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ON GENERALIZED FORMAL POWER SERIES

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In the present paper the usual construction of formal power series is carried over to the case where "exponents" form an ordered loop and "coefficients" form a structure T with two binary operations + , . such that (T,+) is a commutative group with a neutral element 0, $(T-\{0\},.)$ is a quasigroup, and a.0=0.a=0 holds for all $a \in T$. Especially, if the set of coefficients is a commutative Cartesian group, then formal power series also form a commutative Cartesian group. Some of our proofs can be understood as a modern form of the classical proofs given by H. H a h n in /4/ for power series with real coefficients and exponents forming an ordered commutative group. We use transfinite induction in contemporary version given in /2/, p.243.

Power series with exponents in an ordered loop and coefficients in a non-associative ring have been investigated in /3/ by D. Z e l i n s k i.

A loop L is said to be ordered under a linear order $\preccurlyeq$ if the set L is linearly ordered under $\preccurlyeq$ so that $a \preccurlyeq b$ implies $c+a \preccurlyeq c+b$ and $a+c \preccurlyeq b+c$ for all a,b,c in L. Each ordered loop is necessarily infinite. Some types of ordered loops are mentioned in /3/ and /4/.

Let us investigate an ordered loop $(L,+,\preccurlyeq)$ with a neutral element e together with a set T admitting two binary operations +,. such that

(i) (T,+) is an Abelian group with a neutral element 0,
(ii) $(T-\{0\},.)$ is a quasigroup,
(iii) $a.0=0.a=0$ for all $a \in T$.

Denote by D(T,L) the set of all functions from L to T, $f \in T^L$, such that the support $spt(f)=\{x \in L \mid f(x)\neq 0\}$ is well-ordered by the starting order $\preccurlyeq$, that is, each non-empty subset of spt(f) has a smallest element. Obviously $D(T,L) \neq \emptyset$ and the elements of D(T,L) can be interpreted as generalized formal Laurent series with exponents in L and coefficients in T.

Define addition and multiplication on D(T,L) by

$$(f+g)(x) = f(x) + g(x) \text{ for all } x \in L,$$

$$(f.g)(x) = \sum_{y+z=x} f(y).g(z) \text{ for all } x \in L.$$

It must be verified that these operations are well-defined. First, the set spt(f+g) is well-ordered by $\preccurlyeq$ because spt(f) and spt(g) are well-ordered under $\preccurlyeq$, and spt(f+g) is a subset in the ordered set-union $spt(f) \cup spt(g)$. The summation in $\sum_{y+z=x} f(y).g(z)$ is meaningful because there is only a finite number of non-zero terms under the summation sign. In fact, suppose an infinite number of such terms. Then it can be found a countable set $\{(y_j,z_j) \mid j \in \mathbb{N}\}$, $\mathbb{N} = \{1,2,3,\dots\}$, such that $y_j \in spt(f), z_j \in spt(g)$ and $y_j+z_j=x$. Since $\{y_j \mid j \in \mathbb{N}\}$ is an ordered subset in spt(f) under $\preccurlyeq$, we can suppose without loss of generality that $y_j \preccurlyeq y_k$ *) whenever $j < k$. Now $j < k$ implies $z_k \prec z_j$, contrary to the fact that the set $\{z_j \mid j \in \mathbb{N}\}$ is well-ordered under $\preccurlyeq$. Further, spt(f.g) is well-ordered under $\preccurlyeq$. For if spt(f.g) would not be well-ordered by $\preccurlyeq$, it had to contain a non-empty subset without a smallest element, and it could be found a countable set of couples (y_j,z_j) such that $\{y_j \mid j \in \mathbb{N}\} \subseteq spt(f)$, $\{z_j \mid j \in \mathbb{N}\} \subseteq spt(g)$ and $y_k+z_k \prec y_j+z_j$ whenever $j < k$. Let y_{n_1} denotes the smallest element in $\{y_j \mid j \in \mathbb{N}\}$, y_{n_2} the smallest element in $\{y_j \mid j > n_1\}$ etc. For any $j < k$, we have inequalities $y_{n_k} + z_{n_k} \prec y_{n_j} + z_{n_j}$, $y_{n_j} \prec y_{n_k}$, which implies $z_{n_k} \prec z_{n_j}$, a contradiction.

In can be easily seen that (D(T,L),+) is a commutative group with the neutral element o determined by spt(o) = $\emptyset$. From now on we shall use the notation $D(T,L)^* = D(T,L) - \{o\}$.

Lemma. *For all $f,g \in D(T,L)^*$ it is valid the equation*

min spt(f.g)=min spt(f)+ min spt(g).

Proof. *Let c=min spt(f.g), a=min spt(f), b=min spt(g).* Now

$(f.g)(c) = \sum_{\xi + \eta = c} f(\xi).g(\eta)$, $\xi \in spt(f)$, $\eta \in spt(g)$. Thus

$a \preccurlyeq \xi$, $b \preccurlyeq \eta$, and $c \preccurlyeq a+b$. Assume $\tilde{a}+\tilde{b}=a+b$ for some $\tilde{a},\tilde{b} \in L$. If $\tilde{a} \prec a$, then $f(\tilde{a})=0$. The other inequality $a \prec \tilde{a}$ implies $\tilde{b} \prec b$ and consequently $g(\tilde{b})=0$. Therefore $\tilde{a}=a, \tilde{b}=b$, and it must be $(f.g)(a+b)=f(a).g(b) \neq 0$. We get $a+b \in spt(f.g)$, so that $c \preccurlyeq a+b$. Thus $c=a+b$.

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*) If $a \preccurlyeq b$ and $a \neq b$ we shall write $a \prec b$.

Proposition. Let L be an ordered loop and (T,+,.) a structure satisfying (i)-(iii). Then $(D(T,L)^*,.)$ is a quasigroup.

For the proof we shall use the theorem on definitions by transfinite induction in the following form (see /2/, p.243): Let S be a set, α a given ordinal, ϕ a set of all S-valued ξ -sequences[1] for all $\xi < \alpha$, and H: $\phi \longrightarrow S$ a given map. Then there is a unique S-valued (α +1)-sequence U such that

(1) $\quad U(\iota) = H(U|_{W(\iota)})$ for all $\iota \leq \alpha$.

P r o o f. We shall show that equation

(2) $\quad f.h=g$

has a unique solution $h \in D(T,L)^*$ for any given $f,g \in D(T,L)^*$. (The case h.f=g is similar.)

Let $S=L \times T$. Assume α is the least ordinal not corresponding to any well-ordered subset in L. More detailed, if A is the set of all well-ordered subsets of $(L,+,\leq)$ and $\tilde{A}$ the set of their ordinals, then $\tilde{A}$ is equal to $W(\alpha)$ for the above ordinal α . Let $\lambda : L \times L \to L$ be a map given by $\lambda(u,v)=w \Longleftrightarrow u+w=v$ and $\tau : T \times T \to T$ a map given by $\tau(a,b)=c \Longleftrightarrow a.c=b$. Further, let the map G $: \bar{\phi} \longrightarrow T^L$ be given as follows. For every ξ -sequence $P=((a_\iota ,b_\iota))_{\iota<\xi}$ of $\bar{\phi}$ with $\xi \neq 0, a_\iota \in L, b_\iota \in T$ and

(3) $\quad a_{\iota_1} \neq a_{\iota_2}$ whenever $\iota_1 \neq \iota_2$ (i.e., $(a_\iota)_{\iota<\xi}$ injective)

let

$$G(P)(a_\iota)=b_\iota \quad \text{for all } \iota \leq \xi ,$$

$$G(P)(u)=0 \quad \text{for all } u \in L - \{ a_\iota \mid \iota < \xi \},$$

whereas set G(P)=o for all remaining P. Now define our map H

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1) If ξ is an ordinal, then $W(\xi)$ denotes the set of all ordinals less than ξ . If moreover S is a non-empty set, then an S-valued ξ -sequence is a map from $W(\xi)$ to S.

by $H(P)=(\rho(\min spt(f), \min spt(g-f.G(P))$,

$\tau(f(\min spt(f), (g-f.G(P))(\min spt(g-f.G(P))))$

for all $P\in\Phi$ with $g-f.G(P)\neq o$,

and

$H(P)=(e,o)$ for all $P\in\Phi$ with $g-f.G(P)=o$ (recall that e is a neutral element of $(L,+)$). By the Theorem on definitions by transfinite induction, there exists a unique $(\alpha+1)$-sequence $U=((z_\iota, r_\iota))_{\iota<\alpha+1}$ satisfying (1), i.e., $U(\iota)=(z_\iota, r_\iota)= =H((z_\eta, r_\eta)_{\eta<\iota})$. Let us denote $y=\min spt(f)$, $U_\iota=U|W(\iota)$, $h_\iota=G(U_\iota)$, $g_\iota=g-f.h_\iota$ for all $\iota<\alpha+1$. Let $x_\iota=\min spt(g_\iota)$ whenever $g_\iota\neq o$, $\iota<\alpha+1$. Clearly, if x_ι is defined, then $f(y).r_\iota=g_\iota(x_\iota)$ and

(4) $y+z_\iota=x_\iota$.

Furthermore, h_ι : $\begin{cases} h_\iota(z_\eta)=r_\eta \text{ for } \eta<\iota, \\ h_\iota(z)=0 \text{ otherwise.}\end{cases}$

The proof continues in five steps.

S t e p 1. (Auxiliary statements). Let

(5) $\beta\in W(\alpha+1)$ with $U(\iota)\neq(e,o)$ for all $\iota\in W(\beta+1)$.

Then

(6) $g_\iota(x)=0$ for all $x<x_\iota$ and $\iota<\beta+1$,

(7) if $\eta<\iota<\beta+1$ and $x\prec x_\eta$ then $g_\iota(x)=0$,

(8) $g_\iota(x_\eta)=0$ for $\eta<\iota<\beta+1$.

(6) is trivial. By properties (4) and (6),

$$g_\iota(x)=g(x)-(f.h_\iota)(x)=g(x)-\sum_{j=1}^{n(x)} f.(y_{\xi_j}).h_\iota(z_{\xi_j}),$$

where

$y_{\xi_j}+z_{\xi_j}=x$, $y_{\xi_j}\in spt(f)$, and $z_{\xi_j}\in spt(h_\iota)$.

Since $y \prec y_{\xi_j}$ and $y_{\xi_j} + z_{\xi_j} \prec x_\eta = y + z_\eta$, it must be $z_{\xi_j} \prec z_\eta$. Therefore $g_\iota(x) = g_\eta(x) = 0$, which proves (7). Similar considerations lead to (8).

S t e p 2. For an ordinal β satisfying (5) the sequence $(z_\iota)_{\iota < \beta+1}$ is strongly increasing. Because of (4) it suffices to show that $(x_\iota)_{\iota < \beta+1}$ is strongly increasing. To prove it we use transfinite induction: Let β satisfies (5). Set $X = \{\iota \mid \iota \in W(\beta+1)$ and $(\xi < \iota \Rightarrow x_\xi \prec x_\iota)\}$. Assume $\iota < \beta+1$ such that $W(\iota) \subseteq X$. We must show that $\iota \in X$. For this, suppose there exists $\xi < \iota$ such that $x_\iota \preceq x_\xi$. By (7) this assumption implies $g_\iota(x_\iota) = g_\xi(x_\iota) = 0$, which contradicts (6). Hence $x_\xi \not\prec x_\iota$. By (8), $g_\iota(x_\xi) = 0$. But $g_\iota(x_\iota) \neq 0$, thus the equality cannot occur. Therefore $\iota \in X$, indeed. That is, $X \subseteq W(\beta+1)$. Since the other inclusion is trivial, the assumption $\xi < \iota < \beta+1$ implies $x_\xi \prec x_\iota$ as required.

S t e p 3. There exists $\xi \in W(\alpha+1)$ with $U(\xi) = (e,o)$. In fact, suppose $U(\iota) \neq (e,o)$ for all $\iota < \alpha+1$. Then we can express U in the form $U = ((z_\iota, r_\iota))_{\iota < \alpha+1}$, where $(z_\iota)_{\iota < \alpha+1}$ is by step 2 a strictly increasing L-valued $(\alpha+1)$-sequence. But $\{z_\iota \mid \iota < \alpha+1\} \in A$ is a well-ordered subset in L with the ordinal α, a contradiction.

S t e p 4. (Existence of solution) Let $\xi_0 = \min\{\xi \mid U(\xi) = (e,o)\}$. Then $h = h_{\xi_0} = G(U_{\xi_0})$ is a solution of (2). It is clear that $h \in D(T,L)^*$. Since $U(\xi_0) = (e,o)$, the equality g-f.h=o holds.

S t e p 5. (Uniqueness). The proof of unicity is based on unique solvability of all equations (4) and uses transfinite induction again. QED.

Remarks. Many algebraic properties of the original structure (T,+,.) are preserved by passing over to (D(T,L),+,.). If T has associative or commutative multiplication, D(T,L) has the same property. Also distributive laws in (T,+,.) are preserved in D(T,L). If T has an element 1 such that x.1 = = 1.x=x for all $x \in T$, then D(T,L) contains an element j such that j(e)=1, j(x)=0 for all $x \neq e, x \in L$. If T is a ring without divisors of zero, then D(T,L) is also a ring without divisors of zero. If T is a division ring, D(T,L) is a division ring, too. If T is a commutative Cartesian group, then so also is D(T,L). These facts permit a construction of some non-desarguesian projective planes yielding convenient homomorphisms of projective planes.

REFERENCES

/1/ H a h n, H.: Über die nichtarchimedischen Grössensysteme, Sitzungsberichte der Österreichischen Akademie, Wien, 116, (1907), 601-655.

/2/ K u r a t o w s k i, K. - M o s t o w s k i, A.: Set theory, Amsterdam-Warszawa 1967.

/3/ Z e l i n s k i, D.: Non-associative valuations, Bull. Amer.Math.Soc., 54 (1948), 175-183.

/4/ Z e l i n s k i, D.: On ordered loops, Amer.Journal Math., 70 (1948), 681-697.

SHRNUTÍ

ZOBECNĚNÉ FORMÁLNÍ MOCNINNÉ ŘADY

Alena VANŽUROVÁ

Článek vychází z klasické práce H.Hahna (z r.1907) a z článků D.Zelinského (z r.1948). Ukazuje se, že obvyklá konstrukce zobecněných formálních (Laurentových) řad dává užitečný výsledek i v případě, že exponenty jsou prvky z uspořádané lupy a koeficienty tvoří strukturu se dvěma binárními operacemi (sčítání a násobení), přičemž sčítání tvoří abelovskou grupu s neutrálním prvkem 0, násobení tvoří na nenulových prvcích kvazigrupu a součin prvku 0 s libovolným prvkem je opět 0. Zvolíme-li koeficienty v komutativní kartézské grupě, tvoří Laurentovy řady opět komutativní kartézskou grupu. Této skutečnosti lze využít ke konstrukci příkladů netriviálních homomorfismů velmi obecných projektivních rovin.

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РЕЗЮМЕ

ФОРМАЛЬНЫЕ СТЕПЕННЫЕ РЯДЫ

АЛЕНА ВАНЖУРОВА

Статья исходит из работ Хана и Зелинского. Показывается что привычная конструкция степенных рядов дает полезный результат даже в случае когда экспоненты принадлежат упорядоченной лупе и коэффициенты составляют структуру (T, +,.) выполняющую следующие условия: (T, +)-коммутативная группа с неутральным элементом 0,a. = 0.a = 0 для всех $a \in T$ и все ненулевые элементы образуют квазигруппу. Если исходить из коммутативной картезианской группы (T, +,.), то степенные ряды образуют опять коммутативную картезианскую группу. Тот факт можно применить к конструкции примеров гомоморфизмов проективных плоскостей.