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SQUARE ROOTS OF ELEMENTS WITH AN UNBOUNDED SPECTRUM

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Abstract

Let A be a complete locally multiplicatively convex algebra with a unit element e whose topology is given by the family of submultiplicatively seminorms $\{p_\alpha\}_{\alpha\in\Sigma}$ separating points in A. We study the existence of square roots of elements in A. In this paper we will show that for each element of A which posseses a positive (not necessarily bounded) spectrum there exists a square root possesing the positive spectrum, too. Moreover, the spectral radius of the square root s fullfils the condition $|s|_\sigma \leqq \sqrt{|a|_\sigma}$. If, in adition, the algebra A is endowed with the involution and a is selfadjoint, the square root is selfadjoint, too.

1. Notations and preliminaries

All linear spaces are over the complex field C. The reader is assumed to be familiar with the basic concepts concerning the topological algebras, namely the Banach algebras, Banach star algebras, locally multiplicatively convex (lmc) star algebras, including spectra, Gelfand representation for the commutative case and so on. See Bonsall F., Duncan J. (1973), Michael (1952), Najmark (1968), Želazko (1972). Let us now recall some notations and preliminary facts. Let A be a lmc algebra. Without any loss of generality we can assume the topology being given by a family $\{p_\alpha\}_{\alpha\in\Sigma}$ of submultiplicative seminorms on A separating points in A for which Σ is directed in a natural way by the relation $\alpha < \beta$ if and only if p_α is continuous with respect to p_β. It is well known and a widely utilized fact that A is topologically isomorphic to the projective limit of Banach algebras A_α, $\alpha \in \Sigma$, where A_α denotes the completion of the normed algebra $(A/Ker\, p_\alpha, p_\alpha)$. Speaking more closely $\Pi(A) = \varprojlim A_\alpha$, where m denotes the natural isomorphism mapping from A into $\prod_{\alpha\in\Sigma} A_\alpha$, $\Pi(x) = (\pi_\alpha, (x))_{\alpha\in\Sigma}$ and π_α denotes the natural homomorphism mapping from A onto A_α. Hereby the system of Banach algebras

$(A_\alpha, \alpha \in \Sigma)$ forms a projective system with respect to the set of continuous homomorphism mappings $\pi_{\alpha\beta}, : A_\beta \to A_\alpha, \pi_{\alpha\beta}, (\pi_\beta, (x)) = \pi_\alpha(x)$ for each $x \in A$ whenever $\alpha < \beta$. Throughout, the spectrum of an element $x \in A$ will be denoted by $\sigma(x, A)$ pointing out that it is taken with respect to algebra A. The spectral radius will be denoted by $| x |_\sigma^A$. It is obvious that an element $x \in A$ is regular if and only if for each $\alpha \in \Sigma$ $\pi_\alpha(x)$ is regular in the algebra A_α which yields the equality $\sigma(x, A) = \bigcup_{\alpha \in \Sigma} \sigma(\pi_\alpha(x), A_\alpha)$. For the spectral radius then $| x |_\sigma^A = \sup | \pi_\alpha(x) |_\sigma^A$. Let us recall that the spectrum of an element in the Banach algebra is always a compact set of C. Our situation is more general: the spectrum of an element in the lmc algebra need not be necessarily bounded.

1.1. Theorem: Let A be a complete lmc-algebra with a unit element e. Let $N \subset A$ be the set of pairwise commuting elements. Then N is contained in a maximal closed and commutative subalgebra $B \subset A$. Moreover, for each $x \in B$ holds $\sigma(x, B) = \sigma(x, A)$.

Proof: This Theorem may be proved analogous to the proof of the existence of the maximal closed and normal subalgebra containing a given normal set in Banach star algebra, see Pták (1970). It is easy to see that the union of a chain of subsets of pairwise commuting elements in A ordered by the set inclusion $\subseteq$ is also commutative. So, by Zorn's lemma, there exists a maximal commutative subset $B \subset A$ containing the given set N. Obviously, B is a closed subalgebra of A and $e \in B$. If $yB = By$ holds for any $y \in A$ then by the maximality of B $y \in B$. So B is a maximal closed commutative subalgebra containing the given N. Further, if $x \in B$ is a regular element of A, so also is the inverse x^{-1} and the latter belongs to B. By the definition of spectrum this easily yields $\sigma(B, x) = \sigma(A, x)$ for arbitrary $x \in B$.

2. Square roots

All algebras in this section are supposed to be complete lmc algebras with a unit element e.

2.1. Theorem: The following holds for each $a \in A$ such that $\sigma(a) > 0$: Let a positive K be given such that $| a |_\sigma^A \leqq K^{1/2}$. Then there exists a unique square root $s \in A$ of a such that its spectrum is positive and $| s |_\sigma^A \leqq K^{1/2}$, s commuting with a. If, moreover, A is endowed with an involution and a is selfadjoint, so also is s.

Proof: See Štěrbová (1980).

2.2. Theorem: The following holds for each $a \in A$ such that $\sigma(a) > 0$: There exists a square root $s \in A$ of the element a such that $\sigma(s) > 0$ and $| s |_\sigma^{A_\alpha} \leqq \sqrt{| a |_\sigma^{A_\alpha}}$ for each $\alpha \in \Sigma$. If, moreover, A is endowed with an involution and a is selfjoint, so also is s.

Proof: Since $\sigma(a) > 0$, the element $(e + a)$ is regular in the algebra A, yielding

$$(e + a)(e + a)^{-1} = (e + a)^{-1}(e + a) = e = e(e + a)^{-1} + a(e + a)^{-1}.$$

We apply the Gelfand representation theory to easily demonstrate

$$0 < \sigma((e + a)^{-1}) < 1, \qquad 0 < \sigma(a(e + a)^{-1}) < 1$$

and so by the preceding theorem there exists the unique square root $h \in A$, regula and positive of the element $u = (e + a)^{-1}$, so that $| h |_{\sigma}^{A_\alpha} \leqq \sqrt{| u |_{\sigma}^{A_\alpha}}$ for each $\alpha \in \Sigma$. Analogous there exists the unique positive and regular square root $k \in \dot{A}$ of the element $v = a(e + a)^{-1}$, so that $| k |_{\sigma}^{A_\alpha} \leqq (| v |_{\sigma}^{A_\alpha})^{1/2}$ for each index $\alpha \in \Sigma$. Thus for arbitrary $\alpha \in \Sigma$ the elements $\pi_\alpha(h)$, $\pi_\alpha(k)$ commute since h, k are contained in the maximal commutative subalgebra of A involving the set $\{a, e\}$. Thus so doe h, k. Further it holds

$$h^2 = (e + a)^{-1}, \qquad k^2 = a(e + a)^{-1}$$

and

$$h^2 + k^2 = u + v,$$

whence

$$h^2 = e - k^2.$$

Making use of the fact that $(e + a) = (h^{-1})^2$ then a simple computation gives

$$a = (h^{-1})^2 - e = (h^{-1})^2 (e - h^2) = (h^{-1})^2 k^2 = (h^{-1}k)^2.$$

We stated that $s = h^{-1}k$ is the square root of the element a required. If, moreover, the algebra A is endowed with an involution with respect to which the element a is selfadjoint, then by the preceding theorem we state that h, k are selfadjoint, too. Since h, k commute, $s = h^{-1}k$ is selfadjoint as well. The rest of the proof is now evident. Q.E.D.

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Souhrn

ODMOCNINY PRVKŮ ALGEBRY S NEOMEZENÝM SPEKTREM

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V práci se dokazuje existence odmocniny prvků s kladným, ne nutně omezeným spektrem v úplné lokálně multiplikativně konvexní algebře. Když je navíc na této algebře definovaná involuce (spojitost, která se nepředpokládá), vůči které je daný prvek samoadjungovaný, je dokázáno, že odmocnina je taktéž samoadjungovaná.

Резюме

О КВАДРАТНЫХ КОРНЯХ ЭЛЕМЕНТОВ СПЕКТРЫ КОТОРЫЕ НЕОГРАНИЧЕНЫ

Д. ШТЕРБОВА

В настоящей статье показывается существование квадратных корней элементов польных полунормированных колец, спектры которых положительны а ограниченность которых не предполагается. Если в кольце определена инволюция, непрерывность которой тоже не предполагается квадратные корни симметрических элементов с положительным спектром также симметрические.