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A NOTE ON LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH THE SAME BASIC CENTRAL DISPERSION OF THE FIRST KIND

MIROSLAV LAITOCH

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Consider a linear differential equation of the 2nd order

$$y'' = q(t) \, . \, y, \tag{q}$$

where $q \in C^{(0)}$ on an interval $j = (-\infty, \infty)$. We assume the solution of the differential equation (q) to be oscillatory on the interval j to both end points of the interval.

Let φ be the basic (z.) central (c.) dispersion of the 1st kind relative to the differential equation (q). Let $c \in j$ be an arbitrary number and let y be such a solution of the differential equation (q) that $y(c) = 0$. The zeros of y are just the numbers $c_\nu = \varphi_\nu(c)$, $\nu = 0, \pm 1, \pm 2, \ldots$

O. Borůvka [1] proved under the above assumptions the following Theorem: The carriers $\bar{q}$ of all differential equations $(\bar{q})$ with the same basic central dispersion of the 1st kind φ, possessed by the differential equation (q) with the carrier q, are given by the formula

$$\bar{q} = q + \frac{p''}{p} + 2\frac{y'}{p} \cdot \frac{p'}{y}, \tag{*}$$

where p is an arbitrary function defined in the interval j having the following five properties:

1° $p(t) \neq 0$ for $t \in j$,
2° $p[\varphi(t)] = p(t)$ for $t \in j$,
3° $p(t) \in C^{(2)}(j)$,
4° $p'(c) = 0$,
5° $\int_c^{\varphi(c)} \left[\frac{1}{p^2(\sigma)} - \frac{1}{p^2(c)}\right] \cdot \frac{d\sigma}{y^2(\sigma)} = 0$

and the value of the last addend on the right in (*) at the point c_ν is defined by the formula $2p''(c_\nu)/p(c_\nu)$.

We present now some findings on the function p discussed in the above theorem.

Theorem 1.

Let $c \in j$ be an arbitrary number and let $f(t)$ be an increasing 1st phase of the differential equation (q), $f(c) = 0$. Let the function $h = h(t)$ satisfy the following conditions;

$0°$ *$h(t)$ is defined on the interval j,*

$A°$ *$h(t + \pi) = h(t)$ for $t \in j$,*

$B°$ *$h(t) \in C^{(0)}$, $h(t) \,.\, \sin^2(t) \in C^{(2)}(j)$,*

$C°$ *$\int_0^\pi h(t)\,dt = 0$,*

$D°$ *$h(t) \,.\, \sin^2(t) < 1$ for $t \in j$.*

Then the function

$$p(t) = \{1 - \sin^2[f(t)] \,.\, h[f(t)]\}^{-1/2}, \; t \in j \qquad (**)$$

possesses the five properties $1^0 - 5^0$ stated in the Theorem of Borůvka. And conversely also, if the function $p = p(t)$ possesses the properties $1^0 - 5^0$ in j, then there exists a function $h = h(t)$ with the properties $0^0 - D^0$ and there holds the equality

$$p(t) = \{1 - \sin^2[f(t)] \,.\, h[f(t)]\}^{-1/2}, \qquad t \in j.$$

Proof. Let us first note that the 1st phase is known that it satisfies the function equation

$$E^0: \quad [f\varphi(t)] - f(t) = \pi \qquad \text{for } t \in j,$$

that

$$F^0: \quad f \in C^{(3)},$$

for f is a solution of the 3rd order nonlinear equation

$$\sqrt{|f'|} \,.\, \left(\frac{1}{\sqrt{|f'|}}\right)'' - f'^2 = q, \qquad q \in C^{(2)}(j),$$

that by assumption

$$G^0: \quad f(c) = 0,$$

that

$$H_1^0: \quad f(t) \text{ maps the interval } j \text{ schlicht on itself,}$$

that

$$H_2^0: \quad f'(t) \neq 0 \qquad \text{for } t \in j.$$

We know also that the solution y discussed in the Theorem of O. Borůvka may be expressed in the form

$$I_0: \quad y = \sin f(t)/\sqrt{|f'(t)|}.$$

Now we prove

Property 1°.

Let $p = p(t)$ be defined by the equation (**). With respect to D^0 and H_2^0 we infer that $p = p(t)$ is a real function different from zero in j.

Property 2°.

With respect to A^0 and E^0 we may say that

$$p[\varphi(t)] = \frac{1}{\sqrt{1 - \sin^2[f(\varphi(t))] \,.\, h[f(\varphi(t))]}} = \frac{1}{\sqrt{1 - \sin^2[f(t) + \pi] \,.\, h[f(t) + \pi]}} =$$

$$= \frac{1}{\sqrt{1 - \sin^2[f(t)] \,.\, h[f(t)]}} = p(t) \qquad \text{for } t \in j.$$

Property 3°.

It is easy to see that with respect to B^0 and F^0 $p(t) \in C^{(2)}(j)$.

Property 4°.

Differentiating (**) we obtain

$$p'(t) = \frac{f'(t)}{2\{1 - \sin^2[f(t)] \,.\, h[f(t)]\}^{3/2}} \times$$
$$\times [2 \sin[f(t)] \,.\, \cos[f(t)] \,.\, h[f(t)] + \sin^2[f(t) \,.\, h'[f(t)]].$$

With respect to G_0 we get $p'(c) = 0$ for the square bracket is equal to zero at the point $t = c$.

Property 5°.

By calculation we find that with respect to (**) H_2^0, E^0 and C^0 is

$$\int_c^{\varphi(c)} \left[\frac{1}{p^2(\sigma)} - \frac{1}{p^2(c)}\right] \frac{\mathrm{d}\sigma}{y^2(\sigma)} = \int_c^{\varphi(c)} [1 - \sin^2[f(\sigma)] \,.\, h[f(\sigma)] - 1] \,.\, \frac{f'(\sigma)}{\sin^2[f(\sigma)]} \,.\, \mathrm{d}\sigma =$$

$$= -\int_c^{\varphi(c)} h[f(\sigma)] \,.\, f'(\sigma) \,.\, \mathrm{d}\sigma = -\int_0^{\pi} h(\tau) \,.\, \mathrm{d}\tau = 0, \qquad \text{for } p(c) = 1.$$

Conversely, let the function $p = p(t)$ have the properties 1°–5°. Clearly, without any loss on generality, we may assume that $p^2(c) = 1$ and so $p^2(t) > 0$ on j with respect to 1°. According to 2° we have $f[\varphi(t)] = f(t) + \pi$ and since $f(c) = 0$, it follows that $f^{-1}(t + \pi) = \varphi[f^{-1}(t)], f^{-1}(c) = 0$.

We prove now Property 0°. Let us define the function $h(t)$ on the interval j by putting

$$h(t) = \begin{cases} \dfrac{1}{\sin^2 t} \,.\left(1 - \dfrac{1}{p^2[f^{-1}(t)]}\right) & \text{for } t \in j,\ t \neq k\pi, \\[2ex] \dfrac{p''(c) \,.\, f^{-1'^2}(c)}{p^3(c)} & \text{for } t = k\pi \end{cases}$$

$$k = 0, \pm 1, \pm 2, \ldots$$

Since

$$\lim_{t\to 0} h(t) = \lim_{t\to 0} \frac{2p'[f^{-1}(t)] \,.\, f^{-1'}(t)}{2 \sin t \,.\, \cos t \,.\, p^3[f^{-1}(t)]} = \frac{f^{-1'}(0)}{p^3(c)} \,.\, \lim_{t\to 0} \frac{p'[f^{-1}(t)]}{\sin t} =$$

$$= \frac{f^{-1'}(0)}{p^3(c)} \,.\, \lim_{t\to 0} \frac{p''[f^{-1}(t)] \,.\, f^{-1'}(t)}{\cos t} = \frac{p''(c) \,.\, f^{-1'^2}(c)}{p^3(c)}$$

and since

$$h(t + \pi) = \frac{1}{\sin^2 (t + \pi)} \cdot \left(1 - \frac{1}{p^2[f^{-1}(t + \pi)]}\right) =$$

$$= \frac{1}{\sin^2 t}\left(1 - \frac{1}{p^2[\varphi(f^{-1}(t))]}\right) = \frac{1}{\sin^2 t}\left(1 - \frac{1}{p^2[f^{-1}(t)]}\right) = h(t)$$

for $t \neq k\pi$, is $h \in C^0(j)$, which proves the first property in B^0 and $h(t + \pi) = h(t)$ for $t \in j$, which proves the property A^0.

The function $h(t) \,.\, \sin^2 t = \left(1 - \dfrac{1}{p^2[f^{-1}(t)]}\right)$ for $t \in j$ and so $h(t) \,.\, \sin^2 t \in C^{(2)}(j)$, which proves the second property in B^0 and $1 - h(t) \,.\, \sin^2 t > 0$ for $t \in j$, which proves the property D^0.

Finally, we verify the property C^0:

$$\int_0^\pi h(t)\,\mathrm{d}t = \int_0^\pi \left(1 - \frac{1}{p^2[f^{-1}(t)]}\right) \cdot \frac{1}{\sin^2 t}\,\mathrm{d}t =$$

$$= \int_{f^{-1}(0)}^{f^{-1}(\pi)} \left(1 - \frac{1}{p^2(s)}\right) \cdot \frac{f'(s)}{\sin^2 f(s)}\,\mathrm{d}s =$$

$$= \int_0^{\varphi(c)} \left(1 - \frac{1}{p^2(s)}\right) \cdot \frac{f'(s)}{\sin^2 f(s)}\,\mathrm{d}s = 0, \qquad \text{for } y = \frac{\sin f(t)}{\sqrt{f'(t)}}$$

is a solution discussed in the Theorem of O. Borůvka.

With the aid of Theorem 1 and the Theorem of O. Borůvka we can prove the following statement.

Theorem 2.

The coefficients $\bar{q}$ of all differential equations $(\bar{q})$ possessing the basic central dispersion of the 1st kind $\varphi = t + \pi$ may be expressed in the form

$$\bar{q} = -1 + \frac{3}{4}\,\frac{[2 \sin (t - c) \,.\, \cos (t - c) \,.\, h(t - c) + \sin^2 (t - c) \,.\, h'(t - c)]^2}{[1 - \sin^2 (t - c) \,.\, h(t - c)]^2} +$$

$$+ \frac{1}{2}\,\frac{[6 \cos^2 (t - c) - 2 \sin^2 (t - c)] \,.\, h(t - c) + 6 \sin (t - c) \cos (t - c) \,.\, h'(t - c) + \sin^2 (t - c) \,.\, h''(t - c)}{1 - \sin^2 (t - c) \,.\, h(t - c)},$$

where the function $h(t)$ satisfies the conditions $0^0 - D^0$ of Theorem 1.

Proof. Putting $y = \sin(t - c)$, then $y' = \cos(t - c)$. With respect to (*) we get then

$$p = [1 - \sin^2(t - c) \,.\, h(t - c)]^{-1/2}$$

$$p' = \frac{1}{2}[1 - \sin^2(t - c) \,.\, h(t - c)]^{-3/2} \times$$
$$\times [2 \sin(t - c) \,.\, \cos(t - c) \,.\, h(t - c) + \sin^2(t - c) \,.\, h'(t - c)],$$

$$p'' = \frac{3}{4}[1 - \sin^2(t - c) \,.\, h(t - c)]^{-5/2} \times$$
$$\times [2 \sin(t - c) \,.\, \cos(t - c) \,.\, h(t - c) + \sin^2(t - c) \,.\, h'(t - c)]^2 +$$
$$+ \frac{1}{2}[1 - \sin^2(t - c) \,.\, h(t - c)]^{-3/2} \times$$
$$\times [2 \cos^2(t - c) \,.\, h(t - c) - 2 \sin^2(t - c) \,.\, h(t - c) +$$
$$+ 2 \sin(t - c) \cos(t - c) \,.\, h'(t - c) + 2 \sin(t - c) \,.\, \cos(t - c) \,.\, h'(t - c) +$$
$$+ \sin^2(t - c) \,.\, h''(t - c)].$$

Therefore

$$p''/p = \frac{3}{4}[2 \sin(t - c) \,.\, \cos(t - c) \,.\, h(t - c) + \sin^2(t - c) \,.\, h'(t - c)]^2 \times$$
$$\times [1 - \sin^2(t - c) \,.\, h(t - c)]^{-2} + \frac{1}{2}[2 \cos^2(t - c) \,.\, h(t - c) -$$
$$- 2 \sin^2(t - c) \,.\, h(t - c) + 4 \sin(t - c) \cos(t - c) \,.\, h'(t - c) +$$
$$+ \sin^2(t - c) \,.\, h''(t - c)]\, [1 - \sin^2(t - c) \,.\, h(t - c)]^{-1},$$

$$p'/p = \frac{1}{2}[2 \sin(t - c) \,.\, \cos(t - c) \,.\, h(t - c) + \sin^2(t - c) \,.\, h'(t - c)] \times$$
$$\times [1 - \sin^2(t - c) \,.\, h(t - c)]^{-1}.$$

Next we have

$$2 \,.\, \frac{y'}{p} \cdot \frac{p'}{y} = \frac{\cos(t - c) \,.\, [2 \cos(t - c) \,.\, h(t - c) + \sin(t - c) \,.\, h'(t - c)]}{[1 - \sin^2(t - c) \,.\, h(t - c)]},$$

$$\frac{p''}{p} + 2 \,.\, \frac{y'}{p} \cdot \frac{p'}{y} =$$

$$= \frac{3}{4} \cdot \frac{[2 \sin(t - c) \,.\, \cos(t - c) \,.\, h(t - c) + \sin^2(t - c) \,.\, h'(t - c)]^2}{[1 - \sin^2(t - c) \,.\, h(t - c)]^2} +$$

$$+ \frac{1}{2} \frac{\begin{array}{c}[6 \cos^2(t - c) \,.\, h(t - c) - 2 \sin^2(t - c)] \,.\, h(t - c) + \\ + 6 \sin(t - c \;.\, \cos(t - c) \,.\, h'(t - c) + \sin^2(t - c) \,.\, h''(t - c)]\end{array}}{[1 - \sin^2(t - c) \,.\, h(t - c)]}$$

whence the statement follows, because $q = -1$.

REFERENCES

[1] Borůvka, O.: *Lineare Differentialtransformationen 2. Ordnung*, VEB DVW, Berlin 1967.

SOUHRN

PŘÍSPĚVEK K LINEÁRNÍM DIFERENCIÁLNÍM ROVNICÍM 2. ŘÁDU S TOUŽ ZÁKLADNÍ CENTRÁLNÍ DISPERSÍ 1. DRUHU

MIROSLAV LAITOCH

Uvažujeme lineární diferenciální rovnici 2. řádu $(q) : y'' = q(t) . y$ se spojitým koeficientem q a s oscilujícími řešeními v intervalu $j = (-\infty, \infty)$ a k diferenciální rovnici (q) příslušnou základní centrální dispersi 1. druhu φ. O. Borůvka v [1] dokázal, že nosiče $(\bar{q})$ všech diferenciálních rovnic $y'' = \bar{q}(t) . y$ s touž základní centrální dispersí 1. druhu φ jsou dány vzorcem

$$\bar{q} = q + \frac{p''}{p} + 2\frac{y'}{p} \cdot \frac{p'}{y},$$

kde y je řešení diferenciální rovnice (q) a funkce p je charakterizována jistými pěti vlastnostmi.

V článku je konstruována funkce p pomocí periodické funkce $h = h(t)$, pro niž jsou nalezeny nutné a postačující podmínky.

РЕЗЮМЕ

ЗАМЕЧАНИЕ К ЛИНЕЙНЫМ ДИФФЕРЕНЦИАЛНЫМ УРАВНЕНИЯМ 2-го ПОРЯДКА С ОДИНАКОВОЙ ЦЕНТРАЛЬНОЙ ДИСПЕРСИЕЙ 1-го РОДА

МИРОСЛАВ ЛАЙТОХ

Рассматривается дифференциальное уравнение 2-го порядка $(q) : y'' = \bar{q}(t)\, y$ с непрерывным коэффициентом q и с осцилирующими решениями в интервале $j = (-\infty, \infty)$ и соответствующая основная центральная дисперсия 1-го рода φ. О. Борувка в [1] доказал, что носители $(\bar{q})$ всех дифференциальных уравнений $y'' = \bar{q}(t) . y$ с одинаковой центральной дисперсией 1-го рода φ определены выражением

$$\bar{q} = q + \frac{p''}{p} + 2\frac{y'}{p} \cdot \frac{p'}{y},$$

где y является решением дифференциального уравнения (q) и функция p характеризуется определенными пятью свойствами.

В настоящей работе конструрируется функция p с помощью периодической функцией $h = h(t)$ для которой найдены необходимые и достаточные условия.