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INCREASE OF ACCURACY IN SOLVING DIFFERENTIAL EQUATIONS WITH SINGULARITIES OF THE TYPE $\frac{0}{0}$ BY A RIGHT CHOICE OF AN ADDITIONAL TERM

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In solving differential equations with singularities of the type $\frac{0}{0}$ on electronic analog computers using a diode (quadratic) multilpier there arises a certain inaccuracy when modelling the expression $z(t) = \frac{f(t)}{g(t)}$ at small values of the numerator and the donominator, that is if $\lim_{t \to 0} f(t) = 0$, $\lim_{t \to 0} g(t) = 0$. This inaccuracy of this multiplier, for it forms the product on the basis of the relation $uv = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2$, where the quadratic depences are approximated by linear depences. Indefinite expressions of the type $z(t) = \frac{f(t)}{g(t)}$, where $\lim_{t \to 0} f(t) = \lim_{t \to 0} g(t) = 0$ for $t \to 0$ are modelled in the form

$$z(t) \doteq \frac{f(t) + az(o)\, e^{-\alpha t}}{g(t) + ae^{-\alpha t}}, \tag{a1}$$

where $z(0) = \lim_{t \to 0} \frac{f(t)}{g(t)}$. Expression $az(0)\, e^{-\alpha t}$ and $ae^{-\alpha t}$ generally being a certain error into the modelled quotient $z(t) = \frac{f(t)}{g(t)}$. Here, two requirements on the magnitude of the coefficient a encounter: if the value of a is small, then the accuracy of approximation of the expression $\frac{f(t)}{g(t)}$ by the expression $\frac{f(t) + az(o)\,e^{-\alpha t}}{g(t) + ae^{-\alpha t}}$ increases, however, if the values of the numerator and the denominator are small, then the inaccuracy in modelling the given quotint may increase and this due to the inaccuracy of the diode multiplier (Cf. [1] and [2]). The task is to determine the optimal value of a for some cases of $f(t)$ and $g(t)$.

Theoretical part

Let us assume the functions $f(t)$ and $g(t)$ in the form of a power series, i.e.

$$f(t) = a_\nu t^\nu + a_{\nu+1} t^{\nu+1} + \ldots + a_{\nu+n} t^{\nu+n}, \tag{1}$$

$$g(t) = b_\mu t^\mu + b_{\mu+1} t^{\mu+1} + \ldots + b_{\mu+n} t^{\mu+n}. \tag{1a}$$

It holds for $t \to 0$ that $f(t) \doteq a_\nu t^\nu$, $g(t) \doteq b_\mu t^\mu$, $\frac{f(t)}{g(t)} \doteq \frac{a_\nu t^\nu}{b_\mu t^\mu}$. We have put from our consideration the case of $\nu \neq \mu$, because by mutual values of ν and μ the quotient has either an improper limit ($\nu > \mu$), thus for $t \to 0$ this quotient cannot be modelled, or it has the limit taking on the value zero ($\nu < \mu$) and the accuracy is increaseable only through a consistent normalization of variables. In [1] and [2] has been shown that in case both quadrators of the multiplier are working in the same sections, there occur marked considerable errors in modelling the quotient. Both quadrators are working in the same sections just form the values of the denominator

$$g(x) = x_j - x_{j+1} \tag{2}$$

or

$$g(x) = -\frac{1}{n} \tag{2a}$$

at a uniform distribution of the break points of the linear functions approximating the quadratic dependences; n is the number of linear sections. Hereafter we shall be concerned with the approximately equidistant distribution of the break points as it is the case with Czechoslovak analog computer MEDA TA, TC and ADT 3000. In these cases $n = 10$ and the quadrators begin to work in the same sections according to (2a) from the value $g(x) \doteq 0.1$ and there may arise marked errors for the values $g(x) < 0.1$ in modelling the quotien. In view of (1) it is necessary to satisfy the condition

$$g(t) + ae^{-\alpha t} \geqq 0.1 \tag{3}$$

in order to secure the corect calculation. (If we now assume that for the stability of the computing setwork the quotient may be positive.) For the sake of simplicity, let us first consider the quotient $z(t) = \frac{a_\nu t^\nu}{b_\nu t^\nu} = \frac{a_\nu}{b_\nu}$, which will be modelled in the form

$$z(t) \doteq \frac{a_\nu t^\nu + a \frac{a_\nu}{b_\nu} e^{-\alpha t}}{b_\nu t^\nu + a e^{-\alpha t}}. \tag{4}$$

With some modification we can write the above expression as

$$z(t) \doteq \frac{a_\nu \left(t^\nu + a \frac{1}{b} e^{-\alpha t} \right)}{b_\nu \left(t^\nu + \frac{a}{b_\nu} e^{-\alpha t} \right)} = \frac{a_\nu}{b_\nu}, \tag{4a}$$

which leads us to the conclusion that, in this case, the magnitude of the coefficient a (not that of coefficient α) does not matter, so that the condition $b_\nu t^\nu + a e^{-\alpha t} \geqq 0.1$ may be secured by a right choice of a and α. We arrive to the same result if a quotient of two polynomials is concerned one of which is a multiple of the other, thence $b_j = k a_j$. Then

$$z(t) \doteq \frac{f(t) + a z(o) e^{-\alpha t}}{g(t) + a e^{-\alpha t}} = \frac{a_\nu t^\nu + a_{\nu+1} t^{\nu+1} + \ldots + a_{\nu+n} t^{\nu+n} + a \frac{a_\nu}{b_\nu} e^{-\alpha t}}{b_\nu t^\nu + b_{\nu+1} t^{\nu+1} + \ldots + b_{\nu+n} t^{\nu+n} + a e^{-\alpha t}} =$$

$$= \frac{a_\nu \left(t^\nu + \frac{a_{\nu+1}}{\alpha_\nu} t^{\nu+1} + \ldots + \frac{a_{\nu+n}}{a_\nu} t^{\nu+n} + a \frac{1}{b_\nu} e^{-\alpha t} \right)}{b_\nu \left(t^\nu + \frac{b_{\nu+1}}{b_\nu} t^{\nu+1} + \ldots + \frac{b_{\nu+n}}{b_\nu} t^{\nu+n} + a \frac{1}{b_\nu} e^{-\alpha t} \right)} =$$

$$= \frac{a_\nu \left(t^\nu + \frac{a_{\nu+1}}{a_\nu} t^{\nu+1} + \ldots + \frac{a_{\nu+n}}{a_\nu} t^{\nu+n} + a \frac{1}{k a_\nu} e^{-\alpha t} \right)}{k a_\nu \left(t^\nu + \frac{k a_{\nu+1}}{k a_\nu} t^{\nu+1} + \ldots + \frac{k a_{\nu+n}}{k a_\nu} t^{\nu+n} + a \frac{1}{k a_\nu} e^{-\alpha t} \right)} = \frac{1}{k} = \frac{\alpha_\nu}{b_\nu}, \tag{4b}$$

so that the quotient $z(t) = \frac{f(t)}{g(t)}$ is modelled without any loss of accuracy even with additional terms $a z(0) e^{-\alpha t}$ and $a e^{-\alpha t}$. Figure 1 shows a general course of the function $g(t)$, of the additional term $a e^{-\alpha t}$ and of the denominator $a e^{-\alpha t} + g(t)$. Relation (3) must hold for the correct calculation. If we consider $f(t)$ and $g(t)$ in the forms of (1) and (1a), $\nu = \mu$, then it holds for $t \to 0$, $f(t) \doteq a_\nu t^\nu$, $g(t) \doteq b_\nu t^\nu$. Let us programme the expression in the form

$$z(t) = \frac{\frac{a_\nu}{b_\nu} t^\nu}{t^\nu} \doteq \frac{\frac{a_\nu}{b_\nu} t^\nu + a \frac{a_\nu}{b_\nu} e^{-\alpha t}}{t^\nu + a e^{-\alpha t}}. \tag{4c}$$

According to the [3] let us choose $\alpha = 10$ and determine such values $a_{\min}$, for some values ν to satisfy condition (3) in case of (4c)

$$t^{\nu} + ae^{-10t} \geqq 0.1. \tag{4d}$$

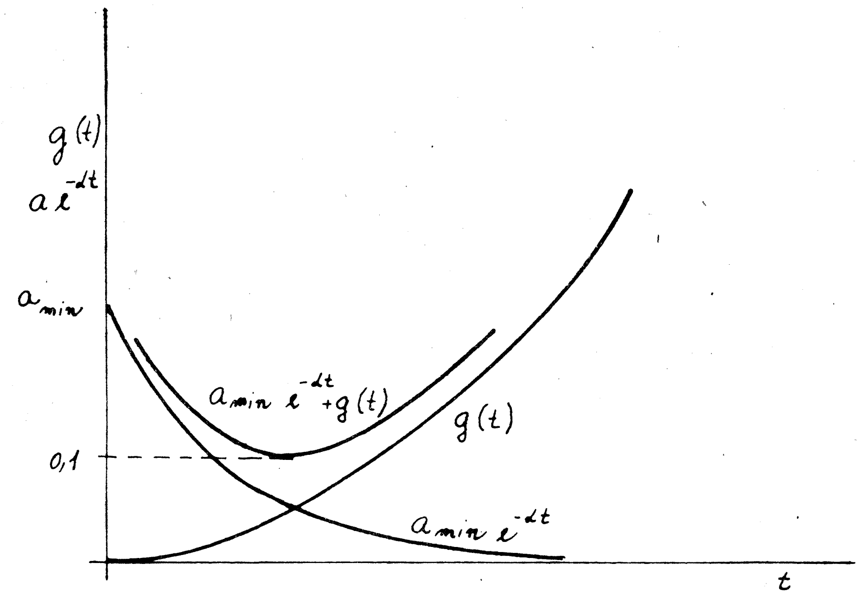

Fig. 1

The course of the function $t^{\nu} + ae^{-10t}$ or $0.5t^{\nu} + ae^{-10t}$ was examined for the values $\nu = 0.5$; 1; 2; 3; 4. The values $a_{\min}$ found for the individual values ν are listed in table 1. In view of the fact that some $a_{\min} > 1$, we set $a_{\min} = 1$ and determine the exponent α such that condition (3: be fulfilled (4d) with $a_{\min} = 1$.

ν	0.5	1	2	3	4	0.5	1	2	3	4
$a_{\min}$	0.1	0.135	1.288	5.565	14.39	0.1	0.1	0.470	2.018	5.707
$a_{\min} = 1$ $\alpha_{\max}$			9.2	6.45	5.28				8.13	6.29
	$b = 0.5$					$b = 1$				

Table 1

The small value of the expression $ae^{-\alpha t}$ is theoretically if importance especially in modelling the quotient of general functions $f(t)$ and $g(t)$ where the error is approximation of the expression (1) is increasing with the value a. For instance, in modelling the expression

$$z = \frac{0{,}5(e^t - e^{-t})}{\sin t} =$$

$$= \frac{0{,}5\left(1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} - 1 + t - \frac{t^2}{2!} + \ldots - (-1)^n \frac{t^n}{n!}\right.}{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!}} =$$

$$= \frac{t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots + \frac{t^{2n+1}}{(2n+1)!}}{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!}}, \tag{5}$$

$z(0) = 1$. By table 1 we have in this case $a_{\min} = 0.1$ if the first terms of the numerator and the denominator are taken into account. The maximal error of approximation of the function z from (5) by the function

$$z_1 = \frac{0{,}5(e^t - e^{-t}) + a_{\min} e^{-10t}}{\sin t + a_{\min} e^{-10t}} \tag{6}$$

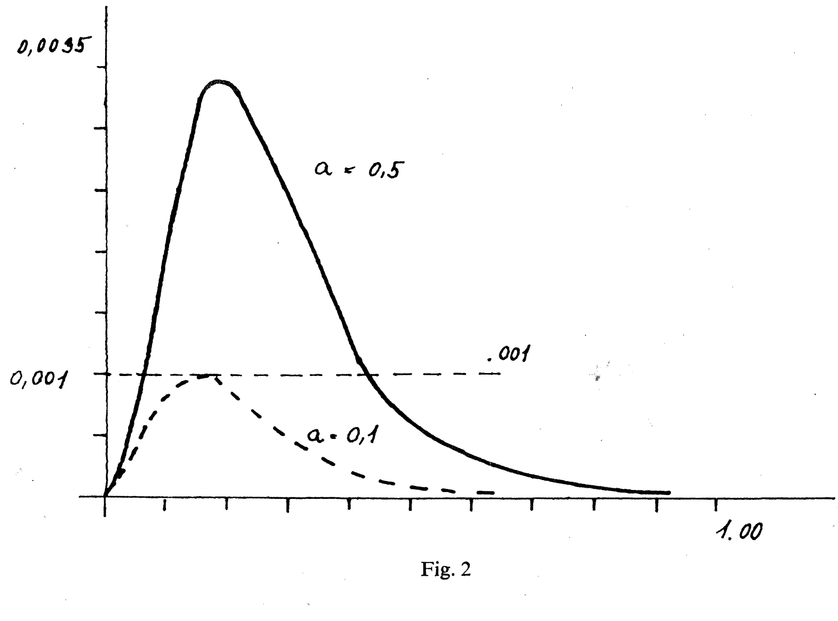

Fig. 2

in case of $a_{\min} = 0.1$ is 0.001 for $t = 0.14$; in case of $a_{\min} = 0.5$ it is 0.003 for $t =$ $= 0.18$ as can be seen in figure 2. It is clear that this error is due to the additional terms and it is with respect to the error of the dividing circuit proper, practically uegligible.

The greatest errors occur in modelling the quotient $z = \frac{mt^{\nu}}{t^{\nu}}$ for $t \to 0$ using the computer MEDA 41TC for $m = 0.138;\ 0.341;\ 0.558;\ 0.776;\ 0.987$. It is thus neccessary that these values are experimentally precised for the above computer.

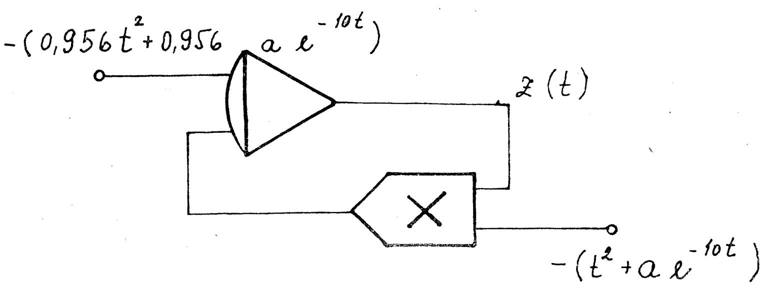

Fig. 3

If we choose $m = 0.987$ (experimentally precised to $m = 0.956$), $\nu = 2$, then in modelling the expression

$$z = \frac{0{,}956t^2}{t^2} \tag{7}$$

in the form of $z \doteq \frac{0{,}956t^2 + 0{,}956ae^{-10t}}{t^2 + ae^{-10t}}$ there occur, according to the programme schema in figure 3, marked errors at the small value a (see table 2), no more considerable errors occur at the value $a_{\min} = 0.476$ $(0.956 \doteq 1)$ corresponding to table 1.

a	0.020	0.200	0.470
$\delta(\bar{z})_{\max}$	−0.145	−0.084	−0.025

Table 2

We meet with the neccessity to choose additional terms in solving some differential equations with variable coefficients and nonlinear differential equations, where the singularities of the type $\frac{0}{0}$ occur. For instance in solving the equation

$$t^2y' - y = 2mt^3 - mt^2 \tag{8}$$

with the initial condition $y(0) = 0$, solution of which is the function $y = mt^2$. we proceed as follows: Equation (8) is programmed in form

$$y' = \frac{y}{t^2} + 2mt - m \tag{8a}$$

by the programme schema in figure 4. Using the computer MEDA 41TC and choosing $m = 0.956$ (theoretical value is $m = 0.987$), we get at the small value a for instance 0.020 the solution with an error of $\delta(y)_{max} = -0.730$, at $a = 0.200$ with

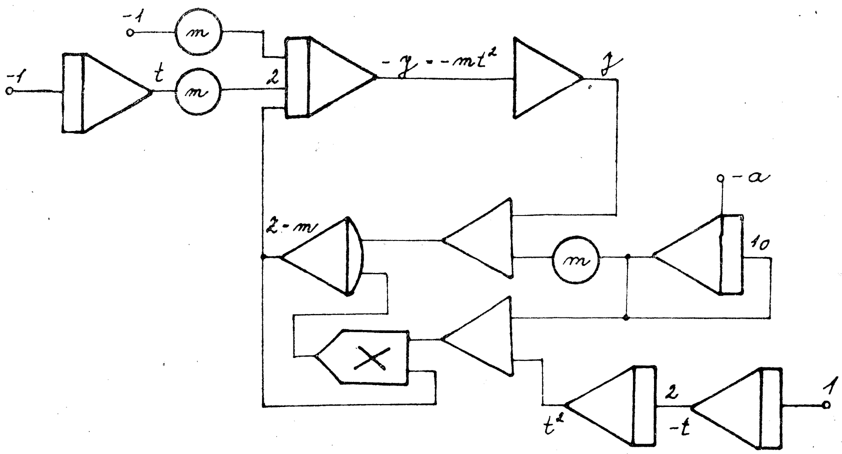

Fig. 4

an error of $\delta(y)_{max} = -0.223$, at $a = 0.470$ by table 1 ($v = 2$) we get the solution with an error of $\delta(y)_{max} = -0.043$. Marked errors being to appear at $a = 0.200$ ($m = 0.956$, $t \in \langle 0; 1 \rangle$).

ПОВЫШЕНИЕ ТОЧНОСТИ РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С СИНГУЛАРИТАМИ ТИПА $\frac{0}{0}$ ВЫГОДНЫМ ПОИСКОМ ДОДАТОЧНОГО ЧЛЕНА

Резюме

В статье описано определение оптимального значения додаточного члена читателя и знаменателья при моделированию неопределенных выражений типа $\frac{0}{0}$ на чс. аналоговых машинах.

ZVÝŠENÍ PŘESNOSTI ŘEŠENÍ DIFERENCIÁLNÍCH ROVNIC SE SINGULARITAMI TYPU $\frac{0}{0}$ VHODNOU VOLBOU PŘÍDAVNÉHO ČLENU

Souhrn

V článku je popsáno určení optimální hodnoty přídavného členu čitatele a jmenovatele při modelování neurčitých výrazů typu $\frac{0}{0}$ na čs. analogových počítačích MEDA 41TA, MEDA 41TC a ADT 3000 s ohledem na větší nepřesnost modelování podílu malých hodnot těchto počítačů, je zvlášť výrazná nepřesnost modelování podílu malých čísel. Jsou určeny minimální hodnoty těchto přídavných členů, při kterých se ještě neprojevuje nepřesnost dělicího obvodu.

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