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ON SOME LIMIT PROPERTIES OF THE REWARD FROM A MARKOV REPLACEMENT PROCESS

PAVLA KUNDEROVÁ

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1. Basic definitions and notations

Let a homogeneous Markov process with rewards $\{X_t, t \geqq 0\}$ (see [5], [6]) describing the evolution of a system in state space $I = \{1, \ldots, r\}$ be defined by exit intensities $(\mu(1), \ldots, \mu(r))$, $0 < \mu(j) < \infty$, $j = 1, \ldots, r$, and by a matrix $\boldsymbol{P} = \| p(i,j) \|_{i,j=1}^{r}$, $p(i, i) = 0$, of transition probabilities in the moment of exit. Let us denote by $\boldsymbol{M} = \| \mu(i,j) \|_{i,j=1}^{r}$ the matrix of transition intensities of the process, where

$$\mu(i,j) = \mu(i)\, p(i,j) \qquad \text{for } i \neq j, \qquad\qquad \mu(i, i) = -\mu(i) = -\sum_{j \neq i} \mu(i, j).$$

The development of the process can be influenced by an action called *replacement*. According to [5] we mean under a replacement of type $(i, +j)$ the instantaneous shift of the trajectory from state i into state j. The information of the evolution of the process up to the m-th state change is given by the sequence of states visited

$$i_0, i_1, \ldots, i_m = j,$$

by the corresponding sojourn times

$$t_0, t_1, \ldots, t_{m-1},$$

and by the sequence

$$\delta_0, \delta_1, \ldots, \delta_{m-1},$$

where $\delta_m = 0$ in case of $i_m \to i_{m+1}$ without interference and $\delta_m = 1$ in case of $i_m \to \to i_{m+1}$ is replacement.

We use in accordance with [5] the notation

$$\omega_m = [i_0, t_0, \delta_0; i_1, t_1, \delta_1; \ldots; i_{m-1}, t_{m-1}, \delta_{m-1}; i_m],$$

or equivalently

$$\omega_m = [\sigma_1, (i_0, i_1); \sigma_2, (i_1, +i_2); \ldots; \sigma_n, (i_{m-2}, i_{m-1}, +i_m)],$$

where $\sigma_0 = 0, \sigma_1, \sigma_2, \ldots$ are the moments in which the trajectory is discontinuous, (i_0, i_1) denotes the transition $i_0 \to i_1, (i_1, +i_2)$ denotes the replacement $i_1 \to \to i_2, \ldots, (i_{m-2}, i_{m-1}, +i_m)$ denotes the transition $i_{m-2} \to i_{m-1}$ and simultaneously in the same moment the replacement $i_{m-1} \to i_m$.

Let us denote the complete history of the process (elementary event)

$$\omega = [i_0, t_0, \delta_0; i_1, t_1, \delta_1; \ldots]$$

or

$$\omega = [\sigma_1, (i_0, i_1); \sigma_2, (i_1, +i_2); \ldots].$$

Under a *replacement policy* (see [5]) we mean a sequence of functions $F = \{{}^mF_k(t/\omega_m)\}$, $k = 1, \ldots, r$; $m = 0, 1, 2, \ldots$ where ${}^mF_k(t/\omega_m)$ is a probability that the maximal sojourn time in i_m will be less than t and the eventual replacement will be into $k \neq i_m$. Let us write

$${}^mF(t/\omega_m) = \sum_{k=1}^{r} {}^mF_k(t/\omega_m), \qquad \text{it can be } {}^mF(\infty/\omega_m) < 1.$$

The distribution of the probability of the initial state $p(1), p(2), \ldots, p(r)$ is arbitrary, but firmly chosen.

Assumption 1. *Consider such replacement policies only, where with probability* 1

a) there exists only a finite number of replacements in every finite interval,

b) there are neither two or more replacements in the same moment.

There is assigned to nearly every ω the trajectory $\{Y_t, t \geqq 0\}$ not left continuous at the time of the transition and not right continuous at the time of replacement. In what follows we denote by

$$Y_t^- = Y_{t-}, t > 0; \; Y_0^- = Y_0; \; Y_t^+ = Y_{t+}, t \geqq 0;$$

$$\mathfrak{B}_t = \sigma a(\{Y_s = j\}, j \in I, s \in \langle 0, t \rangle; \text{ events of zero probability}),$$

$$\mathfrak{B}_t^+ = \bigcap_{s>t} \mathfrak{B}_s,$$

D is a set of couples $(i, +j)$ meaning admissible replacements, $D_i = \{j: (i, +j) \in D\}$,

$r(i,j)$ the reward from the transition (i,j), we set $r(i,i) = 0$; $v(i,j)$ the reward from the replacement $(i, +j)$, we set $v(i,i) = 0$; we do not consider the continuous component of the reward (the reward from the persistence in the state).

A stationary replacement policy f is given by a function $f(j)$ defined on a subset $I_f \subset I$ and taking on values in I such that $f(j) \in D_j$ for $j \in I_f$, $f(j) \neq j$. The replacement policy f is the prescription to realize instantaneously the replacement $j \to f(j)$ whenever there a transition into state $j \in I_f$ occurs. No replacements occur in states $j \notin I_f$.

Let R_t be a reward from the process up to the time t, in accordance with the previous definitions

$$R_t = \sum_{n=0}^{N} [r(Y_{\sigma_n}^-, Y_{\sigma_n}) + v(Y_{\sigma_n}, Y_{\sigma_n}^+)], \qquad \sigma_N \leqq t < \sigma_{N+1}.$$

If the state space of the process under a stationary replacement policy f contains one recurrent class only, then there exists (independently of the initial distribution) the mean reward per a time unit (see [6])

$$\lim_{t \to \infty} \frac{1}{t} E(R_t) = \Theta,$$

and this mean reward Θ is uniquely determined by the system of equations

$$v(j, f(j)) + w(f(j)) - w(j) = 0, \qquad j \in I_f,$$

$$\sum_{k \neq j} \mu(j,k) [r(j,k) + w(k) - w(j)] - \Theta = 0, \qquad j \notin I_f, \tag{1}$$

uniquely determining the numbers $w(j), j = 1, \ldots, r$, up to the additive constant (see [1]).

2. Markov replacement processes as point processes

We can study Markov replacement processes by means of the theory of point processes (see [3]). Let us consider the so called *marked point process*, where the events with marks $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ occur in the moments $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ We identify the marks with numbers $l = 1, \ldots, q$ and investigate the occurrence of the q events in all: "there occurred the event with mark l". Let us denote by $\{\underline{N}_t, t \geqq 0\} = \{({}^1N_t, \ldots, {}^qN_t), t \geqq 0\}$ the counting processes of the events, $\mathscr{F}_t^{\underline{N}} = \sigma a(\underline{N}_s, s \in \langle 0, t \rangle)$, $\mathscr{F}^{\underline{N}} = \{\mathscr{F}_t^{\underline{N}}, t \geqq 0\}$ the nondecreasing system of σ-algebras, $\{\underline{A}_t, t \geqq 0\} = \{({}^1A_t, \ldots, {}^qA_t), t \geqq 0\}$ the compensator (integral of the intensity) of the point process considered with respect to $\mathscr{F}^{\underline{N}}$.

It is known from the theory of the point processes that $\{\underline{M}_t = \underline{N}_t - \underline{A}_t, t \geqq 0\}$ is a martingale with respect to $\mathscr{F}^{\underline{N}}$. We can consider the marked point process as a process with rewards. If ${}^lr$ is a reward from the event with mark l, $\underline{r} = ({}^1r, \ldots, {}^qr)$, then the total reward in the time $\langle 0, t \rangle$ equals to $\underline{r}\underline{N}_t = \sum_{l=1}^{q} {}^lr\,{}^lN_t$.

We now show the continuity between the processes with replacements and the point processes. Let us consider the marked point process with marks $l \sim (i, k)$, $l \sim (i, +k)$, $l \sim (i, k, +k')$ and with rewards ${}^{(i,k)}r = r(i, k)$, ${}^{(i,+k)}r = v(i, k)$, ${}^{(i,k,+k')}r = r(i, k) + v(k, k')$.

Let us define the vector $\underline{v} = ({}^1v, \ldots, {}^qv)$ this way ${}^{(i,k)}v = w(k) - w(i)$, ${}^{(i,+k)}v = w(k) - w(i)$, ${}^{(i,k,+k')}v = w(k') - w(i)$, where the numbers $w(1), \ldots, w(r)$ are the solution of the system of equations (1).

Let $\tilde{F}$ be the so called *policy of destination* in the process with replacements and let it be such a stationary replacement policy f that under it exists one recurrent class only. The compensator of destination ${}^l\tilde{A}_t$ with respect to $\mathscr{F}^N$ has the form ${}^{(i,k)}\tilde{A}_t = 0$ for $i \in I_f$ or $k \in I_f$,

$${}^{(i,k)}\tilde{A}_t = \int_0^t [\chi_{\{i\}}(Y_s^+) + \sum_{j \in f^{-1}(i)} \chi_{\{j\}}(Y_s^+)] \, \mu(i, k) \, ds \qquad \text{for } i \notin I_f, \quad \text{and simultaneous-}$$

ly $k \notin I_f$, $k \neq i$,

${}^{(i,+k)}\tilde{A}_t = 0$, because the marks $(i, +k)$ do not exist if the stationary replacement policy is applied,

${}^{(i,k,+k')}\tilde{A}_t = 0$ for $i \in I_f$ or $k \notin I_f$ or $k' \neq f(k)$,

$${}^{(i,k,+k')}\tilde{A}_t = \int_0^t [\chi_{\{i\}}(Y_s^+) + \sum_{j \in f^{-1}(i)} \chi_{\{j\}}(Y_s^+)] \, \mu(i, k) \, ds \qquad \text{for } i \notin I_f \quad \text{and at the same}$$

time $k \in I_f$ and at the same time $k' = f(k)$, $k \neq i$, $k' \neq k$.

Using the equations of (1), then with some modification

$$(\underline{r} + \underline{v}) \, \underline{\tilde{A}}_t = \sum_{i \notin I_f} \int_0^t \{\sum_{k \neq i} [r(i, k) + w(k) - w(i)] \, \mu(i, k)\} \, \{\chi_{\{i\}}(Y_s^+) + \sum_{j \in f^{-1}(i)} \chi_{\{j\}}(Y_s^+)\} \, ds +$$

$$+ \sum_{i \notin I_f} \int_0^t \{ \sum_{k \in I_f} [v(k, f(k)) + w(f(k)) - w(k)] \, \mu(i, k)\} \, \{\chi_{\{i\}}(Y_s^+) + \sum_{j \in f^{-1}(i)} \chi_{\{j\}}(Y_s^+)\} \, ds =$$

$$= \Theta \sum_{i \notin I_f} \int_0^t \{\chi_{\{i\}}(Y_s^+) + \sum_{j \in f^{-1}(i)} \chi_{\{j\}}(Y_s^+)\} \, ds = \Theta t. \tag{2}$$

We now determine the compensators ${}^lA_t$ in case of the process with replacements being controlled by the replacement policy F, given by the sequence of functions $\{{}^mF_k(t/\omega_m)\}$. Under this replacement policy we investigate again the occurrence of the events of type: "it has the event with mark l occurred", where $l \sim (i, k)$, $l \sim (i, +k)$, $l \sim (i, k, +k')$. Let σ_s be the moment of the last discontinuity of the trajectory in the time $\langle 0, s \rangle$ and let the last change (transition or replacement) in the moment σ_s be the m_s-th change.

Let us assume that there exists the intensity of arrival ${}^lQ_s$ of the arrival of the event with mark l, i.e. ${}^lA_t = \int_0^t {}^lQ_s \, ds$. We determine ${}^lQ_s$ under the following assumptions:

a) by the last change in the moment σ_s the system has passed into the state i,

b) no change has occurred during the time $(\sigma_s, s\rangle$.
Let $\tau(l)$ denote the moment of the occurrence of the event with mark l and $A^i_{(\sigma_s, s\rangle}$ be the event, which means that assumptions a), b) are fulfilled.

Then

$$^{(i,k)}Q_s = \lim_{\Delta \to 0+} \frac{1}{\Delta} P(s < \tau[(i, k)] \leqq s + \Delta) \, . \, [P(\tau[(i, k)] > s)]^{-1} =$$

$$= \left[-\frac{d}{ds} P(\tau[(i, k)] > s) \right] . \, [P(A^i_{(\sigma_s, s\rangle})]^{-1} =$$

$$= \left[-\frac{d}{ds} \int_s^\infty (1 - {}^{m_s}F([u - \sigma_s]^- / \omega_{m_s})) \, \mu(i, k) \, e^{-\mu(i)(u - \sigma_s)} \times \right.$$

$$\left. \times \{1 - {}^{m_s+1}F(0^+ / \omega_{m_s}, u - \sigma_s; 0; k\} \, du \right] [(1 - {}^{m_s}F([s - \sigma_s]^- / \omega_{m_s})) \, e^{-\mu(i)(s - \sigma_s)}]^{-1} =$$

$$= \mu(i, k) \, (1 - {}^{m_s+1}F(0^+ / \omega_{m_s}, s - \sigma_s, 0; k),$$

because following [5], if $\mu(i) < \infty$, there occurs $\omega_{m_s+1} = (\omega_{m_s}, u, 0; k)$, $i_{m_s} = i$ with the probability $\mu(i, k) \, e^{-\mu(i)t}(1 - {}^{m_s}F(t/\omega_{m_s})) \, dt$, $u \in \langle t - dt, t)$ and the probability that the replacement does not occur in the state k in the moment $s - \sigma_s$ is equal to $(1 - {}^{m_s+1}F(0^+/\omega_{m_s}, s - \sigma_s, 0; k))$.

Likewise, we have

$$^{(i,+k)}Q_s = \left[-\frac{d}{ds} P(\tau[(i, +k)] > s) \right] . \, [P(A^i_{(\sigma_s, s\rangle})]^{-1} =$$

$$= \left[-\frac{d}{ds} \int_s^\infty e^{-\mu(i)(u - \sigma_s)} \, d^{m_s}F_k(u - \sigma_s / \omega_{m_s}) \right] [P(A^i_{(\sigma_s, s\rangle})]^{-1} =$$

$$= \left[\frac{d}{ds} {}^{m_s}F_k(s - \sigma_s / \omega_{m_s}) \right] [1 - {}^{m_s}F([s - \sigma_s]^- / \omega_{m_s})]^{-1},$$

because

$$\omega_{m_s+1} = [\omega_{m_s}, u, 1; k], \, i_{m_s} = i$$

occurs with the probability

$$e^{-\mu(i)t} \, d^{m_s}F_k(t/\omega_{m_s}), \qquad u \in \langle t, t + dt),$$

(see [5]).

Finally

$$^{(i,k,+k')}Q_s = \left[-\frac{d}{ds} P(\tau[(i, k, +k')] > s) \right] [P(A^i_{(\sigma_s, s\rangle})]^{-1} =$$

$$= \left[-\frac{d}{ds} \int_s^\infty (1 - {}^{m_s}F([u - \sigma_s]^- / \omega_{m_s})) \, \mu(i, k) \, e^{-\mu(i)(u - \sigma_s)} \times \right.$$

$$\left. \times {}^{m_s+1}F_{k'}(0^+ / \omega_{m_s}, u - \sigma_s, 0; k) \, du \right] . \, [P(A^i_{(\sigma_s, s\rangle})]^{-1} =$$

$$= \mu(i, k) \, {}^{m_s+1}F_{k'}(0^+ / \omega_{m_s}, s - \sigma_s, 0; k),$$

because the probability of the replacement $k \to k'$ occurring in the moment $s - \sigma_s$

(when the transition $i \to k$ realized), is equal to ${}^{m_s+1}F_{k'}(0^+/\omega_{m_s}, s - \sigma_s, 0; k)$. (see [5]). Thus

$$^{(i,k)}A_t = \int_0^t \chi_{\{i\}}(Y_s^+)\, \mu(i, k)\, [1 - {}^{m_s+1}F(0^+/\omega_{m_s}, s - \sigma_s, 0; k)]\, \mathrm{d}s,$$

$$^{(i,+k)}A_t = \int_0^t \chi_{\{i\}}(Y_s^+)\, \frac{1}{1 - {}^{m_s}F([s - \sigma_s]^- /\omega_{m_s})}\, \mathrm{d}^{m_s}F_k(s - \sigma_s/\omega_{m_s}),$$

$$^{(i,k,+k')}A_t = \int_0^t \chi_{\{i\}}(Y_s^+)\, \mu(i, k)\, {}^{m_s+1}F_{k'}(0^+/\omega_{m_s}, s - \sigma_s, 0; k)\, \mathrm{d}s.$$

We calculate

$$(\underline{r} + \underline{v})\, \underline{A}_t = \sum_{i\in I} \sum_{k\neq i} [r(i, k) + w(k) - w(i)]\, {}^{(i,k)}A_t +$$
$$+ \sum_{i\in I} \sum_{k\neq i} [v(i, k) + w(k) - w(i)]\, {}^{(i,+k)}A_t +$$
$$+ \sum_{i\in I} \sum_{k\neq i} \sum_{k'\neq k} [r(i, k) + v(k, k') + w(k') - w(i)]^{(i,k,+k')} A_t =$$
$$= \sum_{i\in I} \sum_{k\neq i} [r(i, k) + w(k) - w(i)] \int_0^t \chi_{\{i\}}(Y_s^+)\, \mu(i, k) (1 - \sum_{k'\neq k} {}^{m_s+1}F_{k'}(0^+/\omega_{m_s}, s - \sigma_s, 0; k)\, \mathrm{d}s +$$
$$+ \sum_{i\in I} \sum_{k\neq i} [v(i, k) + w(k) - w(i)] \int_0^t \chi_{\{i\}}(Y_s^+)\, \frac{\mathrm{d}^{m_s}F_k(s - \sigma_s/\omega_{m_s})}{(1 - {}^{m_s}F([s - \sigma_s]^- /\omega_{m_s}))} +$$
$$+ \sum_{i\in I} \sum_{k\neq i} \sum_{k'\neq k} [r(i, k) + v(k, k') + w(k') - w(i) + w(k) - w(k)] \times$$
$$\times \int_0^t \chi_{\{i\}}(Y_s^+)\, \mu(i, k)\, {}^{m_s+1}F_{k'}(0^+/\omega_{m_s}, s - \sigma_s, 0; k)\, \mathrm{d}s =$$
$$= \sum_{i\in I} \int_0^t \chi_{\{i\}}(Y_s^+)\, [\varphi(i) + \Theta]\, \mathrm{d}s + \sum_{i\in I} \sum_{k\neq i} [v(i, k) + w(k) - w(i)]\, {}^{(i,+k)}A_t +$$
$$+ \sum_{i\in I} \sum_{k\neq i} \sum_{k'\neq k} [v(k, k') + w(k') - w(k)]\, {}^{(i,k,+k')}A_t =$$
$$= \int_0^t \varphi(Y_s)\, \mathrm{d}s + \Theta t + \sum_{i\in I} \sum_{k\neq i} [v(i, k) + w(k) - w(i)]\, {}^{(i,+k)}A_t +$$
$$+ \sum_{i\in I} \sum_{k\neq i} \sum_{k'\neq k} [v(k, k') + w(k') - w(k)]\, {}^{(i,k,+k')}A_t, \tag{3}$$

using the following notation

$$\varphi(i) = \sum_{k\neq i} \mu(i, k)\, [r(i, k) + w(k) - w(i)] - \Theta, \qquad i \in I.$$

In [2] there was defined an auxiliary random process for the study of the processes with replacements

$$M_t = R_t - \Theta t + w(Y_t^+) - w(Y_0) - \int_0^t \varphi(Y_s)\, \mathrm{d}s - \sum_{n=0}^{N} [v(Y_{\sigma_n}, Y_{\sigma_n}^+) + w(Y_{\sigma_n}^+) - w(Y_{\sigma_n})],$$
$$t \geqq 0,\ \sigma_N \leqq t < \sigma_{N+1},$$

showing that $\{M_t, t \geqq 0\}$ is a martingale with respect to $\{\mathfrak{B}_t^+, t \geqq 0\}$ under an arbitrary replacement policy F. By means of the vectors $\underline{r}$, $\underline{v}$ may be expressed the reward from the process with replacements up to the time t

$$R_t = \sum_{l=1}^{q} {}^l r^l N_t = \underline{r}\underline{N}_t \tag{4}$$

and the difference

$$w(Y_t^+) - w(Y_0) = \sum_{n=0}^{N} [w(Y_{\sigma_n}) - w(Y_{\sigma_n}^-) + w(Y_{\sigma_n}^+) - w(Y_{\sigma_n})] = \sum_{l=1}^{q} {}^l v^l N_t = \underline{}\underline{N}_t,$$

$$\sigma_N \leqq t < \sigma_{N+1}. \tag{5}$$

Let us define a random process

$$m_t = (\underline{r} + \underline{v})\, \underline{M}_t = (\underline{r} + \underline{v})\, (\underline{N}_t - \underline{A}_t).$$

Applying relations (2)–(5) we can write

$$m_t = \underline{r}\underline{N}_t - \Theta t + \underline{v}\underline{N}_t - (\underline{r} + \underline{v})\, (\underline{A}_t - \tilde{\underline{A}}_t) =$$

$$= R_t - \Theta t + w(Y_t^+) - w(Y_0) - \int_0^t \varphi(Y_s)\, ds - \sum_{i \in I} \sum_{k \neq i} [v(i, k) + w(k) - w(i)]\, {}^{(i,+k)}A_t -$$

$$- \sum_{i \in I} \sum_{k \neq i} \sum_{k' \neq k} [v(k, k') + w(k') - w(k)]\, {}^{(i,k,+k')}A_t =$$

$$= M_t + \sum_{n=0}^{N} [v(Y_{\sigma_n}, Y_{\sigma_n}^+) + w(Y_{\sigma_n}^+) - w(Y_{\sigma_n})] - \sum_{i \in I} \sum_{k \neq i} [v(i, k) + w(k) - w(i)] \times$$

$$\times ({}^{(i,+k)}N_t - {}^{(i,+k)}M_t) - \sum_{i \in I} \sum_{k \neq i} \sum_{k' \neq k} [v(k, k') + w(k') - w(k)] ({}^{(i,k,+k')}N_t - {}^{(i,k,+k')}M_t) =$$

$$= M_t + \sum_{i \in I} \sum_{k \neq i} [v(i, k) + w(k) - w(i)]\, {}^{(i,+k)}M_t +$$

$$+ \sum_{i \in I} \sum_{k \neq i} \sum_{k' \neq k} [v(k, k') + w(k') - w(k)]\, {}^{(i,k,+k')}M_t, \quad \sigma_N \leqq t < \sigma_{N+1}. \tag{6}$$

Definition. *The replacement policy F will be called the replacement policy with a bounded intensity if under this policy for all $l = 1, \dots, q$*

$${}^l A_t = \int_0^t {}^l Q_s\, ds,\ t \geqq 0,$$

and if there exists a constant $\bar{Q} < \infty$ such that ${}^l Q_t \leqq \bar{Q}$ F-almost sure, $t \geqq 0$.

Lemma 1. *If F is a replacement policy with the bounded intensity, then for any $l = 1, \dots, q$*

$$E({}^l N_t - {}^l N_s)^p \leqq \sum_{k=0}^{\infty} k^p \frac{(\bar{Q}(t-s))^k}{k!} e^{-\bar{Q}(t-s)}, \qquad p = 1, 2, \dots;\ s \leqq t. \tag{7}$$

Proof: the statement is obvious because the constant $\bar{Q}$, whose existence is guaranteed by the assumption, may be taken to be the intensity of the Poisson process. □

In what follows we denote the sum on the right hand side of (7) by the symbol

$$C(p, \bar{Q}, t - s).$$

Lemma 2. *Under an arbitrary replacement policy F with bounded intensity*

$$\lim_{t \to \infty} \frac{m_t}{t} = 0 \qquad F\text{-}almost\ sure. \tag{8}$$

Proof: We start from relation (6) making use of the fact that under every replacement policy F (see [2])

$$\lim_{t \to \infty} \frac{M_t}{t} = 0 \qquad F\text{-almost sure.}$$

Thus it suffices to prove for all $l \sim (i, +k)$, $l \sim (i, k, +k')$

$$\lim_{t \to \infty} \frac{{}^l M_t}{t} = 0 \qquad F\text{-almost sure.}$$

Let l be arbitrary.

a) We first prove that $\lim_{n \to \infty} \frac{{}^l M_n}{n} = 0$ F-almost sure. (9)

Since ${}^l M_0 = {}^l N_0 - {}^l A_0 = 0$ then

$${}^l M_n = \sum_{k=0}^{n-1} ({}^l M_{k+1} - {}^l M_k),$$

and since $\{{}^l M_n, n = 1, 2, \ldots\}$ is a martingale, it suffices for the validity of (9) to prove that (see [4], page 404 D)

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} E({}^l M_{n+1} - {}^l M_n)^2 < \infty. \tag{10}$$

Indeed, since

$$E({}^l M_{n+1} - {}^l M_n)^2 = E[({}^l N_{n+1} - {}^l N_n) - ({}^l A_{n+1} - {}^l A_n)]^2 \leqq 2E({}^l N_{n+1} - {}^l N_n)^2 + $$
$$+ 2E[\int_n^{n+1} {}^l Q_s \, ds]^2 \leqq 2[C(2, \bar{Q}, 1) + \bar{Q}^2],$$

the series (10) is convergent.

b) We now prove that $\lim_{t \to \infty} \frac{{}^l M_t}{t} = 0$ F-almost sure.

Let $n \leqq t < n + 1$, then

$$\left| \frac{{}^l M_t}{t} \right| \leqq \frac{1}{n} \sup_{n \leqq t < n+1} | {}^l M_t - {}^l M_n | + \frac{| {}^l M_n |}{n}, \tag{11}$$

with respect to (9) it suffices to prove that

$$\lim_{n \to \infty} \left[\frac{1}{n} \sup_{n \leqq t < n+1} | {}^l M_t - {}^l M_n | \right] = 0 \qquad F\text{-almost sure.} \tag{12}$$

It is

$$| {}^l M_t - {}^l M_n | \leqq | {}^l N_t - {}^l N_n | + | \int_n^t {}^l Q_s \, ds | \leqq {}^l N_{n+1} - {}^l N_n + \bar{Q}.$$

By Lemma 1 the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} E({}^lN_{n+1} - {}^lN_n)^2$$

is convergent. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} ({}^lN_{n+1} - {}^lN_n)^2$$

is finite F-almost sure. Hence

$$\lim_{n\to\infty} \frac{1}{n} ({}^lN_{n+1} - {}^lN_n) = 0 \qquad F\text{-almost sure.}$$

This completes the proof of (12). □

Theorem 1. *Let* $\{\underline{A}_t, t \geqq 0\}$ *be a compensator of the process under the replacement policy F with a bounded intensity and let* $\{\underline{\tilde{A}}_t, t \geqq 0\}$ *be a compensator of the process with replacements under the stationary policy of destination f. If for all* $l = 1, \ldots, q$

$$\lim_{t\to\infty} \frac{1}{t} ({}^lA_t - {}^l\tilde{A}_t) = 0 \qquad F\text{-almost sure} \quad (F\text{-in probability}) \tag{13}$$

then

$$\lim_{t\to\infty} \frac{R_t}{t} = \Theta \qquad F\text{-almost sure} \quad (F\text{-in probability}).$$

Proof: In the foregoing we derived

$$m_t = \underline{rN}_t - \Theta t + \underline{vN}_t - (\underline{r} + \underline{v})(\underline{A}_t - \underline{\tilde{A}}_t) =$$
$$= R_t - \Theta t + w(Y_t^+) - w(Y_0) - \sum_{l=1}^{q} ({}^lr + {}^lv)({}^lA_t - {}^l\tilde{A}_t).$$

By Lemma 2 we get

$$\lim_{t\to\infty} \frac{m_t}{t} = 0 \qquad F\text{-almost sure.}$$

Since

$$\min_{k,j\in I} \{w(k) - w(j)\} \leqq w(Y_t^+) - w(Y_0) \leqq \max_{k,j\in I} \{w(k) - w(j)\} \tag{14}$$

then

$$\lim_{t\to\infty} \frac{w(Y_t^+) - w(Y_0)}{t} = 0 \qquad F\text{-almost sure.}$$

The statement of the theorem follows thus by immediate applying the assumption of (13). □

Let us denote by D_t the whole sojourn time in states I_f in the interval $\langle 0, t \rangle$, and by O_t the whole number of replacements different from $i \to f(i)$ in $\langle 0, t \rangle$.

Theorem 2. *Let F be a replacement policy with a bounded intensity and f a stationary policy of destination. Let for any $l = 1, \ldots, q$ hold*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} ({}^lA_t - {}^l\tilde{A}_t) = 0 \qquad F\text{-in probability,} \tag{15}$$

and let

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} O_t = 0 \qquad F\text{-in probability,} \tag{16}$$

then

$$\frac{R_t - \Theta t}{\sqrt{t}}$$

has asymptotically normal distribution $N(0, \zeta)$ for $t \to \infty$, where ζ is determined by the following equations

$$\begin{gathered} w_2(f(i)) - w_2(i) = 0, \qquad i \in I_f, \\ \psi(i) + \sum_{k\neq i} \mu(i,k)\,[w_2(k) - w_2(i)] - \zeta = 0,\ i \notin I_f, \end{gathered} \tag{17}$$

involving auxiliary constants $w_2(1), \ldots, w_2(r)$, and where

$$\psi(i) = \sum_{k\neq i} \mu(i,k)\,[r(i,k) + w(k) - w(i)]^2,\ i \notin I.$$

Proof: Assumption (15) with expression

$$m_t = R_t - \Theta t + w(Y_t^+) - w(Y_0) - (\underline{r} + \underline{v})\,(\underline{A}_t - \tilde{\underline{A}}_t)$$

and relation (14) yield

$$\lim_{t\to\infty} \left[\frac{R_t - \Theta t}{\sqrt{t}} - \frac{m_t}{\sqrt{t}}\right] = 0 \qquad F\text{-in probability.} \tag{18}$$

Hereafter we use the relation (see the derivation of (6))

$$\begin{gathered} m_t = M_t + \sum_{n=0}^{N} [v(Y_{\sigma_n}, Y_{\sigma_n}^+) + w(Y_{\sigma_n}^+) - w(Y_{\sigma_n})] - \\ - \sum_{i\in I}\sum_{k\neq i} [v(i,k) + w(k) - w(i)]\,{}^{(i,+k)}A_t - \sum_{i\in I}\sum_{k\neq i}\sum_{k'\neq k} [v(k,k') + w(k') - w(k)]\,{}^{(i,k,+k')}A_t, \\ \sigma_N \leqq t < \sigma_{N+1}. \end{gathered} \tag{19}$$

In paper [2] (Theorem 3, parts II. and IV. of the proof) it was proved that under an arbitrary replacement policy F assuming (16) and

$$\lim_{t\to\infty} \frac{D_t}{\sqrt{t}} = 0 \qquad F\text{-in probability}$$

then

$$\frac{M_t}{\sqrt{t}} \tag{20}$$

has asymptotically normal distribution $N(0, \zeta)$ for $t \to \infty$, where ζ is determined by the equations of (17).

The assumption required is fulfilled. Indeed, since

$$\sum_{k \neq i} ({}^{(i,k)}A_t + \sum_{k'} {}^{(i,k,+k')}A_t) = \int_0^t \chi_{\{i\}}(Y_s^+) (\sum_{k \neq i} \mu(i,k)) \, ds = \\ = \mu(i) \int_0^t \chi_{\{i\}}(Y_s^+) \, ds,$$

and since for $i \in I_f$

$$\sum_{k \neq i} ({}^{(i,k)}\tilde{A}_t + \sum_{k'} {}^{(i,k,+k')}\tilde{A}_t) = 0,$$

(15) results in

$$\lim_{t \to \infty} \frac{D_t}{\sqrt{t}} = \lim_{t \to \infty} \frac{1}{\sqrt{t}} \sum_{i \in I_f} \int_0^t \chi_{\{i\}}(Y_s^+) \, ds = 0 \qquad F \text{ in probability.}$$

If we denote by O_t^* the whole number of such replacements $i \to j$ under the policy F in the interval $\langle 0, t \rangle$, for which $v(i,j) + w(j) - w(i) \neq 0$, then clearly

$$[\min_{i,j \in I} \{v(i,j) + w(j) - w(i)\}] \, O_t^* \leqq \sum_{n=0}^{N} [v(Y_{\sigma_n}, Y_{\sigma_n}^+) + w(Y_{\sigma_n}^+) - w(Y_{\sigma_n})] \leqq \\ \leqq [\max_{i,j \in I} \{v(i,j) + w(j) - w(i)\}] \, O_t^*, \qquad \sigma_N \leqq t < \sigma_{N+1} .$$

Because of $O_t^* \leqq O_t$, then by (16)

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} \sum_{n=0}^{N} [v(Y_{\sigma_n}, Y_{\sigma_n}^+) + w(Y_{\sigma_n}^+) - w(Y_{\sigma_n})] = 0 \qquad F \text{ in probability.} \tag{21}$$

From now on we shall utilize the validity of

$$\sum_{i \in I} \sum_{k \neq i} [v(i,k) + w(k) - w(i)] \, {}^{(i,+k)}\tilde{A}_t + \sum_{i \in I} \sum_{k \neq i} \sum_{k' \neq k} [v(k,k') + w(k') - w(k)] \times \\ \times {}^{(i,k,+k')}\tilde{A}_t = 0,$$

because ${}^{(i,+k)}\tilde{A}_t = 0$ and ${}^{(i,k,+k')}\tilde{A}_t \neq 0$ only in case of $i \notin I_f$ and at the same time $k \in I_f$ and at the same time $k' = f(k)$, but at that time $v(k, k') + w(k') - w(k) = 0$. Assumption (15) yields

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} \{ \sum_{i \in I} \sum_{k \neq i} [v(i,k) + w(k) - w(i)] ({}^{(i,+k)}A_t - {}^{(i,+k)}\tilde{A}_t) + \\ + \sum_{i \in I} \sum_{k \neq i} \sum_{k' \neq k} [v(k,k') + w(k') - w(k)] ({}^{(i,k,+k')}A_t - {}^{(i,k,+k')}\tilde{A}_t) \} = 0 \\ F\text{-in probability,}$$

and thus

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} \{ \sum_{i \in I} \sum_{k \neq i} [v(i,k) + w(k) - w(i)] \, {}^{(i,+k)}A_t + \\ + \sum_{i \in I} \sum_{k \neq i} \sum_{k' \neq k} [v(k,k') + w(k') - w(k)] \, {}^{(i,k,+k')}A_t \} = 0 \qquad F\text{-in probability.} \tag{22}$$

Relations (19) – (22) show that

$$\frac{m_t}{\sqrt{t}}$$

has asymptotically normal distribution $N(0, \zeta)$ for $t \to \infty$, where ζ is determined by the equations of (17).

This assertion together with (18) proves the Theorem 2. □

Remark. *The assumption of* (16) *in Theorem* 2 *can be left out.* Paper [2] shows that to prove the assertion of (20) it suffices only to obey the two following assumptions

$$\lim_{t\to\infty} \frac{O_t}{t} = 0 \qquad F\text{-in probability,} \tag{23}$$

$$\lim_{t\to\infty} \frac{D_t}{t} = 0 \qquad F\text{-in probability,} \tag{24}$$

(see relation (19) in [2]). Both these assumptions follow from (15) of Theorem 2.

a) We verify the validity of (23) as follows: let $l \sim (i, +k)$ denote the replacements in the policy F different from $i \to f(i)$ and let L denote the set of all such replacements. For all $l \in L$ then ${}^l\tilde{A}_t = 0$ and it follows from (15) for this l

$$\lim_{t\to\infty} \frac{{}^lA_t}{t} = 0 \qquad F\text{-in probability.}$$

Using (11) from the proof of Lemma 2 we see that for all $l \in L$

$$\lim_{t\to\infty} \frac{{}^lN_t}{t} = 0 \qquad F \text{ in probability}$$

and thus also

$$\lim_{t\to\infty} \frac{O_t}{t} = \lim_{t\to\infty} \frac{1}{t} \sum_{l\in L} {}^lN_t = 0 \qquad F\text{-in probability.}$$

b) The validity of (24) can be verified analogy with that of the assumption $\frac{1}{\sqrt{t}} D_t \to 0$ in the proof of Theorem 2.

НЕКОТОРЫЕ ПРЕДЕЛЬНЫЕ КАЧЕСТВА ДОХОДА ИЗ ПРОЦЕССА МАРКОВА С ВОССТАНОВЛЕНИЯМИ

Резюме

К исследованию процессов с восстановлениями применена теория точечных процессов ([3]). Рассмотрены меченые точечные процессы с доходами, метки $l = 1, \ldots, q$ такие что $l \sim (i, k)$ переход $i \to k$, $l \sim (i, +k)$ восстановление $i \to k$, $l \sim (i, k, +k')$ переход $i \to k$ и в тот же самый момент восстановление $k \to k'$. Далее рассмотрен компенсатор (интеграл от интенсивности) $\{\underline{\tilde{A}}_t, t \geqq 0\} = \{({}^1\tilde{A}_t, \ldots, {}^q\tilde{A}_t), t \geqq 0\}$ процесса когда процесс управлен такой стационарной стратегией f при которой существует только единственный класс возвратных состояний и компенсатор $\{\underline{A}_t, t \geqq 0\}$ процесса с восстановлениями с общей стратегией F. Определена стратегия восстановления с ограниченной интенсивностью.

Пусть R_t доход из процесса в течение интервала $\langle 0, t \rangle$ и Θ средний доход за единицу времени. Теорема 1 приводит достаточное условие для того, чтобы

$$\lim_{t \to \infty} \frac{1}{t} R_t = \Theta .$$

Теорема 2 устанавливает условия при которых имеет $\frac{R_t - \Theta t}{\sqrt{t}}$ при $t \to \infty$ асимптотическое распределение $N(0, \zeta)$, где ζ некоторая константа.

NĚKTERÉ LIMITNÍ VLASTNOSTI VÝNOSU Z MARKOVOVA PROCESU S OBNOVAMI

Souhrn

Ke studiu procesů s obnovami je užita teorie bodových procesů (viz [3]). Uvažují se značkované bodové procesy s výnosy, značky $l = 1, \ldots, q$ takové, že $l \sim (i, k)$ je přechod $i \to k$, $l \sim (i, +k)$ obnova $i \to k$, $l \sim (i, k, +k')$ přechod $i \to k$ a v témže okamžiku obnova $k \to k'$. Dále se vyšetřuje kompensátor (integrál z intensity) $\{\underline{\tilde{A}}_t, t \geqq 0\} = \{({}^1\tilde{A}_t, \ldots, {}^q\tilde{A}_t), t \geqq 0\}$ procesu, je-li proces s obnovami řízen takovou cílovou stacionární strategií f, při níž existuje pouze jedna třída rekurentních stavů a kompensátor $\{\underline{A}_t, t \geqq 0\}$ procesu s obnovami s obecnou strategií obnovy F. Je definována strategie obnovy s ohraničenou intensitou.

Nechť R_t je výnos z procesu za dobu $\langle 0, t \rangle$ a Θ průměrný výnos za jednotku času (viz [6]). Věta 1 formuluje postačující podmínky k tomu, aby

$$\lim_{t \to \infty} \frac{R_t}{t} = \Theta .$$

Věta 2 uvádí podmínky, za nichž má $\frac{R_t - \Theta t}{\sqrt{t}}$ pro $t \to \infty$ asymptoticky rozdělení $N(0, \zeta)$, kde ζ je jistá konstanta.

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