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ON THE GENERALIZED FLOQUET THEORY OF DISCONJUGATE DIFFERENTIAL EQUATIONS $y'' = q(t)\, y$

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1. INTRODUCTION

The Floquet theory of a differential equation

$$\text{(q)} \qquad y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}),$$

($\mathbf{R} = (-\infty, \infty)$), describes the properties of solutions of (q) when the function q is periodic, usually with period π: $q(t + \pi) = q(t)$ for $t \in \mathbf{R}$. The whole theory is based on the fact that with every solution $u(t)$ of (q) also $u(t + \pi)$ is a solution of this equation.

By the Floquet theory there is (uniquely) associated a quadratic algebraic equation to every equation (q) possessing a π-periodic coefficient q whose roots – the so-called characteristic multipliers of (q) – play an important role in investigating the properties of solutions of (q). In [2–5, 10, 11, 15, 16] are expressed the characteristic multipliers of (q) by means of phases and central dispersions of (q) under the assumption that (q) is bothside oscillatory (on $\mathbf{R}$). There are also investigated the bothside oscillatory equations of the type (q) with given characteristic multipliers. Under the assumption that (q) is nonoscillatory (then necessarily disconjugate) on $\mathbf{R}$, the characteristic multipliers of (q) are expressed in [17] by means of hyperbolic and parabolic phases of this equation.

Borůvka [1] investigated all functions X – the so-called dispersions (of the 1st kind) of (q) – characterized by a property that the function $\frac{uX(t)}{\sqrt{|X'(t)|}}$ is a solution of this equation (generally on a subinterval of $\mathbf{R}$) for every solution u of (q). On this basis the Floquet theory was generalized by Laitoch [9] even for equations of the type (q), whose coefficient is not generally a π-periodic function. To every bothside oscillatory equation (q) and to every dispersion X of (q), $X \neq \mathrm{id}_{\mathbf{R}}$, may be uniquely associated a quadratic algebraic equation, whose roots are called the characteristic multipliers of (q) relative to the dispersion X. These roots are expressed by means of phases and dispersions of (q) in [12, 13].

Our object is now to express the characteristic multipliers of (q) relative to the dispersion X in assuming that (q) is disconjugate on $\mathbf{R}$, making use of dispersions and hyperbolic and parabolic phases of (q). There is also described a structuse of disconjugate equations of type (q) with given characteristic multipliers relative to the same dispersion X. Finally, there is described a structure of dispersions of (q) relative to which this equation has given characteristic multipliers. This article generalizes the results of [17] where the coefficient q of the disconjugate equation (q) is supposed to be a π-periodic function and $X = t + \pi$.

2. BASIC DEFINITIONS, NOTATIONS AND RELATIONS

In what follows we investigate equations (q) disconjugate on $\mathbf{R}$, that is, every nontrivial solution of (q) has at most one zero on $\mathbf{R}$. Trivial solutions are excluded from our considerations.

Convention.

f^{-1} will denote the inverse function (if any) to f. Let $\mathrm{S} \subset \mathbf{R}$. Then id_{S} will denote the identity mapping of S. Composite functions such as $\beta[X(t)]$, $\beta_1(h)$ will be written in short $\beta X(t)$, $\beta_1 h$.

In accordance with [1, 5] we say that a function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha \in C^0(\mathbf{R})$ is a (first) phase of (q) if there exist independent solutions u, v of (q) satisfying

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{for } t \in \mathbf{R} - \{t \in \mathbf{R};\, v(t) = 0\}.$$

Any phase α of a (disconjugate) equation (q) has the following properties:

$$\alpha \in C^3(\mathbf{R}), \qquad \alpha'(t) \neq 0, \qquad |\lim_{t\to-\infty} \alpha(t) - \lim_{t\to\infty} \alpha(t)| \leqq \pi,$$

$$-\{\alpha, t\} - \alpha'^2(t) = q(t),$$

where $\{\alpha, t\} = \dfrac{1}{2}\dfrac{\alpha'''(t)}{\alpha'(t)} - \dfrac{3}{4}\left(\dfrac{\alpha''(t)}{\alpha'(t)}\right)^2$ is the Schwarzian derivative of the function α.

$\mathfrak{E}$ denotes a set of the phases of the differential equation $y'' = -y$. The set $\mathfrak{E}$ is a group with respect to the composition of functions. It holds for every $\varepsilon \in \mathfrak{E}$ that $\varepsilon(t + \pi) = \varepsilon(t) + \pi \,.\, \operatorname{sign} \varepsilon'$. The function $\varepsilon \in C^0(\mathbf{R})$ belongs to $\mathfrak{E}$ exactly if there exist numbers a_{ij} $(i, j = 1, 2)$, $\det (a_{ij}) \neq 0$, such that

$$\operatorname{tg} \varepsilon(t) = \frac{a_{11} + a_{12} \operatorname{tg} t}{a_{21} + a_{22} \operatorname{tg} t}$$

for $t \in \mathbf{R}$, where the expressions on both sides of the last formula are meaningful. If α is a phase of (q), then $\mathfrak{E}\alpha := \{\varepsilon\alpha;\, \varepsilon \in \mathfrak{E}\}$ is the set of phases of (q).

A function $X \in C^3(\mathrm{S})$, $X'(t) \neq 0$ for $t \in \mathrm{S} \subset \mathbf{R}$ which is a solution (on S) of the nonlinear differential equation

$$\text{(qq)} \qquad -\{X, t\} + X'^2 \,.\, q(X) = q(t)$$

is called the dispersion (of the 1st kind) of (q). If X is a dispersion of (q) defined on S, then the function $\dfrac{uX(t)}{\sqrt{|X'(t)|}}$ is for every solution u of (q) a solution of (q) (on S). The function $X(t) := t + \pi$, $t \in \mathbf{R}$, is a dispersion of (q) exactly if q is a π-periodic function. The dispersions of (q) are not generally defined on $\mathbf{R}$. Let us say that the dispersion X of (q) is complete if it defined on $\mathbf{R}$ and $X(\mathbf{R}) = \mathbf{R}$.

Let α be a phase of (q). A function X is a dispersion of (q) on S if there exists $\varepsilon \in \mathfrak{E}$: $X(t) = \alpha^{-1}\varepsilon\alpha(t)$ for $t \in \mathrm{S}$ and conversely, for every $\varepsilon \in \mathfrak{E}$ the composite function $\alpha^{-1}\varepsilon\alpha$ is a dispersion of (q) (on an interval, where the composite function $\alpha^{-1}\varepsilon\alpha$ is defined). Let X be a dispersion of (q) and let $X(t) = t$ on an interval S. Then it follows from the existence and uniqueness theorem of solutions of (qq) that $X = \mathrm{id}_{\mathrm{R}}$.

We say that (q) is generally (specially) disconjugate (on $\mathbf{R}$) if for a (and then for every) phase α of (q) is $|\lim_{t\to-\infty} \alpha(t) - \lim_{t\to\infty} \alpha(t)| < \pi$ ($|\lim_{t\to-\infty} \alpha(t) - \lim_{t\to\infty} \alpha(t)| = \pi$). The equation (q) is specially disconjugate exactly if there exists a unique (up to the multiplicative constant) solution u of (q) satisfying $u(t) \neq 0$ for $t \in \mathbf{R}$.

All the foregoing definitions and results are given in [1, 5].

Say (in accordance with [6]) that a function $\beta \in C^0(\mathrm{S})$, $\mathrm{S} \subset \mathbf{R}$, is a (first) hyperbolic phase of (q) on S if there exist independent solutions u, v of (q) satisfying $|u(t)| < |v(t)|$ for $t \in \mathrm{S}$ and

$$\operatorname{tgh} \beta(t) = \frac{u(t)}{v(t)} \qquad \text{for } t \in \mathrm{S}.$$

Then $\beta \in C^3(\mathrm{S})$, $\beta'(t) \neq 0$, $-\{\beta, t\} + \beta'^2(t) = q(t)$ for $t \in \mathrm{S}$. The equation (q) is generally disconjugate exactly if there exists a hyperbolic phase β of (q) on $\mathbf{R}$ for which $\beta(\mathbf{R}) = \mathbf{R}$.

Say (in accordance with [7, 8]) that a function $\gamma \in C^0(\mathrm{S})$, $\mathrm{S} \subset \mathbf{R}$, is a (first) parabolic phase of (q) on S if there exist independent solutions u, v of (q) satisfying $v(t) \neq 0$ for $t \in \mathrm{S}$ and

$$\gamma(t) = \frac{u(t)}{v(t)} \qquad \text{for } t \in \mathrm{S}.$$

Then $\gamma \in C^3(\mathrm{S})$, $\gamma'(t) \neq 0$ and $-\{\gamma, t\} = q(t)$ for $t \in \mathrm{S}$. The equation (q) is specially disconjugate exactly if there exists a parabolic phase γ of (q) on $\mathbf{R}$ for which $\gamma(\mathbf{R}) = \mathbf{R}$.

Let $\mathfrak{G}$ be a set of functions f such that $f \in C^3(\mathbf{R})$, $f(\mathbf{R}) = \mathbf{R}$ and $f'(t) \neq 0$ for $t \in \mathbf{R}$. The set $\mathfrak{G}$ is a group with respect to the composition of functions and $\mathfrak{E}$ is a subgroup of $\mathfrak{G}$.

3. PREPARATORY, LEMMAS

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be a complete dispersion of the disconjugate equation (q) and let u, v be its independent solutions. Then $\frac{uX(t)}{\sqrt{|X'(t)|}}, \frac{vX(t)}{\sqrt{|X'(t)|}}$ are also independent solutions of (q) on $\mathbf{R}$ and there exist therefore real numbers a_{ij} $(i, j = 1, 2)$, $\det a_{ij} \neq 0$:

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = a_{11}u(t) + a_{12}v(t),$$

$$\frac{vX(t)}{\sqrt{|X'(t)|}} = a_{21}u(t) + a_{22}v(t).$$

Let a solution z of (q) exist such that $\frac{zX(t)}{\sqrt{|X'(t)|}} = \lambda \,.\, z(t)$ for $t \in \mathbf{R}$, where λ is a (generally complex) number. Then λ is a root of equation

$$\varrho^2 - (a_{11} + a_{22})\,\varrho + \det a_{ij} = 0. \qquad (1)$$

The coefficients of (1) are independent of the choice of the independent solutions u, v of (q). Equation (1) is called the characteristic equation of (q) relative to the dispersion X and its roots are called characteristic multipliers of (q) relative to the dispersion X (see [12]). If there does not exist any solution z of (q) possessing the above properties, we say that (q) does not possess any characteristic multipliers relative to the dispersion X. Analogous to the proof of Lemma 4 [12] we may show: $\det a_{ij} = \operatorname{sign} X'$.

Let ϱ_{-1}, ϱ_1 be the characteristic multipliers of (q) relative to the dispersion X. Then it follows from [9] the existence of the independent solutions u, v of (q) satisfying either

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \varrho_{-1} \,.\, u(t), \qquad \frac{vX(t)}{\sqrt{|X'(t)|}} = \varrho_1 \,.\, v(t), \qquad \varrho_{-1} \,.\, \varrho_1 = \operatorname{sign} X' \qquad (2)$$

or

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \varrho_1 \,.\, u(t), \qquad \frac{vX(t)}{\sqrt{|X'(t)|}} = u(t) + \varrho_1 \,.\, v(t), \qquad \varrho_{-1} = \varrho_1, \qquad \varrho_1^2 = 1. \qquad (3)$$

Lemma 1.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be a complete dispersion of (q). *Then the equation* (q) *relative to the dispersion X has the characteristic multipliers only if* $\operatorname{sign} X' = 1$. *These roots are then real and positive. If* 1 *is a characteristic multiplier of* (q) *relative to the dispersion X, then there exist independent solutions u, v of* (q) *satisfying* (3).

Proof. Let $X \neq \mathrm{id}_{\mathbf{R}}$ be a complete dispersion of (q). Let (q) has characteristic multipliers relative to the dispersion X denoted by ϱ_{-1}, ϱ_1. Let $\operatorname{sign} X' = -1$. Then

$\varrho_{-1} \cdot \varrho_1 = -1$. Hence there exist independent solutions u, v of (q) for which (2) holds and consequently also

$$uX(t) \,.\, vX(t) = X'(t) \,.\, u(t) \,.\, v(t), \qquad t \in \mathbf{R}. \tag{4}$$

Let t_0 be an arbitrary number for which $X(t_0) \neq t_0$. Now, by our assumption, $X'(t_0) < 0$ and from (4) follows the existence of at least one zero of the function $u \,.\, v$ on the closed interval with the boundary points t_0 and $X(t_0)$. We deduce, using our assumption $X \neq \mathrm{id}_{\mathbf{R}}$, that $X(t) \not\equiv t$ on any interval – thus (q) is oscillatory, which contradicts our assumption.

Let sign $X' = 1$. If ϱ_{-1}, ϱ_1 are complex numbers, then analogous to [12] we can prove that they are equal to $e^{\pm a\pi i}$, where $0 < a < 1$ and there exists a phase α of (q) and an integer n: $\alpha X(t) = \alpha(t) + (2n + a)\,\pi$, $t \in \mathbf{R}$. However then $\alpha(\mathbf{R}) = \mathbf{R}$ and therefore (q) is oscillatory. Consequently the equation (q) relative to the dispersion X may have real characteristic multipliers only.

Suppose $\varrho_{-1} < 0$, $\varrho_1 < 0$. Then there necessarily exists a solution u of (q): $\dfrac{uX(t)}{\sqrt{|X'(t)|}} = \varrho \,.\, u(t)$, $t \in \mathbf{R}$, where $\varrho(< 0)$ is one of the numbers ϱ_{-1}, ϱ_1. Let t_0 be an arbitrary number, $X(t_0) \neq t_0$. Then the solution u has at least one zero in the closed interval with end points t_0 and $X(t_0)$, which conflicts with our assumptoin on disconjugacy of (q).

Let us assume finally that $\varrho_{-1} = \varrho_1 = 1$ and that there exist independent solutions u, v of (q) satisfying (2). Then we have for every solution z of (q) that $\dfrac{zX(t)}{\sqrt{|X'(t)|}} = z(t)$ for $t \in \mathbf{R}$. Let $X(t_0) \neq t_0$ and let z_1 be a solution of (q), $z_1(t_0) = 0$. Then $z_1 X(t_0) = 0$, hence z_1 has at least two zero, which is a contradiction.

Remark 1.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be a complete dispersion of (q). It becomes evident from Lemma 1 that the investigation of the characteristic multipliers of (q) is meaningful only in increasing complete dispersions. The characteristic multipliers of (q) relative to the given dispersions are expressable in the form ϱ, ϱ^{-1}, where $\varrho \geqq 1$.

In the following two lemmas we investigate a set of all increasing complete dispersions of (q). We show that this set is always dependet on at least one parameter and is therefore "sufficiently rich".

Lemma 2.

Let (q) *be a generally disconjugate equation. Then the set of increasing complete dispersions of* (q) *form a group dependent on one parameter.*

Proof. Let (q) be a generally disconjugate equation. Then, by the Theorem [1, p. 82], there exists a phase α of (q): $\alpha(\mathbf{R}) = \left(0, \dfrac{\pi}{2}\right)$. Let us put $\mathfrak{E}_1 := \left\{\varepsilon \in \mathfrak{E},\ \varepsilon(0) = 0,\ \varepsilon\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}\right\}$. Since $\alpha^{-1}\mathfrak{E}\alpha$ is the set of dispersions of (q), it is obvious that $\alpha^{-1}\mathfrak{E}_1\alpha$ is the set of increasing complete dispersions of (q). It follows from the definition of group $\mathfrak{E}$ and from the set $\mathfrak{E}_1$: if $\varepsilon \in \mathfrak{E}_1$, then there exists a number $k > 0$ and

$$\operatorname{tg} \varepsilon(t) = k \,.\, \operatorname{tg} t, \qquad t \in \mathbf{R} - \left\{\frac{\pi}{2} + j\pi, j = 0, \pm 1, \pm 2, \ldots\right\} \tag{5}$$

and conversely: if $k > 0$ is a number, $\varepsilon \in C^0(\mathbf{R})$ meets (5) and $\varepsilon(0) = 0$, then $\varepsilon \in \mathfrak{E}_1$. Consequently the set of increasing complete dispersions of (q) is dependent on one (positive) parameter. It remains to prove that the set $\alpha^{-1}\mathfrak{E}_1\alpha$ is a group. It suffices to show that $\mathfrak{E}_1$ is a subgroup of the group $\mathfrak{E}$. Let $\varepsilon_1, \varepsilon_2 \in \mathfrak{E}_1$. Then there exist positive numbers k_1, k_2: $\operatorname{tg} \varepsilon_1(t) = k_1 \,.\, \operatorname{tg} t$, $\operatorname{tg} \varepsilon_2(t) = k_2 \,.\, \operatorname{tg} t$. From $\operatorname{tg} \varepsilon_1\varepsilon_2(t) = k_1 \,.\, \operatorname{tg} \varepsilon_2(t) = k_1k_2 \,.\, \operatorname{tg} t$, $\operatorname{tg} \varepsilon_1^{-1}(t) = k_1^{-1} \,.\, \operatorname{tg} t$ then it follows $\varepsilon_1^{-1}, \varepsilon_1\varepsilon_2 \in \mathfrak{E}_1$, which was to be proved.

Corollary 1.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ is an increasing complete dispersion of a generally disconjugate equation (q). *Then $X(t) \neq t$ for $t \in \mathbf{R}$.*

Proof. Let α be a phase of a generally disconjugate equation (q), $\alpha(\mathbf{R}) = \left(0, \frac{\pi}{2}\right)$ and let $\mathfrak{E}_1$ be similarly defined as in the proof in Lemma 2. Then there exists $\varepsilon \in \mathfrak{E}_1$ such that $X = \alpha^{-1}\varepsilon\alpha$. Evidently $X(t_0) = t_0$ exactly if $\varepsilon(t_1) = t_1$ for $t_1 := \alpha(t_0) \in \left(0, \frac{\pi}{2}\right)$. Since $\operatorname{tg} \varepsilon(t) = k \,.\, \operatorname{tg} t$ for a positive number k, $k \neq 1$, we obtain $\operatorname{tg} t_1 = \operatorname{tg} \varepsilon(t_1) = k \,.\, \operatorname{tg} t_1$, which is a contradiction.

Lemma 3.

Let (q) *be a specially disconjugate equation. Then the set of increasing complete dispersions of* (q) *form a group depending on two parameters.*

Proof. Let (q) be a specially disconjugate equation. By the Theorem [1, p. 82] there exists a phase α of (q) from which $\alpha(\mathbf{R}) = (0, \pi)$. Let us put $\mathfrak{E}_2 := \{\varepsilon \in \mathfrak{E}, \varepsilon(0) = 0, \operatorname{sign} \varepsilon' = 1\}$. Then $\alpha^{-1}\mathfrak{E}_2\alpha$ is the set of increasing complete dispersions of (q). If $\varepsilon \in \mathfrak{E}_2$, then there exist numbers $k_1 > 0$, k_2:

$$\operatorname{tg} \varepsilon(t) = \frac{\operatorname{tg} t}{k_1 + k_2 \operatorname{tg} t} \tag{6}$$

(for all $t \in \mathbf{R}$ where the expressions on both sides of (6) are meaningful) and also reversely: if $k_1 > 0$, k_2 are arbitrary numbers and $\varepsilon \in C^0(\mathbf{R})$ meets (6) and $\varepsilon(0) = 0$, then $\varepsilon \in \mathfrak{E}_2$. Therefore the set of increasing complete dispersions of (q) depends on two parameters. Let $\varepsilon_1, \varepsilon_2 \in \mathfrak{E}_2$. Then there exist numbers $k_1 > 0$, k_2, $k_3 > 0$, k_4 : $\operatorname{tg} \varepsilon_1(t) = \dfrac{\operatorname{tg} t}{k_1 + k_2 \operatorname{tg} t}$, $\operatorname{tg} \varepsilon_2(t) = \dfrac{\operatorname{tg} t}{k_3 + k_4 \operatorname{tg} t}$. From the equalities $\operatorname{tg} \varepsilon_1\varepsilon_2(t) = \dfrac{\operatorname{tg} \varepsilon_2(t)}{k_1 + k_2 \operatorname{tg} \varepsilon_2(t)} = \dfrac{\operatorname{tg} t}{k_1k_3 + (k_1k_4 + k_2) \operatorname{tg} t}$, $\operatorname{tg} \varepsilon_1^{-1}(t) = \dfrac{\operatorname{tg} t}{k_1^{-1} - k_1^{-1}k_2 \operatorname{tg} t}$,

$\varepsilon_1\varepsilon_2(0) = \varepsilon_2(0) = 0, \varepsilon_1^{-1}(0) = 0$ then follows that ε_1^{-1}, $\varepsilon_1\varepsilon_2 \in \mathfrak{E}_2$ and consequently $\mathfrak{E}_2$ is a subgroup of the group $\mathfrak{E}$ and $\alpha^{-1}\mathfrak{E}_2\alpha$ is a group.

Corollary 2.

Let (q) *be a specially disconjugate equation. Then the set of increasing complete dispersions X of* (q), *for which $X(t) \neq t$ for $t \in \mathbf{R}$, depends on one parameter.*

Proof. Let (q) be a specially disconjugate equation and α be one of its phases, $\alpha(\mathbf{R}) = (0, \pi)$. Let $\mathfrak{E}_2$ be similarly defined as in the proof of Lemma 3 and $\varepsilon \in \mathfrak{E}_2$. Then there exist numbers $k_1 > 0$, k_2, for which (6) holds and $X := \alpha^{-1}\varepsilon\alpha$ is an increasing complete dispersion of (q). It is easy to see that $X(t) \neq t$ for $t \in \mathbf{R}$ exactly if $\varepsilon(t) \neq t$ for $t \in (0, \pi)$ which obviously occurs if and only if $k_1 = 1$ and $k_2 \neq 0$.

Corollary 3.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be an increasing complete dispersion of a specially disconjugate equation (q). *Then the equation $X(t) - t = 0$ has at most one root on $\mathbf{R}$.*

Proof. Let α be a phase of a specially disconjugate equation (q), $\alpha(\mathbf{R}) = (0, \pi)$ and let $X = \alpha^{-1}\varepsilon\alpha$, where $\varepsilon \in \mathfrak{E}_2$. Then for $t_0 \in \mathbf{R}$ we have $X(t_0) = t_0$ exactly if $\varepsilon(t_1) = t_1$ for $t_1 := \alpha(t_0) \in (0, \pi)$. To prove our assertion of Corollary 3 it suffices to show that the equation $\varepsilon(t) = t$ has at most one root on interval $(0, \pi)$. Since $\varepsilon \in \mathfrak{E}_2$, there exist numbers $k_1 > 0$, k_2 such that (6) holds. If $k_1 = 1$ and $k_2 \neq 0$, then it follows from the proof of Corollary 2 that $\varepsilon(t) \neq t$ for $t \in (0, \pi)$. Let $k_1 > 0$, $k_2 = 0$. By our assumption $X \neq \mathrm{id}_{\mathbf{R}}$ and therefore $k_1 \neq 1$ and $\varepsilon(t) = t$ exactly for $t = \frac{\pi}{2}$. Let $0 < k_1 \neq 1$, $k_2 \neq 0$. Then the equation $\varepsilon(t) = t$ has the solution t_1 on the interval $(0, \pi)$ if and only if $\mathrm{tg}\, t_1 = (1 - k_1)\, k_2^{-1}$. Thus the equation $X(t) - t = 0$ has at most one root on $\mathbf{R}$.

4. THEOREMS ON THE EXPRESSION OF THE CHARACTERISTIC MULTIPLIERS OF A DISCONJUGATE EQUATION (q) RELATIVE TO THE DISPERSION X

Theorem 1.

Let X be an increasing complete dispersion of a disconjugate equation (q), *$X(t) \neq t$ for $t \in \mathbf{R}$. Then:*

a) numbers ϱ, ϱ^{-1}, where $\varrho > 1$, are the characteristic multipliers of (q) *relative to the dispersion X precisely if* (q) *is generally disconjugate and there exists a hyperbolic phase β of* (q) *on $\mathbf{R}$:*

$$\beta X(t) = \beta(t) + a, \qquad t \in \mathbf{R}, \tag{7}$$

where $a = \ln \varrho (> 0)$,

b) the equation (q) *relative to the dispersion X has a double characteristic multiplier* ($\asymp 1$) *precisely if* (q) *is specially disconjugate and there exists a parabolic phase γ*

of (q) *on* **R**:

$$\gamma X(t) = \gamma(t) + 1, \qquad t \in \mathbf{R}. \tag{8}$$

Proof. Let X be an increasing complete dispersion of a disconjugate equation (q) and let $X(t) \neq t$ for $t \in \mathbf{R}$. Let next ϱ, ϱ^{-1} ($\varrho \geqq 1$) be the characteristic multipliers of (q) relative to the dispersion X.

a) ($\Rightarrow$) Let $\varrho > 1$. Then there exist independent solutions u, v of (q) satisfying

$$\frac{uX(t)}{\sqrt{X'(t)}} = \varrho \,.\, u(t), \qquad \frac{vX(t)}{\sqrt{X'(t)}} = \varrho^{-1} \,.\, v(t), \qquad t \in \mathbf{R}. \tag{9}$$

Evidently the solutions u, v do not have any zero and we may without any loss of generality suppose that $u(t) > 0$, $v(t) > 0$ for $t \in \mathbf{R}$ and $| uv' - u'v | = 2$. Then the equation (q) is generally disconjugate and according to Lemma 2 [17] there exists a hyperbolic phase β of (q) on **R**, so that

$$u(t) = \frac{e^{\beta(t)}}{\sqrt{|\beta'(t)|}}, \qquad v(t) = \frac{e^{-\beta(t)}}{\sqrt{|\beta'(t)|}}, \qquad t \in \mathbf{R}. \tag{10}$$

From (9) it follows

$$\frac{uX(t)}{vX(t)} = \varrho^2 \frac{u(t)}{v(t)}$$

and further

$$e^{2\beta X(t)} = \varrho^2 e^{2\beta(t)} = e^{2(\beta(t)+a)},$$

where $a = \ln \varrho (> 0)$. Thus $\beta X(t) = \beta(t) + a$ for $t \in \mathbf{R}$.

($\Leftarrow$) Let a hyperbolic phase β of (q) exist satisfying (7), where $a = \ln \varrho > 0$. Then (q) is generally disconjugate. Let the functions u, v be defined (10). Then u, v are independent solutions of (q) and

$$\frac{uX(t)}{\sqrt{X'(t)}} = \frac{e^{\beta X(t)}}{\sqrt{|\beta'X(t)\,X'(t)|}} = \frac{e^{\beta X(t)}}{\sqrt{|(\beta X(t))'|}} = \frac{e^{\beta(t)+a}}{\sqrt{|\beta'(t)|}} = e^a \frac{e^{\beta(t)}}{\sqrt{|\beta'(t)|}} = \varrho \,.\, u(t),$$

$$\frac{vX(t)}{\sqrt{X'(t)}} = \frac{e^{-\beta X(t)}}{\sqrt{|\beta'X(t)\,X't))|}} = \frac{e^{-\beta X(t)}}{\sqrt{|\beta(X(t)')|}} = \frac{e^{-\beta(t)-a}}{\sqrt{|\beta'(t)|}} = \mathrm{e}^{-a} \frac{e^{-\beta(t)}}{\sqrt{|\beta'(t)|}} = \varrho^{-1} \,.\, v(t).$$

From the above it follows that ϱ, ϱ^{-1} are the characteristic multipliers of (q) relative to the dispersion X, $\varrho > 1$.

b) ($\Rightarrow$). Let $\varrho = 1$. According to Lemma 1 there exist then independent solutions u, v of (q):

$$\frac{uX(t)}{\sqrt{X'(t)}} = u(t), \qquad \frac{vX(t)}{\sqrt{X'(t)}} = u(t) + v(t), \qquad t \in \mathbf{R}.$$

Hereby necessarily $u(t) \neq 0$ for $t \in \mathbf{R}$. Let us put $\gamma(t) := \dfrac{v(t)}{u(t)}$. Then γ is a parabolic phase of (q) on **R**, $\gamma X(t) = \dfrac{vX(t)}{uX(t)} = \dfrac{u(t) + v(t)}{u(t)} = \dfrac{v(t)}{u(t)} + 1 = \gamma(t) + 1$. The parabolic phase γ of (q) meets (8) and since $\gamma(\mathbf{R}) = \mathbf{R}$, it is evident that (q) is a specially disconjugate equation.

($\Rightarrow$) Let a parabolic phase γ of a specially disconjugate equation (q) exist, satisfying (8). Let us put $u(t) := \dfrac{\gamma(t)}{\sqrt{|\gamma'(t)|}}$, $v(t) := \dfrac{1}{\sqrt{|\gamma'(t)|}}$, $t \in \mathbf{R}$. Then u, v are independent solutions of (q) and it follows from

$$\frac{uX(t)}{\sqrt{X'(t)}} = \frac{\gamma X(t)}{\sqrt{|\gamma' X(t)\, X'(t)|}} = \frac{\gamma X(t)}{\sqrt{|(\gamma X(t))'|}} = \frac{\gamma(t)}{\sqrt{|\gamma'(t)|}} + \frac{1}{\sqrt{|\gamma'(t)|}} = u(t) + v(t),$$

$$\frac{vX(t)}{\sqrt{X'(t)}} = \frac{1}{\sqrt{|\gamma' X(t)\, X'(t)|}} = \frac{1}{\sqrt{|(\gamma X(t))'|}} = \frac{1}{\sqrt{|\gamma'(t)|}} = v(t),$$

that (q) relative to the dispersion X has a double characteristic multiplier (= 1).

Theorem 2.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be an increasing complete dispersion of a disconjugate equation (q) *and let $t_0 \in \mathbf{R}$: $X(t_0) = t_0$ exist. Then* (q) *is specially disconjugate and $\sqrt{X'(t_0)}$, $1/\sqrt{X'(t_0)}$ are the characteristic multipliers of* (q) *relative to the dispersion X and $X'(t_0) \neq 1$.*

Proof. Let $X \neq \mathrm{id}_{\mathbf{R}}$ be an increasing complete dispersion of a disconjugate equation (q) and let $t_0 \in \mathbf{R}$: $X(t_0) = t_0$ exist. By Corollary 1 the equation (q) is then specially disconjugate and by Corollary 3 there exists a single number t_0 of the above property. There exists at the same time one and only one (up to a multiplicative constant) solution u of (q): $u(t) \neq 0$ for $t \in \mathbf{R}$. Since $\dfrac{uX(t)}{\sqrt{X'(t)}}$ is also a solution of (q) without any zero on $\mathbf{R}$, there exists a number b:

$$\frac{uX(t)}{\sqrt{X'(t)}} = b \,.\, u(t), \qquad t \in \mathbf{R}. \tag{11}$$

The number b is necessarily equal to one of the characteristic multipliers of (q) relative to the dispersion X. Since $u(t_0) \neq 0$, it follows, writting t_0 for t in (11), that $b = \sqrt{X'(t_0)}$ and therefore $\sqrt{X'(t_0)}$, $1/\sqrt{X'(t_0)}$ are characteristic multipliers of (q) relative to the dispersion X. Suppose that $X'(t_0) = 1$, which implies that (q) relative, to the dispersion X has the double characteristic multiplier (= 1). There exists then according to Lemma 1, a solution v to the solution u, for which

$$\frac{uX(t)}{\sqrt{X'(t)}} = u(t),$$

$$\frac{vX(t)}{\sqrt{X'(t)}} = u(t) + v(t).$$

Writting t_0 for t in the last equality, we obtain $v(t_0) = u(t_0) + v(t_0)$, thus $u(t_0) = 0$, which is a contradiction.

5. THE STRUCTURE OF THE DISCONJUGATE EQUATIONS (q) WITH GIVEN CHARACTERISTIC MULTIPLIERS RELATIVE TO THE DISPERSION X

Let X be an increasing complete dispersion of a disconjugate equation (q). Let us put $\mathscr{S}_X := \{\alpha \in \mathfrak{G}, \alpha X = X\alpha\}$. Similarly as in Lemma 2 [13] we can prove that $\mathscr{S}_X$ is a subgroup of the group $\mathfrak{G}$.

Definition 1.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be an increasing complete dispersion of a disconjugate equation (q_1). *Say that equations* (q_1) *and* (q_2) *relative to the dispersion X have the same behaviour if:*

(i) *they have the same dispersion X,*

(ii) *both are either specially or generally disconjugate and*

(iii) *they have the same characteristic multipliers relative to the dispersion X.*

Theorem 3.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ be an increasing complete dispersion of a generally disconjugate equation (q_1) *and let β_1 be a hyperbolic phase of* (q_1) *on* $\mathbf{R}$:

$$\beta_1 X(t) = \beta_1(t) + a, \qquad a > 0, \qquad t \in \mathbf{R}. \tag{12}$$

Then (q_1) *and* (q_2) *relative to the dispersion X have the same behaviour precisely if a hyperbolic phase β_2 of* (q_2) *is expressable in the form*

$$\beta_2 = \beta_1 h,$$

where $h \in \mathscr{S}_X$.

Proof. Let $X \neq \mathrm{id}_{\mathbf{R}}$ is an increasing complete dispersion of a generally disconjugate equation (q_1) and let a hyperbolic phase β_1 of (q_1) satisfy (12).

($\Rightarrow$) Let (q_1) and (q_2) relative to the dispersion X have the same behaviour. According to Theorem 1 there exists a hyperbolic phase β_2 of (q_2) on $\mathbf{R}$: $\beta_2 X(t) = \beta_2(t) + a$ $t \in \mathbf{R}$. Putting $h := \beta_1^{-1}\beta_2$, then $\operatorname{sign} h' = 1$, $hX = \beta_1^{-1}\beta_2 X = \beta_1^{-1}(\beta_2 + a) =$, $= X\beta_1^{-1}\beta_2 = Xh$ and consequently $h \in \mathscr{S}_X$.

($\Leftarrow$) Let $h \in \mathscr{S}_X$ and $\beta_2 := \beta_1 h$ be a hyperbolic phase of (q_2). Then $\beta_2 X(t) = \beta_1 hX(t) = \beta_1 Xh(t) = \beta_1 h(t) + a = \beta_2(t) + a$ and $q_2(t) = -\{\beta_2, t\} + \beta_2'^2(t) = -\{\beta_2 X, t\} + \beta_2'^2(t) = -\{\beta_2, X(t)\} \,.\, X'^2(t) - \{X, t\} + \beta_2'^2(t) = [q_2 X(t) - \beta_2'^2 X(t)]\, X'^2(t) - \{X, t\} + \beta_2'^2(t) = -\{X, t\} + X'^2(t) \,.\, q_2 X(t)$. Thus (q_2) has the dispersion X and (q_1) and (q_2) relative to the dispersion X have the same behaviour.

Theorem 4.

Let X be an increasing complete dispersion of a specially disconjugate equation (q_1), *$X(t) \neq t$ for $t \in \mathbf{R}$. Let γ_1 be a parabolic phase of* (q_1) *on* $\mathbf{R}$ *such that*

$$\gamma_1 X(t) = \gamma_1(t) + 1, \qquad t \in \mathbf{R}. \tag{13}$$

Then (q_1) *and* (q_2) *relative to the dispersion X have the same behaviour exactly if* *a parabolic phase* γ_2 *of* (q_2) *is expressable in the form*

$$\gamma_2 = \gamma_1 h,$$

where $h \in \mathcal{S}_X$.

Proof. Let X be an increasing complete dispersion of a specially disconjugate equation (q_1), $X(t) \neq t$ for $t \in \mathbf{R}$. Let γ_1 be a parabolic phase of (q_1) on $\mathbf{R}$ satisfying (13).

($\Rightarrow$) Let (q_1) and (q_2) relative to the dispersion X have the same behaviour. According to Theorem 1 there exists then a parabolic phase γ_2 of (q_2): $\gamma_2 X(t) = \gamma_2(t) + 1$ for $t \in \mathbf{R}$. Putting $h := \gamma_1^{-1}\gamma_2$, then $hX = \gamma_1^{-1}\gamma_2 X = \gamma_1^{-1}(\gamma_2 + 1) = X\gamma_1^{-1}\gamma_2 = Xh$ and consequently $h \in \mathcal{S}_X$.

($\Leftarrow$) Let $h \in \mathcal{S}_X$ and $\gamma_2 := \gamma_1 h$ be a parabolic phase of (q_2). Then $\gamma_2(\mathbf{R}) = \mathbf{R}$ and therefore (q_2) is a specially disconjugate equation. It follows from $\gamma_2 X = \gamma_1 hX = \gamma_1 Xh = h\gamma_2 + 1$ and $q_2(t) = -\{\gamma_2, t\} = -\{\gamma_2 X, t\} = -\{\gamma_2, X(t)\} . X'^2(t) - \{X, t\} = X'^2(t) . q_2 X(t) - \{X, t\}$ that (q_2) has the dispersion X and that (q_1) and (q_2) relative to the dispersion X have the same behaviour.

Theorem 5.

Let $X \neq \mathrm{id}_{\mathbf{R}}$ *be an increasing complete dispersion of a specially disconjugate equation* (q_1) *and let there exist a number* t_0 *such that* $X(t_0) = t_0$. *Then* (q_1) *and* (q_2) *relative to the dispersion X have the same behaviour if and only if* $q_1 = q_2$.

Proof. Let the assumptions of Theorem 5 be satisfied. According to Corollary 3 $X(t) \neq t$ for $t \in \mathbf{R} - \{t_0\}$. Let ($q_1$) and ($q_2$) relative to the dispersion X have the same behaviour. Then

$$-\{X, t\} + X'^2(t) . q_1 X(t) = q_1(t),$$
$$-\{X, t\} + X'^2(t) . q_2 X(t) = q_2(t).$$

Herefrom $X'^2(t)[q_1 X(t) - q_2 X(t)] = q_1(t) - q_2(t)$ and $X'(t)\sqrt{|q_1 X(t) - q_2 X(t)|} = \sqrt{|q_1(t) - q_2(t)|}$. Integrating the last equality from t_0 to t we get

$$\int_{t_0}^{t} \sqrt{|q_1 X(s) - q_2 X(s)|}\, X'(s)\, \mathrm{d}s = \int_{t_0}^{t} \sqrt{|q_1(s) - q_2(s)|}\, \mathrm{d}s$$

and on making use of the substitution method in the integral on the left side of the last formula we obtain

$$\int_{t_0}^{X(t)} \sqrt{|q_1(s) - q_2(s)|}\, \mathrm{d}s = \int_{t_0}^{t} \sqrt{|q_1(s) - q_2(s)|}\, \mathrm{d}s$$

and

$$\int_{t}^{X(t)} \sqrt{|q_1(s) - q_2(s)|}\, \mathrm{d}s = 0, \qquad t \in \mathbf{R}.$$

Since X is not identically equal to t on any interval, it follows from the last equality that $q_1 = q_2$.

If $q_1 = q_2$, it becomes evident that (q_1) and (q_2) relative to the dispersion X have the same behaviour.

6. THE STRUCTURE OF DISPERSIONS OF THE CONJUGATE EQUATION (q) RELATIVE TO WHICH THIS EQUATION HAS GIVEN CHARACTERISTIC MULTIPLIERS

Let $\varrho > 1$ *and let* (q) *be a generally disconjugate equation. Then there exist exactly two increasing complete dispersions* X_1, X_{-1} $(\neq \mathrm{id}_{\mathbf{R}})$ *of* (q), $X_1 \neq X_{-1}$, *relative to which the equation* (q) *has the characteristic multipliers* ϱ, ϱ^{-1}.

Proof. Let the assumptions of Theorem 6 be satisfied. Then there exists a phase α of (q) meeting $\alpha(\mathbf{R}) = (0, \pi/2)$. Let $\varepsilon_1 \in C^0(\mathbf{R})$, $\varepsilon_{-1} \in C^0(\mathbf{R})$, $\varepsilon_1(0) = \varepsilon_{-1}(0) = 0$, $\operatorname{tg} \varepsilon_1(t) = \varrho^2 . \operatorname{tg} t$, $\operatorname{tg} \varepsilon_{-1}(t) = \varrho^{-2} . \operatorname{tg} t$. Let us put $X_1 := \alpha^{-1}\varepsilon_1\alpha$, $X_{-1} := \alpha^{-1}\varepsilon_{-1}\alpha$. Then X_1 and X_{-1} are increasing complete dispersions of (q). If we put $\bar{u}(t) := \dfrac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $\bar{v}(t) := \dfrac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $t \in \mathbf{R}$, then $\bar{u}, \bar{v}$ are independent solutions of (q) and it follows from

$$\frac{\bar{u}X_i(t)}{\sqrt{X_i'(t)}} = \frac{\sin \alpha X_i(t)}{\sqrt{|\alpha' X_i(t)\, X_i'(t)|}} = \frac{\sin \varepsilon_i\alpha(t)}{\sqrt{|(\varepsilon_i\alpha(t))'|}} = \varrho^i \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}} = \varrho^i . \bar{u}(t),$$

$$\frac{\bar{v}X_i(t)}{\sqrt{X_i'(t)}} = \frac{\cos \alpha X_i(t)}{\sqrt{|\alpha' X_i(t)\, X_i'(t)|}} = \frac{\cos \varepsilon_i\alpha(t)}{\sqrt{|(\varepsilon_i\alpha(t))'|}} = \varrho^{-i} \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}} = \varrho^{-i} . \bar{v}(t),$$

$$(i = -1, 1),$$

that (q) relative to the dispersion X_1 and X_{-1} has the characteristic multipliers ϱ and ϱ^{-1}.

Let Y be an increasing complete dispersion of (q) and let ϱ and ϱ^{-1} be the characteristic multipliers of (q) relative to the dispersion Y. Then there exist independent solutions u, v of (q) satisfying

$$\frac{uY(t)}{\sqrt{Y'(t)}} = \varrho . u(t), \qquad \frac{vY(t)}{\sqrt{Y'(t)}} = \varrho^{-1} . v(t), \qquad t \in \mathbf{R}, \tag{14}$$

and thus also

$$\frac{uY^{-1}(t)}{\sqrt{Y^{-1\prime}(t)}} = \varrho^{-1} . u(t), \qquad \frac{vY^{-1}(t)}{\sqrt{Y^{-1\prime}(t)}} = \varrho . v(t), \qquad t \in \mathbf{R}.$$

Therefore ϱ, ϱ^{-1} are the characteristic multipliers of (q) relative to the dispersion Y^{-1}. It follows from (14) and from Corollary 1 that $Y(t) \neq t$ and $u(t)\, v(t) \neq 0$ for $t \in \mathbf{R}$. We can assume without any loss of generality that $u(t) > 0$, $v(t) > 0$. Let $\alpha_1 \in C^0(\mathbf{R})$, $0 < \alpha_1(t) < \dfrac{\pi}{2}$ and $\operatorname{tg} \alpha_1(t) = \dfrac{u(t)}{v(t)}$ for $t \in \mathbf{R}$. Then α_1 is a phase of (q) and we get from (14):

$$\operatorname{tg} \alpha_1 Y(t) = \varrho^2 . \operatorname{tg} \alpha_1(t), \qquad t \in \mathbf{R}, \tag{15}$$

$\alpha_1(\mathbf{R}) = \left(0, \frac{\pi}{2}\right)$. Hence, there exist $\varepsilon_2 \in \mathfrak{E}_1$ (the set $\mathfrak{E}_1$ was defined in the proof of Lemma 2) and a number $k > 0$ such that $\alpha_1 = \varepsilon_2\alpha$, $\operatorname{tg} \varepsilon_2(t) = k \cdot \operatorname{tg} t$. Since $Y = \alpha_1^{-1}\varepsilon_3\alpha_1$ for an $\varepsilon_3 \in \mathfrak{E}_1$, we see that $Y = \alpha^{-1}\varepsilon\alpha$, where $\varepsilon := \varepsilon_2^{-1}\varepsilon_3\varepsilon_2$. For the proof of Theorem 6 it suffices to show that $\varepsilon = \varepsilon_1$. Since $\operatorname{tg} \alpha_1 Y = \operatorname{tg} \varepsilon_2\alpha\alpha^{-1}\varepsilon\alpha = \operatorname{tg} \varepsilon_2\varepsilon\alpha = k \cdot \operatorname{tg} \varepsilon\alpha$, $\varrho^2 \cdot \operatorname{tg} \alpha_1 = \varrho^2 \cdot \operatorname{tg} \varepsilon_2\alpha = k\varrho^2 \cdot \operatorname{tg} \alpha$, it follows from (15) that $\operatorname{tg} \varepsilon = \varrho^2 \cdot \operatorname{tg} t$. From the last formula and from the equalities $\varepsilon(0) = \varepsilon_1(0) = 0$ we get $\varepsilon = \varepsilon_1$, which was to be demonstrated.

Theorem 7.

Let $\varrho > 1$ and (q) be a specially disconjugate equation. Then there exists a set of increasing complete dispersions of (q) dependent on a single parameter relative to which (q) has the characteristic multipliers ϱ, ϱ^{-1}.

Proof. Let $\varrho > 1$ and (q) be a specially disconjugate equation. Let α be a phase of (q), $\alpha(\mathbf{R}) = (0, \pi)$. Let the set $\mathfrak{E}_2$ be defined analogous to the proof of Lemma 3. Let finally ϱ, ϱ^{-1} be the characteristic multipliers of (q) relative to an increasing complete dispersion $X \neq \mathrm{id}_{\mathbf{R}}$. According to Theorem 2 there exists then a number t_0: $X(t_0) = t_0$ and $\sqrt{X'(t_0)}$, $1/\sqrt{X'(t_0)}$ are the characteristic multipliers of (q) relative to the dispersion X. Our object now is to find all the increasing complete dispersions Y of (q) having such a property that $Y(t_1) = t_1$ and $\sqrt{Y'(t_1)}$ is equal to one of the numbers ϱ, ϱ^{-1} in a number $t_1 = t_1(Y)$. According to Corollaries 2 and 3 and by their proofs, Y is a increasing complete dispersion of (q) and there exists (a single) number t_1: $Y(t_1) = t_1$ if and only if $Y = \alpha^{-1}\varepsilon\alpha$, where $\varepsilon(0) = 0$, $\operatorname{tg} \varepsilon(t) = \dfrac{\operatorname{tg} t}{k_1 + k_2 \operatorname{tg} t}$ and there is either $k_1 > 0$, $k_2 = 0$ or $0 < k_1 \neq 1$, $k_2 \neq 0$. Hereby $Y(t_1) = t_1$ exactly if $\varepsilon(t_2) = t_2 (\in (0, \pi))$ for $t_2 := \alpha(t_1)$ and it holds: $t_2 = \frac{\pi}{2}$ for $k_1 > 0$, $k_2 = 0$ and t_2 for $0 < k_1 \neq 1$, $k_2 \neq 0$ is one and only one solution of the equation $\operatorname{tg} t = (1 - k_1) k_2^{-1}$ (on $(0, \pi)$). By a calculation we can verify that $Y'(t_1) = \varepsilon'(t_2)$ and $\varepsilon'(t_2) = k_1^{-1}$ for $t_2 = \frac{\pi}{2}$ and $\varepsilon'(t_2) = k_1$ for $t_2 \neq \frac{\pi}{2}$. Let $\mathfrak{E}_3 \subset \mathfrak{E}$ be a set of those ε satisfying

$$\operatorname{tg} \varepsilon(t) = \frac{\operatorname{tg} t}{\varrho^i + k_2 \operatorname{tg} t}$$

where $k_2 \in \mathbf{R}$ and $i = \pm 1$. Then $\alpha^{-1}\mathfrak{E}_3\alpha$ is the set of increasing complete dispersions of (q) relative to which this equation has the characteristic multipliers ϱ, ϱ^{-1}.

Remark 1.

From Corollary 2 and from Theorem 2 then follows the existence of a set $\mathfrak{D}$ of increasing dispersions of the specially disconjugate equation (q) which is dependent on one parameter relative to which (q) has a double characteristic multiplier ($= 1$). If $\mathfrak{E}_4$ is a set of those $\varepsilon \in \mathfrak{E}$, $\varepsilon(0) = 0$ satisfying (6) with $k_1 = 1$ and $k_2 \neq 0$ and if α is a phase of (q) such that $\alpha(\mathbf{R}) = (0, \pi)$, then $\mathfrak{D} = \alpha^{-1}\mathfrak{E}_4\alpha$.

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SOUHRN

ZOBECNĚNÁ FLOQUETOVA TEORIE DISKONJUGOVANÝCH DIFERENCIÁLNÍCH ROVNIC $y'' = q(t)\, y$

SVATOSLAV STANĚK

Jsou vyšetřovány rovnice typu

(q) $$y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}),$$

které jsou diskonjugované na $\mathbf{R}(= (-\infty, \infty))$. Nechť X je disperse (1. druhu) rovnice (q), $X(\mathbf{R}) = \mathbf{R}$, $X(t) \not\equiv t$. Pak pro každé řešení u rovnice (q) je také $uX(t)/|X'(t)|^{1/2}$ řešením této rovnice. Řekneme, že (obecně komplexní) číslo λ je charakteristickým kořenem rovnice (q) při dispersi X, jestliže existuje netriviální řešení z rovnice (q): $zX(t)/|X'(t)|^{1/2} = \lambda \,.\, z(t)$, $t \in \mathbf{R}$. V práci je uvedeno vyjádření charakteristických kořenů rovnice (q) při dispersi X užitím dispersí a hyperbolických a parabolických fází rovnice (q). Je popsána struktura rovnic typu (q), které při téže dispersi X mají předepsané charakteristické kořeny a dále je popsána struktura dispersí rovnice (q) při nichž má tato rovnice předepsané charakteristické kořeny.

РЕЗЮМЕ

ОБОБЩЕННАЯ МЕТОДА ФЛОКЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ $y'' = q(t)y$ БЕЗ СОПРЯЖЕННЫХ ТОЧЕК

СВАТОСЛАВ СТАНЕК

Изучается уравнение типа

(q) $$y'' = q(t)\, y, \; q \in C^0(\mathbf{R})$$

без сопряженных точек на $\mathbf{R}$ $(=(-\infty, \infty))$. Пусть X – дисперсия (1-го рода) уравнения (q), $X(\mathbf{R}) = \mathbf{R}$, $X(t) \not\equiv t$. Тогда для любого решения u уравнения (q) функция $uX(t)/\sqrt{|X'(t)|}$ является тоже решением этого уравнения. (Вообще комплексное) число λ называется характеристическим корнем уравнения (q) при дисперсии X, если существует нетривиальное решение z уравнения (q) : $zX(t)/\sqrt{|X'(t)|} = \lambda \,.\, z(t)$, $t \in \mathbf{R}$. В работе приводятся выражения характеристических корней уравнения (q) при дисперсии X с помощью дисперсий и гиперболических и параболических фаз уравнения (q). Приводится описание структуры уравнений типа (q), которые при такой же дисперсии X имеют предписанные характеристические корни и дальше описана структура дисперсий уравнения (q) при которых это уравнение имеет предписанные характеристические корни.