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ON SOME PROPERTIES OF SOLUTIONS OF THE DISCONJUGATE EQUATION $y'' = q(t)y$ WITH AN ALMOST PERIODIC COEFFICIENT q

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1. Introduction and résumé of results

M a r k u s and M o o r e /6/ investigated the properties of solutions of the equation $y''= (a+b.q_1(t))y$, where $a, b \in R$ and q_1 is a (real) almost periodic function. A special consideration was devoted to the above equation being disconjugate. In studying the properties of their solutions the authors started from the associated Riccati equation $u'+ u^2 = a+b.q_1(t)$. The distributions of zeros of solutions and the derivative of solutions in oscillatory equations are discussed in /8/ - /10/.

The present article follows the results of /6/. The problem under consideration is the disconjugate equation $(q): y''= q(t)y$, where q means a (real) almost periodic function. As is well known (see /1/ or /2/), every disconjugate homogeneous linear differential equation of the second order is either generally disconjugate or specially disconjugate. It is proved for the specially disconjugate equation (q) that there exists a unique almost periodic solution of the associated Riccati equation (Theorem 1), and conditions are given (necessary and sufficient) for the equation (q) to be generally disconjugate or specially disconjugate. These conditions are expressed either through a certain form of a solution of the equation (q) (Theorem 2) or through a certain form of the coefficient q (Theorem 6) or through a certain form of the phase of the equation (q) (Theorems 9, 10). Next, there are given necessary and sufficient conditions for the existence of an (up to a multiplicative constant necessarily unique) almost periodic solution of the specially disconjugate equation (q) (Theorems 3, 4) without using the Floquet representation (see, say /5/), which is not applicable here. Theorem 8 deals with an analogous problem for the generally disconjugate equation.

Suppose $H\{q\}$ is the hull generated by q. Then the specially disconjugate equation (q) has an almost periodic solution exactly if all equations (q^*) have almost periodic solutions, where $q^* \in H\{q\}$ (Theorem 5) and if a is a characteristic exponent related to the generally disconjugate equation (q), then a is a characteristic exponent related to any equation (q^*), $q^* \in H\{q\}$ (Theorem 7).

2. Basic concepts and lemmas

In this article we exclude trivial solutions of the equation

$$y'' = p(t)y, \quad p \in C^0(R). \qquad (p)$$

We say, that (p) is an oscillatory (disconjugate) equation if every solution of (p) has $\pm\infty$ as the cluster points of its zeros (if every solution of (p) has one zero at most).

Definition 1 (/1/, /2/). **Suppose (p) is a disconjugate equation. Assume that (p) is a generally disconjugate equation exactly if there exist two independent solutions of (p) having no zero on R. Assume that (p) is a specially disconjugate equation exactly if there exists a unique solution of (p) (up to a multiplicative constant) having no zero on R.**

Say conformably with /1/ and /2/ that a function $\alpha \in C^0(R)$ is a (first) phase of (p) if there exist independent solutions u, v of (p) such that

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \quad \text{for } t \in R - \{t;\ v(t) = 0\} .$$

In then follows from /1/, /2/ that $\alpha \in C^3(R)$, $\alpha'(t) \neq 0$ and $p(t) = -\{\alpha, t\} - \alpha'^2(t)$ for $t \in R$, where $\{\alpha, t\} :=$
$= \frac{1}{2}\,\frac{\alpha'''(t)}{\alpha'(t)} - \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2$.

Letting E be the set of phases of the equation $y'' = -y$ yields that $E\alpha := \{\varepsilon\alpha,\ \varepsilon \in E\}$ is the set of phases of (p), where α is a phase of (p).

The equations (p) is a specially disconjugate equation exactly if

$$\left| \lim_{t\to -\infty} \alpha(t) - \lim_{t\to \infty} \alpha(t) \right| = \pi$$

for some (and then for every) phase α of (p).

The equation (p) is a generally disconjugate equation exactly if

$$\left| \lim_{t\to -\infty} \alpha(t) - \lim_{t\to \infty} \alpha(t) \right| < \pi$$

for some (and then for every) phase α of (p).

The solutions of (p) stand in the following relations to the solutions of the Riccati equation

$$u' + u^2 = p(t), \quad p \in C^0(R). \tag{1}$$

If $y=y(t)$ is a solution of (p) and $y(t) \neq 0$ for $t \in j$, where $j \subset R$ is an interval, then the function $u(t) := \frac{y'(t)}{y(t)}$, $t \in j$, is a solution of (1) on j and also reversely, if $u=u(t)$ is a solution of (1) on $j \subset R$, then there exists just one (up to a multiplicative constant) solution $y=y(t)$ of (p), $y(t) \neq 0$ for $t \in j$ such that $u(t) = \frac{y'(t)}{y(t)}$ for $t \in j$.

Lemma 1. *Suppose $p \in C^0(R)$, $|p(t)| \leq M$ for $t \in R$, where $M \in R$. If a solution u of (1) is defined on R, then*

$$|u(t)| < M, \quad t \in R.$$

Proof. The proof follows from the remark in /6/ page 101.

Definition 2 (/3/, /4/). *A function f is called almost periodic exactly if $f \in C^0(R)$ and if there exists a number L $(=L(\varepsilon) > 0)$ to every number $\varepsilon > 0$ such that a number τ:*

$$|f(t+\tau) - f(t)| < \varepsilon \quad \text{for } t \in R$$

exists on every interval of type $[x, x+L)$ $(x \in R)$.

The set of almost periodic functions will be denoted by S. Every almost periodic function f is bounded and uniformly continuous on R with the mean value equal to $M\{f\}$, where $M\{f\} := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)dt$. By the Bohr-Bochner Theorem (see /3/, /4/) $f \in S$ exactly if $f \in C^0(R)$ and for every sequence $\{h_n\}$, $h_n \in R$, we may choose a subsequence uniformly convergent on R from the sequence of functions $\{f(t+h_n)\}$.

Suppose that $f \in S$ and $H\{f\}$ is the hull generated by f that is $g \in H\{f\}$ exactly if $\lim_{n\to\infty} f(t+h_n) = g(t)$ uniformly on R for any sequence $\{h_n\}$, $h_n \in R$. Then $H\{f\} \subset S$ and if $g \in H\{f\}$ then $H\{g\} = H\{f\}$.

Lemma 2 (/6/). *The equation*

$$y'' = q(t)y, \quad q \in S, \tag{q}$$

is either oscillatory or disconjugate.

Lemma 3 (/6/). *The equation (q) is generally disconjugate exactly if there exist almost periodic solutions $u = \psi_1(t)$, $u = \psi_2(t)$ of*

$$u' + u^2 = q(t), \tag{2}$$

$M\{\psi_1\} = -M\{\psi_2\} \neq 0$.

Lemma 4 (/6/). *Suppose (q) is a generally disconjugate equation and $u = \psi_1(t)$, $u = \psi_2(t)$ are almost periodic solutions of (2), $\psi_1 \neq \psi_2$, $a := M\{\psi_1\} > 0$. Setting*

$$\varphi_1(t) := \psi_1(t) - a, \quad \varphi_2(t) := \psi_2(t) + a \text{ for } t \in R,$$

yields $\int_0^t (\varphi_1(s) + \varphi_2(s))ds \in S$ *and the functions*

$$y_1(t) := e^{at}\exp\left(\int_0^t \varphi_1(s)ds\right),$$

$$y_2(t) := e^{-at}\exp\left(\int_0^t \varphi_2(s)ds\right), \quad t \in R,$$

are independent solutions of (q).

Definition 3 (/6/, p.113). *Let (q) be a generally discojugate equation and $a > 0$ be the number occurring in Lemma 4, called* the characteristic exponent of (q).

Lemma 5. Suppose $q \in S$ and for a sequence $\{h_n\}$, $h_n \in R$,

$$\lim_{n\to\infty} q(t+h_n) = p(t)$$

uniformly on R. Let for a sequence $\{t_n\}$, $t_n \in R$, and for sequences $\{h_n^{(1)}\}$, $\{h_n^{(2)}\}$ chosen from the sequence $\{h_n\}$

$$\lim_{n\to\infty} q(t+t_n+h_n^{(1)}) = p_1(t), \quad \lim_{n\to\infty} q(t+t_n+h_n^{(2)}) = p_2(t)$$

uniformly on R. This yields $p_1 = p_2$.

Proof. We proceed in the same manner as we did in proving Theorem 1 /8/. Let $\varepsilon > 0$ be an arbitrary number. Then there exists a positive integer N_1 such that

$$\left|q(t+h_n) - q(t+h_m)\right| < \frac{\varepsilon}{3}$$

for $m,n > N_1$ and $t \in R$. It then follows that

$$\left|q(t+h_n^{(1)}) - q(t+h_n^{(2)})\right| < \frac{\varepsilon}{3}$$

and also

$$\left|q(t+t_n+h_n^{(1)}) - q(t+t_n+h_n^{(2)})\right| < \frac{\varepsilon}{3} \tag{3}$$

for $n > N_1$ and $t \in R$. Further, there exists a positive integer N_2 such that

$$\left|q(t+t_n+h_n^{(1)})-p_1(t)\right| < \frac{\varepsilon}{3}, \quad \left|q(t+t_n+h_n^{(2)})-p_2(t)\right| < \frac{\varepsilon}{3} \tag{4}$$

for $n > N_2$ and $t \in R$. Setting $N = \max(N_1, N_2)$ yields from (3) and (4) for $n > N$ and $t \in R$

$$|p_1(t)-p_2(t)| \leq |q(t+t_n+h_n^{(1)}) - p_1(t)| +$$

$$+ |q(t+t_n+h_n^{(2)})-p_2(t)| + |q(t+t_n+h_n^{(1)}) -$$

$$- q(t+t_n+h_n^{(2)})| < \varepsilon \; .$$

Thus $|p_1(t) - p_2(t)| < \varepsilon$ for $t \in R$ and with respect to the arbitrariness of ε we obtain $p_1 = p_2$.

Lemma 6. It holds:

(i) If (q) is a generally disconjugate equation, then, for every $q^* \in H\{q\}$, the equation (q^*) is also generally disconjugate;

(ii) If (q) is a specially disconjugate equation, then, for every $q^* \in H\{q\}$, the equation (q^*) is also specially disconjugate.

Proof. The proof follows form the Bohr-Bochner Theorem and from Theorem 1 /6/.

Lemma 7. Suppose $\varphi, \varphi' \in S$ and $M\{\varphi\} = 0$. Then the function

$$\int_0^t \exp\left(\int_0^s \varphi(\tau)d\tau\right)ds \tag{5}$$

is mapping R onto R.

Proof. Let us set $q(t) := -\frac{1}{2}\varphi'(t) + \frac{1}{4}\varphi^2(t)$, $t \in R$. Then $q \in S$ and the function $-\frac{1}{2}\varphi(t)$ is a solution of (2). The function $y(t) = \exp(-\frac{1}{2}\int_0^t \varphi(s)ds)$ is a solution of (q) and with respect to $-\frac{1}{2}\varphi \in S$ and $M\{-\frac{1}{2}\varphi\} = 0$, and by Lemma 3, (q) is a specially disconjugate equation. Hence

$$\int_{-\infty}^{0} \frac{ds}{y^2(s)} = \infty, \quad \int_{0}^{\infty} \frac{ds}{y^2(s)} = \infty$$ and the function

$$\int_0^t \frac{ds}{y^2(s)} = \int_0^t \exp\left(\int_0^s \varphi(\tau)d\tau\right)ds$$ is mapping R onto R.

Lemma 8. Suppose that φ, $\varphi' \in S$ and $M\{\varphi\} \neq 0$. Then function (5) is mapping R onto j, $j \neq R$.

Proof. Setting in analogy to the proof of the preceding Lemma $q(t) := -\frac{1}{2}\varphi'(t) + \frac{1}{4}\varphi^2(t)$, $t \in R$, then $q \in S$ and the function $y(t) = \exp(-\frac{1}{2}\int_0^t \varphi(s)ds)$ is a solution of (q). The equation (q) is disconjugate and is necessarily generally disconjugate with respect to Lemma 3. Then necessarily at least one of the improper integrals $\int_{-\infty}^{0} \frac{ds}{y^2(s)}$, $\int_{0}^{\infty} \frac{ds}{y^2(s)}$ converges, whence immediately follows the statement of the Lemma.

Lemma 9. Suppose a is the characteristic exponent of the generally disconjugate equation (q). Let (q) have solutions of the form

$$e^{-at}\phi_1(t),\ e^{at}\phi_2(t),\ \text{where } \phi_1,\ \phi_2 \in S.$$

Then there exist functions φ_1, φ_2 such that φ_1, φ_2, φ_1', $\varphi_2' \in S$, $M\{\varphi_1\} = M\{\varphi_2\} = 0$ and

$$\phi_i(t) = k_i.\exp\left(\int_0^t \varphi_i(s)ds\right) \text{ for } t \in R, \text{ where } k_i \in R\ (i=1,2).$$

Proof. By Lemma 4 there exist functions φ_1, φ_2 such that $\varphi_1, \varphi_2, \varphi'_1, \varphi'_2 \in S$, $M\{\varphi_1\} = M\{\varphi_2\} = 0$ and the functions $e^{-at}\exp(\int_0^t \varphi_1(s)ds)$, $e^{at}\exp(\int_0^t \varphi_2(s)ds)$ are independent solutions of (q). Let (q) have a solution of the form $e^{-at}\phi_1(t)$ where $\phi_1 \in S$. Then there exist numbers a_1, a_2:

$$e^{-at}\phi_1(t) = a_1 e^{-at}\exp(\int_0^t \varphi_1(s)ds) + a_2 e^{at}\exp(\int_0^t \varphi_2(s)ds).$$

Since $\lim_{t\to\infty} e^{-a}\exp(\frac{1}{t}\int_0^t \varphi_1(s)ds) = e^{-a} < 1$,

$\lim_{t\to\infty} e^{a}\exp(\frac{1}{t}\int_0^t \varphi_2(s)ds) = e^{a} > 1$, it turns out that

$\lim_{t\to\infty} e^{-at}\exp(\int_0^t \varphi_1(s)ds) = 0$ and $\lim_{t\to\infty} e^{at}\exp(\int_0^t \varphi_2(s)ds) = \infty$.

Therefore $a_2 = 0$ and $\phi_1(t) = a_1\exp(\int_0^t \varphi_1(s)ds)$. Likewise

we can prove the equality $\phi_2(t) = k_2\exp(\int_0^t \varphi_2(s)ds)$, where

k_2 is an appropriate number.

Remark 1. Suppose (q) is a generally disconjugate equation and a is its characteristic exponent. Since (q) has solutions of the form $e^{-at}\phi_1(t)$, $e^{at}\phi_2(t)$, with at least one of the functions ϕ_1, ϕ_2 belonging to S, then, respecting the proof of Lemma 9 and Lemma 4, it follows that necessarily both functions ϕ_1, ϕ_2 are lying in S already.

3. Main results

Theorem 1. *The equation (q) is a specially disconjugate exactly if there exists just one almost periodic solution u of (2) and* $M\{u\} = 0$.

Proof. ($\Longrightarrow$) If there exists just one almost periodic solution u of (2), then, respecting Lemma 2 and Lemma 3, equation (q) is specially disconjugate and it follows from Theorem 15 /6/ that $M\{u\} = 0$.

($\Longleftarrow$) Let (q) be a specially disconjugate equation. Then there exists just one solution u of (2) defined on R being bounded by Lemma 1. Assume next that $u \in S$. Then there exists a sequence $\{h_n\}$, $h_n \in R$, such that $\lim_{n\to\infty} q(t+h_n) = p(t)$ uniformly on R, without any possibility to choose a subsequence uniformly convergent on R from the sequence of functions $\{u(t+h_n)\}$. Thus there exist a number $a > 0$, increasing sequence of positive integer $\{k_n\}$, $\{r_n\}$, and a sequence $\{t_n\}$ ($t_n \in R$, $|t_n| \to \infty$ for $n \to \infty$):

$$|u(t_n+h_{k_n}) - u(t_n+h_{r_n})| \geqq a, \; n = 1,2, \ldots . \qquad (6)$$

Since the sequences $\{u(t_n+h_{k_n})\}$, $\{u(t_n+h_{r_n})\}$ are bounded, it may be realized (in passing to suitable subsequences, whereby, for brevity, the same notation of indexes is retained) that

$$\lim_{n\to\infty} q(t+t_n+h_{k_n}) = p_1(t), \; \lim_{n\to\infty} q(t+t_n+h_{r_n}) = p_2(t)$$

uniformly on R and

$$\lim_{n\to\infty} u(t_n+h_{k_n}) = \alpha, \quad \lim_{n\to\infty} u(t_n+h_{r_n}) = \beta,$$

where $\alpha, \beta \in R$ and because of (6) we get

$$|\alpha - \beta| \geq a. \tag{7}$$

By Lemma 5 (in setting $h_n^{(1)} = h_{k_n}$, $h_n^{(2)} = h_{r_n}$, $n = 1,2,\dots$) we have $p_1 = p_2(=:s)$. Since $u(t+t_n+h_{k_n})$ and $u(t+t_n+h_{r_n})$ are the solutions of the equations

$$u' + u^2 = q(t+t_n+h_{k_n})$$

and

$$u' + u^2 = q(t+t_n+h_{r_n}),$$

respectively, we see that

$$\lim_{n\to\infty} u(t+t_n+h_{k_n}) = u_1(t), \ \lim_{n\to\infty} u(t+t_n+h_{r_n}) = u_2(t)$$

uniformly on every compact interval, where u_1, u_2 are the solutions defined on R of the following equation

$$u' + u^2 = s(t)$$

satisfying the initial conditions $u_1(0) = \alpha$, $u_2(0) = \beta$. By Lemma 6 (p) is a specially disconjugate equation, hence there exists just one solution of (8) defined on R. This yields $u_1 = u_2$, consequently $\alpha = \beta$, which, however, contradicts

(7). Therefore u is an almost periodic solution of (q) and by Theorem 15 /6/ $M\{u\} = 0$.

Remark 2. Theorem 1 makes precise the statement of Theorem 15 /6/ by which there exists at most just one almost periodic solution of a specially disconjugate equation (q) whose mean value then necessarily vanishes.

Theorem 2. (q) *is a specially disconjugate equation exactly if the function*

$$y(t) := \exp\left(\int_0^t \varphi(s)ds\right), \quad t \in R, \qquad (9)$$

is a solution of (q), where φ, $\varphi' \in S$ *and* $M\{\varphi\} = 0$.

Proof. ($\Longleftarrow$) Let the function y defined as in (9) be a solution of (q), where $\varphi, \varphi' \in S$ and $M\{\varphi\} = 0$. The function $u = \varphi(t)$ is an almost periodic solution of (2) and from Lemma 4 it follows that (q) is not a generally disconjugate equation, consequently it is necessarily specially disconjugate.

($\Longrightarrow$) Let (q) be a specially disconjugate equation. By Theorem 1 there exists a solution φ of (2) such that φ, $\varphi' \in S$, $M\{\varphi\} = 0$. Then $\dfrac{y'(t)}{y(t)} = \varphi(t)$ for $t \in R$ and any solution y of (q), $y(0) = 1$ and this by integration yields (9).

Definition 3 (/6/ page 119). *Let (q) be a specially disconjugate equation. 0 is called the characteristic exponent of (q).*

Remark 3. We known from the Floquet theory (see e.g. /5/) that the specially disconjugate eguation (q) with a periodic coefficient q has just one (up to the multiplicative constant) periodic (and so also bounded) solution. If the coefficient q of (q) is not a periodic function, then the situation is more complicated. This becomes apparent from the example on page 119 /6/ with a concrete equation (q), $q \in S$, specially disconjugate, whereby its all solutions are unbounded, i.e. this equation has not almost periodic solutions.

Corollary 1. *Let (q) be a specially disconjugate equation having an almost periodic solution y. Then*

$$y(t) = k.\exp\left(\int_0^t \varphi(s)ds\right), \quad t \in R, \tag{10}$$

where $k \in R$, $\varphi, \varphi' \in S$ *and* $M\{\varphi\} = 0$.

Proof. By Theorem 2 there exists such a function φ that $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$ and the function $y_1(t) := \exp\left(\int_0^t \varphi(s)ds\right)$, $t \in R$, is a solution of (q). Set $y_2(t) := y_1(t) \int_0^t \frac{ds}{y_1^2(s)}$, $t \in R$. Suppose (q) has an almost periodic solution. To prove the corollary it is sufficient to show that $y_2 \notin S$. In the contrary case y_2 is a bounded function and since

$$\int_{-\infty}^0 \frac{ds}{y_1^2(s)} = \infty, \quad \int_0^\infty \frac{ds}{y_1^2(s)} = \infty$$

then $\lim_{t\to\pm\infty} y_1(t) = 0$ and therefore $\lim_{t\to\pm\infty} \varphi(t)y_1(t) = \lim_{t\to\pm\infty} y_1'(t) = 0$. From the other side

$$\lim_{t\to\pm\infty} \frac{1}{y_2(t)} = \lim_{t\to\pm\infty} \frac{\frac{1}{y_1(t)}}{\int_0^t \frac{ds}{y_1^2(s)}} = -\lim_{t\to\pm\infty} y_1'(t) = 0,$$

which contradicts the boundedness of the function y_2.

Corollary 2. *Suppose (q) is a specially disconjugate equation having an almost periodic solution y. Then*

$$|y(t)| \geq m \quad \text{for} \quad t \in R,$$

where m is a positive constant.

Proof. Suppose $y \in S$ is a solution of a specially disconjugate equation (q). By Corollary 1, y may be written as

$$y(t) = k.\exp\left(\int_0^t \varphi(s)ds\right), \quad \text{where } \varphi, \varphi' \in S, \; M\{\varphi\} = 0 \text{ and}$$

$k \in R$. Without any loss of generality we may assume $k = 1$. Let $\inf_{t \in R} |y(t)| = 0$. Then there exists a sequence $\{t_n\}$, $t_n \in R$, such that $\lim_{n \to \infty} y(t_n) = 0$, which yields $\lim_{n \to \infty} y'(t_n) = \lim_{n \to \infty} \varphi(t_n)y(t_n) = 0$. Set $z_n(t) := y(t+t_n)$, $t \in R$, $n = 1, 2, \ldots$. Then z_n is a solution of the equation $y'' = q(t+t_n)y$, $\lim_{n \to \infty} z_n(0) = \lim_{n \to \infty} z_n'(0) = 0$. With the assumption of y, $q \in S$ hence it can be assumed without loss of generality that $\lim_{n \to \infty} z_n(t) = \lim_{n \to \infty} y(t+t_n) = z(t)$, $\lim_{n \to \infty} q(t+t_n) = q^*(t)$ uniformly on R. Then z, $q^* \in S$ and z is the solution of (q^*) satisfying the initial conditions $z(0) = 0$, $z'(0) = 0$. This yields $z(t) \equiv 0$ and then naturally $\lim_{n \to \infty} y(t+t_n) = 0$ uniformly on R. Hence $y(t) \equiv 0$, which is a contradiction. Consequently $\inf_{t \in R} |y(t)| > 0$.

Theorem 3. *Suppose (q) is a specially disconjugate equation. The equation (q) has an almost periodic solution y exactly if y may be written in the form of (10), where* $\int_0^t \varphi(s)ds$, $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$ *and* $k \in R$.

Proof. ($\Longrightarrow$) Letting y be an almost periodic solution of (q) yields that by Corollary 1 we may use the form (10), where $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$ and $k \in R$. It then follows from

Corollary 2 that the function $\int_0^t \varphi(s)ds$ is bounded from below and from the boundedness of y we find that this function is also bounded from above. Hence, the function $\int_0^t \varphi(s)ds$ is bounded and thus also almost periodic.

($\Longleftarrow$) Let there exist such a function φ that $\int_0^t \varphi(s)ds$, $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$ and let y be defined by (10), where $k \in R$, is a solution of (q). It then follows from the properties of almost periodic functions that $y \in S$, i.e. (q) has an almost periodic solution.

Theorem 4. (q) is a specially disconjugate equation having an almost periodic solution exactly if

$$q(t) = \varphi'(t) + \varphi^2(t) \quad \text{for } t \in R, \tag{11}$$

where

$$\int_0^t \varphi(s)ds,\ \varphi,\ \varphi' \in S \quad \text{and} \quad M\{\varphi\} = 0. \tag{12}$$

Proof. ($\Longrightarrow$) Let (q) be a specially disconjugate equation having an almost periodic solution y. By Theorem 2 y may be written in the form (10), where φ satisfies (12). From the equality $q(t) = \dfrac{y''(t)}{y(t)}$ we obtain (11).

($\Longleftarrow$) Let a function φ satisfy (12) and let q be defined by (11). Then $q \in S$ and the function $\exp(\int_0^t \varphi(s)ds) \in S$ be its solution. Then (q) is a specially disconjugate equation as it follows from Theorem 2.

Theorem 5. Suppose (q) is a specially disconjugate equation. If it has an almost periodic solution, then the equation (q^*) has also an almost periodic solution for every $q^* \in H\{q\}$.

Proof. Let (q) have an almost periodic solution y and let $q^* \in H\{q\}$ i.e. $q^*(t) = \lim_{n\to\infty} q(t+h_n)$ uniformly on R, where $\{h_n\}$ is an appropriate sequence of numbers. By Theorem 3, y may be written in the form of (10), where φ satisfies (12) and $k \in R$. Since $y' = \varphi . y$, y' and $y''(=qy)$ are almost periodic functions and passing to a suitable sequence selected from $\{h_n\}$ it my be realized (for brevity, the same notation of indexes is retained) that

$$\lim_{n\to\infty} q(t+h_n) = q^*(t), \qquad \lim_{n\to\infty} y(t+h_n) = y^*(t),$$

$$\lim_{n\to\infty} y'(t+h_n) = s_1(t), \qquad \lim_{n\to\infty} y''(t+h_n) = s_2(t),$$

uniformly on R. Evidently $y^*(t) \not\equiv 0$, $y^* \in S$ and by a limiting process ($n \to \infty$) in the equalities $y(t+h_n) - y(h_n) =$

$$= \int_0^t y'(s+h_n)ds, \quad y'(t+h_n) - y'(h_n) = \int_0^t y''(s+h_n)ds,$$

$y''(t+h_n) = q(t+h_n)y(t+h_n)$ we obtain $s_1 = y^{*\prime}$, $s_2 = y^{*\prime\prime}$, $y^{*\prime\prime} = q . y^*$. Thus y^* is a solution of (q) and this solution is an almost periodic function.

From Theorem 5 immediately follows

Corollary 3. Suppose (q) is a specially disconjugate equation. If (q) has not any almost periodic solution, then likewise (q^*) has not any almost periodic solution for every $q^* \in H\{q\}$.

Theorem 6. (q) is a generally disconjugate equation and a is its characteristic exponent exactly if

$$q(t) = \varphi'(t) + (\varphi(t) - a)^2 \quad \text{for } t \in R, \tag{13}$$

where $\varphi, \varphi' \in S$ *and* $M\{\varphi\} = 0$.

Proof. ($\Longrightarrow$) Let (q) be a generally disconjugate equation and a be its characteristic exponent. By Lemma 3 and Lemma 4 there exists such a solution ψ of (2) that $M\{\psi\} = -a$ and the function $y(t) := \exp(\int_0^t \psi(s)ds)$, $t \in R$, is a solution of (q). Setting $\varphi := \psi + a$, then $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$ and from the equality $q(t) = \frac{y''(t)}{y(t)}$ then $q(t) = \psi'(t) + \psi^2(t) = \varphi'(t) + (\varphi(t) - a)^2$.

($\Longleftarrow$) Let there exist such a function φ that φ, $\varphi' \in S$, $M\{\varphi\} = 0$ and let equality (13) be valid. Then $q \in S$ and the function $\psi := \varphi - a$ is a solution of (2). Since $M\{\psi\} = -a$, it follows from Lemmas 3 and 4 that (q) is generally disconjugate equation and a is its characteristic exponent.

Theorem 7. Let (q) be a disconjugate equation and a be its characteristic exponent. Then the number a is the characteristic exponent of any equation (q^*), $q^* \in H\{q\}$.

Proof. If (q) is a specially disconjugate equation, then the statement of the Theorem immediately follows from Lemma 6.

Suppose (q) is a generally disconjugate equation. By Lemma 4 there exist functions ψ_1, $\psi_2 \in S$, $M\{\psi_1\} = a$, $M\{\psi_2\} = -a$, being solutions of (2). Setting $\varphi_1 := \psi_1 - a$, $\varphi_2 := \psi_2 + a$, then the functions $y_1(t) := e^{at}\exp(\int_0^t \varphi_1(s)ds)$, $y_2(t) := e^{-at}\exp(\int_0^t \varphi_2(s)ds)$, $t \in R$, are independent solutions of (q).

Suppose $q^* \in H\{q\}$. Then there exists such a sequence $\{h_n\}$, $h_n \in R$, that $q^*(t) = \lim_{n \to \infty} q(t+h_n)$ uniformly on R. Passing to a suitable sequence $\{h_{k_n}\}$ selected from the sequence $\{h_n\}$ we arrive at $\lim_{n \to \infty} \psi_i(t+h_{k_n}) = \psi_i^*(t)$ uniformly on R, i=1,2. The functions $\psi_i^* \in S$ are solutions of $u' + u^2 = q^*(t)$, $M\{\psi_1^*\} = a$, $M\{\psi_2^*\} = -a$. Setting $\varphi_1^* := \psi_1^* - a$, $\varphi_2^* := \psi_2^* + a$ yields $M\{\varphi_1^*\} = M\{\varphi_2^*\} = 0$ and the functions $y_1^*(t) :=$

$$:= e^{at}\exp(\int_0^t \varphi_1^*(s)ds),\ y_2^*(t) := e^{-at}\exp(\int_0^t \varphi_2^*(s)ds),\ t \in R,$$

are (independent) solutions of (q^*) - consequently a is the characteristic exponent of (q^*).

<u>Remmark 4.</u> Suppose the coefficient q of the generally disconjugate equation (q) is periodic and a is the characteristic exponent of this equation. From the Floquet theory (see /5/) then follows the existence of such solutions y_1, y_2 of (q) that the functions $e^{at}y_1(t)$, $e^{-at}y_2(t)$ are periodic (hence also bounded). If the coefficient q of (q) is not a periodic function, then the situation is more complicated as may be seen from the example on page 113 /6/, with a concrete form of a generally disconjugate equation whose characteristic exponent is equal to a and for every solution y of this equation the functions $e^{at}y(t)$, $e^{-at}y(t)$ are unbounded (hence, they do not belong to the set S).

Theorem 8. Suppose a is a positive number, (q) is a generally disconjugate equation, a is its characteristic exponent having two independent solutions of the form

$$e^{-at}\phi_1(t),\ e^{at}\phi_2(t),\ \text{where } \phi_1,\ \phi_2 \in S\ , \qquad (14)$$

exactly if equality (13) is valid, where φ satisfies the assumptions of (12).

Proof. ($\Longrightarrow$) Suppose (q) is a generally disconjugate equation, a is the characteristic exponent of (q) and (q) has two independent solutions of the form of (14). It then follows from Lemma 9 that $\phi_i(t) = k_i \exp(\int_0^t \varphi_i(s)ds)$ for $t \in R$, where $\varphi_i,\ \varphi_i' \in S$, $M\{\varphi_i\} = 0$, $k_i \in R$ (i=1,2). Letting ϕ_1 be a solution of (q_1) yields $q_1 = \varphi_1' + \varphi_1^2$, $q_1 \in S$ and it follows from Theorem 2 that (q_1) is a specially disconjugate equation and from Theorem 4 $\int_0^t \varphi_1(s)ds \in S$. Then a calculation shows that $q(t) = q_1(t) + a^2 - 2a\,\varphi_1(t) = \varphi_1'(t) + (\varphi_1(t) - a)^2$, hence (12) and (13) are valid, where we put $\varphi := \varphi_1$.

($\Longleftarrow$) Suppose the function φ satisfies the assumptions of (12) and the coefficient q of (q) is defined by (13). Put $q_1 := \varphi' + \varphi^2$. By Theorem 4 (q_1) is a specially disconjugate equation with an almost periodic solution $\exp(\int_0^t \varphi(s)ds)$. It follows from (13) that the function $\varphi_1 := \varphi - a \in S$ is a solution of (2). Since $M\{\varphi_1\} = -a$, (q) is a generally disconjugate equation and there exists such a solution $\varphi_2 \in S$ of (2) that $M\{\varphi_2\} = a$, which follows from Lemma 3. Consequently a is a characteristic exponent of (q) and the remaining part of the statement follows immediately from Lemma 4.

Theorem 9. Suppose (q) is a specially disconjugate equation. Then there exists such a function φ that $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$ and the function

$$\alpha(t) = \operatorname{arctg} \int_0^t \exp\left(\int_0^s \varphi(\tau)d\tau\right)ds, \quad t \in R, \tag{15}$$

is a phase of (q). And vice versa, if the function α having the form (15), where $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$, is the phase of (p), then $p \in S$ and (p) is a specially disconjugate equation.

Proof. Let (q) be a specially disconjugate equation. By Theorem 2 there exists such a function φ_1 that $\varphi_1, \varphi_1' \in S$, $M\{\varphi_1\} = 0$ and $y_1(t) := \exp\left(\int_0^t \varphi_1(\tau)d\tau\right)$, $t \in R$, is a solution of (q). If we put $y_2(t) := y_1(t)\int_0^t \frac{ds}{y^2(s)}$, $t \in R$, y_2 is a solution of (q) and the function $\alpha(t) = \operatorname{arctg}\frac{y_2(t)}{y_1(t)} =$

$$= \operatorname{arctg}\int_0^t \frac{ds}{y_1^2(s)} = \operatorname{arctg}\int_0^t \exp\left(\int_0^s \varphi(\tau)d\tau\right)ds, \text{ where}$$

$\varphi := -2\varphi_1$, is a phase of (q). Obviously $\varphi, \varphi' \in S$ and $M\{\varphi\} = 0$.

Let the function α be defined by (15), where $\varphi, \varphi' \in S$, $M\{\varphi\} = 0$, is a phase of (p). By Lemma 7 $\alpha(R) = (-\frac{\pi}{2}, \frac{\pi}{2})$, hence (p) is a specially disconjugate equation and it follows from the equalities

$$p(t) = -\{\alpha, t\} - \alpha'^2(t) = -\frac{1}{2}\psi'(t) + \frac{1}{4}\psi^2(t)$$

that $p \in S$.

Theorem 10. Suppose (p) is a generally disconjugate equation and a is its characteristic exponent. Then there exists such a function ψ that $\psi, \psi' \in S$, $M\{\psi\} = 2a$ and the function

$$\alpha(t) = \operatorname{arctg} \int_0^t \exp\left(\int_0^s \psi(\tau)d\tau\right)ds, \quad t \in R, \tag{16}$$

is a phase of (q). And also conversely, if the function α is defined as in (16), where $\psi, \psi' \in S$, $M\{\psi\} = 2a$ ($\neq 0$), is a phase of (p), then $p \in S$, (p) is a generally disconjugate equation and $|a|$ is its characteristic exponent.

Proof. ($\Longrightarrow$) Let (q) be a generally disconjugate equation and a be its characteristic exponent. By Lemma 4 there exists such a function ψ_1, that $\psi_1, \psi_1' \in S$, $M\{\psi_1\} = -a$ and $y_1(t) := \exp\left(\int_0^t \psi_1(s)ds\right)$, $t \in R$, is a solution of (q). Setting

$$y_2(t) := y_1(t)\int_0^t \frac{ds}{y_1^2(s)}, \quad t \in R,$$

yields that y_2 is a solution of (q) and analogous to the first part of the proof of Theorem 9 we also show that the function $\alpha(t) = \operatorname{arctg} \int_0^t \exp\left(\int_0^s \psi(\tau)d\tau\right)ds$, where $\psi := -2\psi_1$ is a phase of (q). Obviously $\psi, \psi' \in S$ and $M\{\psi\} = -2M\{\psi_1\} = 2a$.

($\Longleftarrow$) Let the function α be defined as in (16), where $\psi, \psi' \in S$ and $M\{\psi\} = 2a$ ($\neq 0$) is a phase of (p). By Lemma 8 $\alpha(R) \subsetneq (-\frac{\pi}{2}, \frac{\pi}{2})$, hence (p) is a generally disconjugate equation and it follows from the equalities

$$p(t) = -\{\alpha, t\} - \alpha'^2(t) = -\frac{1}{2}\psi'(t) + \frac{1}{4}\psi^2(t)$$

that $p \in S$. If we put $\psi_1 := -\frac{1}{2}\psi$, then ψ_1 is an almost periodic solution of the equation $u' + u^2 = p(t)$, $|M\{\psi_1\}| = \frac{1}{2}|M\{\psi\}| = |a|$. Thus $|a|$ is a characteristic exponent of (p).

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SOUHRN

O NĚKTERÝCH VLASTNOSTECH ŘEŠENÍ DISKONJUGOVANÉ ROVNICE $y'' = q(t)y$ SE SKOROPERIODICKÝM KOEFICIENTEM q

SVATOSLAV STANĚK

V práci jsou vyšetřovány vlastnosti řešení diskonjugované rovnice

$$y'' = q(t)y, \quad q \in C^0(R), \qquad (q)$$

kde q je skoroperiodická funkce. Každá diskonjugovaná rovnice (q) je buď obecně diskonjugovaná (tzn. existují dvě nezávislá řešení této rovnice, která nemají na R nulový bod) a nebo speciálně diskonjugovaná (tzn. existuje až na multiplikativní konstantu jediné řešení rovnice (q), které nemá na R nulový bod). Je dokázáno, že rovnice (q) je speciálně diskonjugovaná právě když přidružená Riccatiho rovnice $u' + u^2 = q(t)$ má jediné skoroperiodické řešení (věta 1). Dále jsou uvedeny nutné a postačující podmínky, aby rovnice (q) byla buď obecně diskonjugovaná a nebo speciálně diskonjugovaná. Tyto podmínky jsou vyjádřeny buď prostřednictvím jistého tvaru řešení rovnice (q) (věta 2), nebo prostřednictvím jistého tvaru koeficientu q (věta 6) a nebo prostřednictvím fáze rovnice (q) (věta 9 a věta 10). Uvedeny jsou nutné a postačující podmínky, aby existovalo skoroperiodické řešení speciálně diskonjugované rovnice (q) (věta 3 a věta 4). Obdobná úloha pro obecně diskonjugovanou rovnici (q) je vyšetřena ve větě 8.

Nechť $H\{q\}$ je obal generovaný skoroperiodickou funkcí q. Pak speciálně diskonjugovaná rovnice (q) má skoroperiodické řešení právě když mají skoroperiodické řešení všechny rovnice (q^*), kde $q^* \in H\{q\}$ (věta 5) a je-li a charakteristický exponent obecně diskonjugované rovnice (q), pak je a charakteristický exponent každé rovnice (q^*), kde $q^* \in H\{q\}$ (věta 7).

РЕЗЮМЕ

ОБ НЕКОТОРЫХ СВОЙСТВАХ РЕШЕНИЙ УРАВНЕНИЯ $y'' = q(t)\,y$ БЕЗ СОПРЯЖЕННЫХ ТОЧЕК С ПОЧТИ-ПЕРИОДИЧЕСКИМ КОЭФФИЦИЕНТОМ q

СВАТОСЛАВ СТАНЕК

В работе изучаются свойства решений уравнения

$$y'' = q(t)y, \quad q \in C^0 (R), \qquad (q)$$

без сопряженных точек, где q - почти-периодическая функция. Всякое уравнение (q) без сопряженных точек есть или общее уравнение без сопряженных точек (ОУСТ) (существуют две независимые решения этого уравнения, которые не имеют нуля на R) или специальное уравнение без сопряженных точек (СУСТ) (существует до мультипликативной постоянной единственное решение уравнения (q), которое не имеет нуля на R). Доказано, что уравнение (q) есть СУСТ тогда и только тогда, когда уравнение Риккати $u' + u^2 = q(t)$ имеет единственное почти-периодическое решение (теорема 1). Далее приводятся условия, которые необходимые и достаточные для того, чтобы уравнение (q) было или ОУСТ или СУСТ. Эти условия выражаются при помощи или некоторой формы решения уравнения (q) (теорема 2), или некоторой формы коэффициента q (теорема 6), или фазы уравнения (q) (теорема 9 и теорема 10). Приводятся необходимые и достаточные условия существования почти-периодического решения СУСТ (q) (теорема 3 и теорема 4). Аналогичная задача для ОУСТ (q) решена в теореме 8.

Пусть $H\{q\}$ - оболочка порожденная почти-периодической функцией q. Тогда СУСТ (q) имеет почти-периодическое решение тогда и только тогда, когда имеют почти-периодическое решение все уравнения (q^*), $q^* \in H\{q\}$ (теорема 5). Пусть а характеристический мультипликатор ОУСТ (q). Тогда а - характеристический мультипликатор каждого уравнения (q^*), $q^* \in H\{q\}$ (теорема 7).

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