{-1}$

by the formula

$$\tilde{\underline{u}}(t) = B g(t)\, \tilde{\underline{U}}[k(t)], \quad t \quad j_1$$

is the existence of

1. a bijection $T = X(t)$, $X: j_1 \to j_2$, $X \in C^{(2)}(j_1)$,

2. an integer ℓ, for which $h(t) = H[X(t)] + \ell\pi$ for $t \in j_1$:

P r o o f. The above theorem is a modification of Theorem 2.8. in case of the space S_{pq}, S_{PQ} .

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KANONICKÝ PROSTOR SPOJITÝCH FUNKCÍ DIMENZE 2

Souhrn

Článek je věnován studiu globální transformace dvourozměrných regulárních a silně regulárních prostorů spojitých funkcí z geometrického hlediska. Význačnou roli zde má tzv.

kanonický prostor spojitých funkcí, který umožňuje charakterizovat studované prostory spojitých funkcí.

Definuje se fáze α uspořádané dvojice funkcí y_1, y_2 v silně regulárním prostoru S s definičním intervalem j jako každá spojitá funkce v j, která v j vyhovuje funkční rovnici $\mathrm{tg}\,\alpha(t) = y_1(t)/y_2(t)$. Hledá se vztah mezi první fází α a parametrizací h, kterou je zprostředkována globální transformace kanonického prostoru spojitých funkcí na prostor S a ukazuje se, že platí $\alpha_k(t) = h(t) + k\pi$, k celé, $\alpha_0 = \alpha(t)$. Dokazuje se nutná a postačující podmínka pro existenci globální transformace silně regulárních prostorů S_1 a S_2 dimenze 2.

Získané výsledky jsou aplikovány na prostory řešení lineárních diferenciálních rovnic obecného a Sturmova tvaru.

КАНОНИЧЕСКОЕ ПРОСТРАНСТВО НЕПРЕРЫВНЫХ ФУНКЦИЙ РАЗМЕРНОСТИ 2

Резюме

Настоящая статья посвящена изучению глобального преобразования двухмерных регулярных и сильно регулярных пространств непрерывных функций с геометрической точки зрения. Особую роль здесь играет т.н. каноническое пространство непрерывных функций, которое дает возможность характеризировать изучаемые пространства непрерывных функций.

Определяется первая фаза α упорядоченной пары функций y_1, y_2 в сильно регулярном пространстве S с интервалом определения j как любая непрерывная функция в j, которая в j удовлетворяет функциональному уравнению $\mathrm{tg}\,\alpha(t) = y_1(t)/y_2(t)$. Ищется соотношение между первой фазой α и

параметризацией h , с помощью которой осуществляется глобальная трансформация канонического пространства непрерывных функций на пространство S и показывается, что имеет место $\alpha_k(t)=h(t)+k\pi$, k целое, $\alpha_0 = \alpha(t)$.

Доказывается необходимое и достаточное условие для существования глобальной трансформации сильно регулярных пространств S_1 и S_2 размерности 2.

Полученные результаты применяются в теории пространств решений линейных дифференциальных уравнений общего типа и типа Штурма.

Author's address:
RNDr. Jitka Laitochová, CSc.
pedagogická fakulta
Univerzity Palackého
Žerotínovo náměstí 2
771 40 Olomouc
ČSSR /Czechoslovakia/

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