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ASYMPTOTIC FORMULAS FOR SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Preliminaries

Let R^n denote the n-dimensional vector space with a norm $\|\cdot\|$, $R_+ = [0,\infty)$ and $R = (-\infty, \infty)$. Define the operator

$$\sigma(u;t) = \begin{cases} u(t) & \text{for } t \in R_+ \\ 0 & \text{for } t < 0. \end{cases}$$

Let the function $f: R_+ \times R^n \to R^n$ satisfy the Carathéodory local conditions, let the matrix $A: R_+ \to R^n$ be local integrable and let $h: R_+ \to R$ be a continuous function, where $h(t) \leq t$ for $t \in R_+$.

We will consider the following initial problem

$$(1.1) \qquad y' = A(t)y + f(t, \sigma(y; h(t)))$$

(1.2) $y(0) = y_0$, y_0 - a constant vector.

Let $X(t)$ be a fundamental matrix for

(1.3) $x' = A(t)x$

such that $X(0) = I$, I is an identity matrix. Let $X^{-1}(t)$ denote the inverse matrix to $X(t)$ and $X[h(t)]$ denote the value of $X(t)$ at the point $h(t)$ for $t \in R_+$. If $h(t) < 0$, then $X[h(t)] = I$.

Define a function α by

$$\alpha(t) = \| X[h(t)] \| \qquad \text{for } t \in R_+ .$$

The solution of (1.1), (1.2) is understood to be maximally extended to the right.

Symbol $L(R_+)$ will denote the space of Lebesgue integrable functions on R_+ and $L(R_+,K)$ will denote the space of Lebesgue integrable functions on R_+ with the property that the integral of these functions is bunded with some constant $K > 0$.

In this paper we will investigate sufficient conditions for that:

a) all solutions of (1.1), (1.2) exist on R_+ under small initial conditions $\| y_0 \|$ and are of the asymptotic representation

   (1.4) $y(t) = X(t)[c + o(1)]$ as $t \to \infty$,

   where c is a constant vector,

b) the family of solutions of the form (1.4) of (1.1) is stable (in some sense) with regard to small changes of both initial conditions and right-hand side of (1.1),

c) any solution of (1.1), (1.2) has the form (1.4) under arbitrary initial conditions $\|y_0\|$.

Analogous problems were solved in /2, 3, 4/ for functional differential equations and in /1/ and /6/ for systems of ordinary differential equations.

2. Main results

Lemma. Let on the set $R_+ \times R^n$

$$\|X^{-1}(t)f(t,\alpha(t)\varrho)\| \leq \omega(t,\|\varrho\|) \tag{2.1}$$

hold, where $\omega : R_+^2 \to R_+$ is a nondecreasing function in the second argument and

$$\omega(t,s) \in L(R_+) \qquad \text{for } s \in R_+. \tag{2.2}$$

Further, let y(t) be a solution of (1.1), (1.2) defined on R_+ satisfying the inequality

$$\lambda(t) = \|X^{-1}(t)y(t)\| < k \qquad \text{for } t \in R_+. \tag{2.3}$$

where $k > 0$ is a constant.

Then the solution y(t) has the asymptotic form (1.4).

P r o o f. By substitution (see e.g. /4/)

$$y(t) = X(t)z(t) \tag{2.4}$$

every initial problem (1.1), (1.2) transforms into

$$z'(t) = X^{-1}(t)f(t,\sigma[Xz;\, h(t)]) \tag{2.5}$$

(2.6) $$z(0) = y_0 \; .$$

From (2.5), (2.6) it follows with regard to (2.1) and (2.3) that

(2.7) $$\|z'(t)\| = \|X^{-1}(t)f(t, \sigma[Xz;\ h(t)])\| \leq$$

$$\leq \omega(t, \alpha^{-1}(t)\,\sigma[Xz;\ h(t)]) \leq$$

$$\leq \omega(t, \sigma(\lambda;\ h(t))) \leq \omega(t,k) \; ,$$

hence we get that $\|z'(t)\| \in L(R_+)$.

According to integrability of $z'(t)$ on R_+ we obtain

(2.8) $$z(t) = y_0 + \int_0^t z'(s)ds \; .$$

If we choose

$$c = y_0 + \int_0^{\infty} z'(t)dt \; ,$$

then we have from (2.8) and (2.4) that $y(t)$ is of form (1.4). Thus the lemma is proved.

Theorem 2.1. Let (2.1. hold on $R_+ \times R^n$, where $\omega\ R_+^2 \to R_+$ is a nondecreasing function in the second argument and integrable on R_+ in the first argument. Further let there exist such a number $k > 0$ that

(2.9) $$\omega(t,k) \in L(R_+,\ k - \|y_0\|) \; .$$

Then every solution $y(t)$ of (1.1), (1.2) exists on R_+ and is of the asymptotic form (1.4).

P r o o f. Let $y(t)$ be an arbitrary solution of (1.1), (1.2). We show that this solution exists on R_+ and estimate (2.3) holds for it. Suppose that (2.3) does not hold. Then there exists such $t_o > 0$ that

$$\lambda(t) < k \qquad \text{for } 0 \leq t < t_o \,, \tag{2.10}$$

and

$$\lambda(t_o) = k \,. \tag{2.11}$$

With regard to (2.1) and (2.10) for $t \in [0, t_o)$ (2.7) holds. Therefore we have from (2.8) and (2.9)

$$\lambda(t_o) = \| X^{-1}(t_o)y(t_o) \| = \| y_o + \int_0^{t_o} z'(s)ds \| \leq \|y_o\| + \int_0^{t_o} \|z'(s)\| ds \leq$$

$$\leq \| y_o \| + \int_0^{\infty} \omega(t,k)dt \leq \| y_o \| + k - \| y_o \| = k \,.$$

This is a contradiction to (2.11). Therefore the solution $y(t)$ is defined on R_+ and estimate (2.3) holds for it. The fact that $y(t)$ is of the asymptotic form (1.4) follows from the above lemma.

It is easy to see that the problem (1.1), (1.2) can besides solutions of the form (1.4) have even solutions which are of another asymptotic form. Therefore it is worth asking a question on stability of family of solutions of form (1.4) with regard to small changes of initial conditions and of the right-hand side of the equation. The answer is involved in the following theorem.

Consider together with (1.1), (1.2) even initial problem

$$(2.12) \qquad u' = A(t)u + \tilde{f}(t, \mathcal{T}[u; h(t)])$$

$$(2.13) \qquad u(0) = u_0 \ ,$$

where A: $R_+ \to R^n$ is the matrix defined above and $\tilde{f}: R_+ x R^n \to R^n$ fulfils locally Carathéodory condition.

T h e o r e m 2.2. Let (2.1) hold on the set $R_+ x R^n$ and

$$(2.14) \qquad \| X^{-1}(t)[\tilde{f}(t, \varrho) - f(t, \varrho)]\| \leq g(t),$$

where $\omega : R_+^2 \to R_+$ is a nondecreasing function in the second argument fulfilling (2.2), g: $R_+ \to R_+$ is an integrable function. Further let (1.1), (1.2) have the unique solution y(t) defined on R_+ possessing the asymptotic form (1.4).

Then there exists such a $\delta > 0$ that if

$$(2.15) \qquad \| y_0 - u_0 \| \leq \delta \quad \text{and} \quad g \in L(R_+, \delta) \ ,$$

then every solution u(t) of (2.12), (2.13) exists on R_+ satisfying

$$(2.16) \qquad u(t) = X(t)[\tilde{c} + o(1)] \qquad \text{as } t \to \infty \ .$$

P r o o f. It follows from (1.4) that there exists such a sufficiently large number $k > 0$ that (2.3) holds. As (2.2) is true, we can choose such a sufficiently large t_0 that

$$k_1 = k + \int_{t_0}^{\infty} \omega(t, 3k)dt < 3k \ .$$

With regard to the uniqueness of the solution of (1.1), (1.2) there exists $\delta > 0$ such that if (2.15) holds then the existence interval of the solution u(t) of (2.12), (2.13) contains the interval $[0, t_o]$ and

$$(2.17) \qquad \| X^{-1}(t)u(t) \| < k \quad \text{holds for} \quad 0 \leq t \leq t_o \ .$$

Without loss of generality we can put

$$(2.18) \qquad k_1 + \delta < 3k \ .$$

Let u(t) be an arbitrary solution of (2.12), (2.13). We show that this solution exists on R_+ and satisfies the inequality

$$(2.19) \qquad \tilde{\lambda}(t) = \|X^{-1}(t)u(t)\| < 3k \qquad \text{for } t \in R_+ \ .$$

Suppose that (2.19) does not hold. Then there exists such $t_1 > t_o$ that

$$(2.20) \qquad \tilde{\lambda}(t) < 3k \ , \qquad t_o \leq t < t_1$$

and

$$(2.21) \qquad \tilde{\lambda}(t_1) = 3k \ .$$

Let u(t) = X(t)v(t). Then we have from (2.1), (2.15) and (2.20) that

$$\|v'(t)\| \leq \|X^{-1}(t)[\tilde{f}(t,\varrho) - f(t,\varrho)]\| + \|X^{-1}(t)f(t,\varrho)\| \leq g(t) +$$

$$+ \ \omega(t,3k)$$

for $t_o \leq t \leq t_1$.

From the last inequality according to (2.15), (2.17) and (2.18) we obtain

$$\tilde{\lambda}(t_1) \leq \tilde{\lambda}(t_o) + \int_{t_o}^{t_1} \| v'(t) \| dt \leq \tilde{\lambda}(t_o) + \int_{t_o}^{\infty} [g(t) + \omega(t,3k)] dt \leq$$

$$k + \delta + \int_{t_o}^{\infty} \omega(t,3k)dt \leq k_1 + \delta < 3k .$$

This is a contradiction to (2.21). Thus it is proved that the solution u(t) exists on R_+ and the estimate (2.19) holds for it.

The fact that u(t) has the asymptotic form (2.16) follows from the lemma and the theorem is proved.

T h e o r e m 2.3. Let on $R_+ x R^n$ inequality (2.1) hold, where $\omega : R_+^2 \to R_+$ is an integrable function in the first argument on R_+, continuous and nondecreasing in the second argument. Moreover, let all solutions of

(2.22) $$r'(t) = \omega(t, r(t))$$

be bounded on R_+. Then for an arbitrary constant vector y_o every solution of (1.1), (1.2) exists on R_+ and has the asymptotic form (1.4).

P r o o f. Let $r^*(t)$ be upper solution of (2.22) under the initial condition

$$r(0) = r_o = \| y_o \| .$$

Then

(2.23) $$k = \sup_{t \in R_+} r^*(t) < \infty .$$

Let y(t) be an arbitrary solution of (1.1), (1.2) defined on $[0, t^*)$ and let

$$\lambda^*(t) = \max_{0 \le s \le t} \| X^{-1}(s) y(s) \| \qquad \text{for} \quad t \in [0, t^*) .$$

Then with regard to (2.1) and (2.4)

$$\lambda^*(t) \le r_0 + \int_0^t \omega(s, \lambda^*(s)) ds \qquad \text{for} \quad t \in [0, t^*)$$

holds.

From the last inequality it follows (see e.g. [5] lemma 4.4) that

(2.24) $$\lambda^*(t) \le r^*(t) \qquad \text{for} \qquad t \in [0, t^*) .$$

From (2.23) and (2.24) we get $t^* = +\infty$ and

(2.25) $$\lambda^*(t) \le k \qquad \text{for } t \in R_+ ,$$

which proves that y(t) exists on R_+. The fact that y(t) is of the asymptotic form (1.4) follows from the lemma. Our theorem is thus proved.

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SHRNUTÍ

ASYMPTOTICKÉ VZORCE PRE RIEŠENIA FUNKCIONÁLNYCH DIFERENCIÁLNYCH ROVNÍC

JÁN FUTÁK

V práci sú uvedené postačujúce podmienky pre to, aby:

a) každé riešenie úlohy (1.1), (1.2) pri malých začiatočných podmienkach $\|y_0\|$ existovalo na R_+ a malo asymptotický tvar

(1.4) $y(t) = X(t)[c + o(1)]$ ak $t \to \infty$,

kde c je konštantný vektor,

b) množina riešení tvaru (1.4) rovnice (1.1) bola v istom zmysle stabilná vzhľadom na malé zmeny začiatočných hodnôt i pravej strany rovnice,

c) každé riešenie začiatočnej úlohy (1.1), (1.2) pri ľubovoľných začiatočných podmienkach $\|y_0\|$ malo tvar (1.4).

РЕЗЮМЕ

АСИМПТОТИЧЕСКИЕ ФОРМУЛЫ ДЛЯ РЕШЕНИЙ ФУНКЦИОНАЛЬНО ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

ЯН ФУТАК

В статье приведены достаточные условия для того, чтобы:

а) произвольное решение задачи (1.1), (1,2) при малых начальных данных // y_0 // существовало на R_+ и допускало представление

(1, 4) $\qquad y(t) = X(t)\ [c + o(1)] \qquad t \to \infty$,

где c - константный вектор,

б) семейство решений вида (1, 4) уравнения (1, 1) было в некотором смысле устойчиво относительно малых возмущений начальных данных и правой части уравнения,

в) каждое решение задачи (1, 1), (1, 2) при любых //y_0// имело вид (1, 4).

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