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SQUARE ROOTS AND QUASI-SQUARE ROOTS IN LOCALLY MULTIPLICATIVELY CONVEX ALGEBRAS

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In this paper we will show the existence of quasi-square root of element in a complete locally multiplicatively convex algebra which possesses spectrum contained in the interior of the complex unit disc. In the case of spectrum being also positive we state a simple relation between square roots and quasi-square roots which enables us to show the existence of the unique positive square root of an element with positive bounded spectrum. If, moreover, the algebra is endowed by an involution and the elements in our consideration are selfadjoint then the quasi-square roots (square roots) can be choosen selfadjoint too.

1. Introduction

The notion of semi-normed algebra was introduced by R. Arens as a natural generalisation of Banach algebras. They are called locally multiplicatively convex algebras by E. A. Michael [3]. Several properties of Banach algebras have been proved also for semi-normed algebras [3], [7], [8]. The aim of this paper is to study the existence of square roots for elements of these algebras. In the theory of Banach algebras the existence of square roots [1], [2], [5] plays an important role in problems concerning the spectral properties of elements in the non-commutative case. Speaking more closely the non-commutative case does not admitt the use of Gelfand transform in general and so the "square root" technique together with some other algebraic tools (as e.g. the polynomial identity for spectra) work. Let's mention that algebras studied in this paper are not assumed to posses a countable base of uniformity and so we cannot use the wellknown Mittag–Leffler theorem to construct the square root (quasi-square root) from its projections. So we have to find some refined methods of proofs.

2. Preliminaries

The reader is assumed to be familiar with the basic concepts concerning topological algebras, namely the Banach algebras, including spectra, Gelfand representation for the commutative case and so on. All of them as well as proofs can be found in [1] for Banach algebras and in [3], [5], [9] for the semi-normed algebras. Let's recall now some notations and facts which we shall use in this paper. An involution defined on algebra A is a mapping $x \to x^*$ of A onto itself such that the following holds for each pair $x, y \in A$ and for each complex λ:

(i) $x^{**} = x$,

(ii) $(\lambda x)^* = \bar{\lambda} x^*$,

(iii) $(x + y)^* = x^* + y^*$,

(iv) $(xy)^* = y^* x^*$.

A $*$-algebra is an algebra endowed by an involution. An element $x \in A$ is said to be regular, (selfadjoint) respectively if it holds that there exists an inverse to x ($x^* = x$) respectively. A topological algebra is said to be semi-normed, or locally multiplicatively-convex if its topology can be given by mean of a family $\{p_\alpha\}_{\alpha \in \Sigma}$ of semi-norms on A which separates points of A. The class of all locally multiplicatively convex algebras will be denoted by LMC. The spectrum of an element $x \in A$ will be denoted by $\sigma(x)$. If it is necessary to specify the algebra with respect to which the spectrum is taken we shall use the notation $\sigma(A, x)$. The spectral radius of an element $x \in A$ is denoted by $| x |_\sigma$ and it is defined as $| x |_\sigma = \sup \{| \lambda | : \lambda \in \sigma(x)\}$. Let's mention that the last number is not necessarily finite if A is semi-normed. The unit element of A (if exists) will be denoted by e and will be left in expressions like $\lambda - x$. If we set $N_\alpha = \{x \in A : p_\alpha(x) = 0\}$ for some $\alpha \in \Sigma$ we obtain a closed ideal in A. Let A_α denotes the Banach algebra obtained by the completion of the normed algebra $(A/N_\alpha, p_\alpha)$. By π_α we denote the natural homomorphism mapping from A into A_α. Let's denote by π the mapping $\pi : A \to \prod_{\alpha \in \Sigma} A_\alpha$, $\pi(x) = (\pi_\alpha(x))_{\alpha \in \Sigma}$ where $\prod_{\alpha \in \Sigma} A_\alpha$ is the cartesian product of spaces A_α endowed by the product topology and coordinatewise defined operations. This map is a topological isomorphism. If A is complete the image $\pi(A)$ is a closed subalgebra in $\prod_{\alpha \in \Sigma} A_\alpha$.

Let now A be a complete algebra from LMC with a system of semi-norms $\{p_\alpha\}_{\alpha \in \Sigma}$ as mentioned above. Write $\alpha < \beta$ for each $\alpha, \beta \in \Sigma$ if p_α is continuous with respect to p_β. This relation makes of Σ a directed set since we can assume without the loss of generality that the maximum of a finite number of members from Σ is again from Σ. If $\alpha < \beta$ we define a map $\pi_{\alpha\beta}$ from the algebra $(A/N_\beta, p_\beta)$ into $(A/N_\alpha, p_\alpha)$ by $\pi_{\alpha\beta}(\pi_\beta(x)) = \pi_\alpha(x)$. This map is a continous homomorphism of A/N_β onto A/N_α and thus it can be extended to a homomorhpism of A_β into A_α. This extended mapping will be denoted also by $\pi_{\alpha\beta}$. It's obvious that for each $\alpha, \beta, \gamma \in \Sigma$ such that $\alpha < \beta < \gamma$ holds $\pi_{\alpha\gamma} = \pi_{\alpha\beta} , \pi_{\beta\gamma}$. So we obtained a projective system of Banach spaces $(A_\alpha, \alpha \in \Sigma)$

with respect to the set of continuous homomorphisms $(\pi_{\alpha\beta}, \alpha < \beta)$. This enables us to construct the projective limit denoted by $\varprojlim A_\alpha$ of this system i.e. the subspace of $\prod_{\alpha \in \Sigma} A_\alpha$ formed by all those "sequences" $(x_\alpha)_{\alpha \in \Sigma}$ that for each $\alpha, \beta \in \Sigma, \alpha < \beta$ holds $\pi_{\alpha\beta}(x_\beta) = x_\alpha$. It's wellknown [3], [9] that $\pi(A) = \varprojlim A_\alpha$ and so we have that each complete algebra from LMC is topologicaly isomorphic to the projective limit of Banach algebras. We can identify A and the projective limit $\varprojlim A_\alpha$. This yields that an element $x \in A$ is regular iff for each $\alpha \in \Sigma$ its projection $\pi_\alpha(x)$ is regular in A_α and we easily see for each $x \in A$ the equality $\sigma(x, A) = \bigcup_{\alpha \in \Sigma} \sigma(\pi_\alpha(x), A_\alpha)$. For the spectral radius holds $| x |_\sigma = \sup | \pi_\alpha(x) |_\sigma$ where the last term is taken in A_α. We got that the spectrum $\sigma(x, A)$ is a nonempty, in general unbounded set of the complex plane. The mentioned topological isomorphism yields also that a sequence $(x_\alpha)_{\alpha \in \Sigma} \in \prod_{\alpha \in \Sigma} A_\alpha$ belongs to A iff for each pair $\alpha, \beta \in \Sigma$ such that $\alpha < \beta$ holds $\pi_{\alpha\beta}(x_\beta) = x_\alpha$.

3. Quasi-square roots and square roots

3.1. Definition: Let A be an arbitrary algebra. For each pair of its members x, y we define their quasi-product $x \,.\, y$ by formula

$$x \,.\, y = x + y - xy,$$

where xy means the usual algebra product in A.

3.2. Definition: Given $x \in A$ a quasi-square root of x is an element $y \in A$ with $y \,.\, y = 2y - y^2 = x$. A square root of x is an element $z \in A$ with $zz = x$.

Let A be a topological algebra. Given $a \in A$ we denote by $B(a)$ the least closed subalgebra of A containing a. $B(a)$ exists [1] and it is formed by the closure of the set of all polynomials $P_n(a)$. Obviously each pair of elements $x, y \in B(a)$ is a commuting pair. The subalgebra $B(a)$ is contained in the maximal commutative subalgebra $C(a)$ of A containing the element a.

Recall now the original quasi-square root theorem for Banach algebras which is due to J W. Ford [1], [2].

3.3. Theorem: Let A be a Banach algebra and let be $a \in A$ with $| a |_\sigma < 1$. Then there exists the unique quasi-square root x of a in A with $| x |_\sigma < 1$. Moreover is $x \in B(a)$.

Proof: See for [1] p. 44.

We shall make a substantial use of this result. In the rest of this paper denotes A a complete algebra from LMC endowed by topology which is given by a directed set of pseudonorms $\{p_\alpha, \alpha \in \Sigma\}$. For each $a \in A$ and for each $\alpha \in \Sigma$ we denote by a_α the projection $\pi_\alpha(a)$.

3.4. Proposition: Let $a \in A$ and $\alpha \in \Sigma$ are such that $| a_\alpha |_\sigma < 1$. Then there exists the unique quasisquare root $x \in A_\alpha$ of a_α so that $| x |_\sigma < 1$. Further holds $(e_\alpha - a_\alpha) = = (e_\alpha - x)^2$, $\operatorname{Re} \sigma(e_\alpha - x) > 0$ and $x \in B(a_\alpha)$.

Proof: The existence of unique $x \in B(a_\alpha)$ is a consequence of 3.3. and for the rest we can easily see:

$$(e_\alpha - a_\alpha) = (e - a)_\alpha = e_\alpha - 2x + x^2 = (e_\alpha - x)^2 .$$

By the condition $| x |_\sigma < 1$ and by the polynomial identity for spectrum we get

$$\operatorname{Re} \sigma(e_\alpha - x) = \operatorname{Re} \sigma(e_\alpha) - \operatorname{Re} \sigma(x) = 1 - \operatorname{Re} \sigma(x) > 0 .$$

Q.E.D.

3.5. Proposition: Let be $a \in A$ and $\alpha \in \Sigma$ such that $\sigma(a_\alpha) \geqq 0$ and $| a_\alpha |_\sigma < 1$. Then for the quasi-square root $x \in A_\alpha$ of a_α holds $\sigma(e_\alpha - x) > 0$.

Proof: It is immediately seen by 3.4. that $\operatorname{Re} \sigma(e_\alpha - x) > 0$. Further holds $(e_\alpha - a_\alpha) = (e_\alpha - x)^2$. From the fact that $\sigma(a_\alpha) \geqq 0$ follows $\sigma(e_\alpha - a_\alpha) = \sigma((e - a)_\alpha) > 0$ and so $\sigma((e_\alpha - x)^2) > 0$. By the polynomial identity for spectrum we get

$$\sigma((e_\alpha - x)^2) = (\sigma(e - x))^2 > 0 .$$

Now, we conclude by the elementary properties of complex numbers that

$$\sigma(e_\alpha - x) > 0.$$

Q.E.D.

Propositions 3.4. and 3.5. base a simple relation in the case of positiveness of spectrum between the square roots and the quasi-square roots as showes the next proposition.

3.6. Proposition: Let be $a \in A$ and $\alpha \in \Sigma$. Then the following holds:

(i) Provided $\sigma(a_\alpha) > 0$ and $| a_\alpha |_\sigma < 1$ there exists the unique square root $y \in A_\alpha$ of a_α for which holds $| y |_\sigma < 1$ and $\sigma(y) > 0$. Moreover is $y \in B(a_\alpha)$.

(ii) Provided the existence of a positive K so that $| (a_\alpha) |_\sigma < K$ and $\sigma(a_\alpha) > 0$ there exists the unique square root $y \in A_\alpha$ of a_α such that $| y |_\sigma < K^{1/2}$ and $\sigma(y) > 0$. Moreover is, again, $y \in B(a_\alpha)$.

(iii) The square root y from (i) is the unique one which posses a positive spectrum and for which holds $| y |_\sigma \leqq (| a_\alpha |_\sigma)^{1/2}$.

Proof: To prove (i) we apply 3.4. and 3.5. on the element $(e_\alpha - a_\alpha)$, for which holds that $\sigma(e_\alpha - a_\alpha) > 0$ and $| e_\alpha - a_\alpha |_\sigma < 1$. There exists the unique quasi-square root $x \in B(a_\alpha)$ for $(e_\alpha - a_\alpha)$ in A_α so that $| x |_\sigma < 1$. We obtain

$$a_\alpha = (e_\alpha - (e_\alpha - a_\alpha)) = (e_\alpha - x)^2 = e_\alpha - 2x + x^2 = e_\alpha - x \, . \, x.$$

Now we set $y = (e_\alpha - x)$ and the last element is obviously a square root of a_α. By the polynomial identity for spectra we obtain $\sigma(y^2) = (\sigma(y))^2$ and so we get that $| y |_\sigma < 1$. It remains to prove that y is the unique square root of a_α in A_α with positive spectrum and satisfying the condition $| y |_\sigma < 1$. Let's assume the converse and let $o \in A_\alpha$ be such that

$$1 > \sigma(e_\alpha - o) > 0.$$

and

$$(e_\alpha - o) \, . \, (e_\alpha - o) = e_\alpha - o^2 = e_\alpha - a_\alpha.$$

The last equality implies that $(e_\alpha - o)$ is a quasi-square root of $(e_\alpha - a_\alpha)$ and so by 3.4· we get $(e_\alpha - o) = (e_\alpha - y)$ and thus $o = y$.

Q.E.D.

To prove (ii) it's enough to apply (i) on the element a/K.

Q.E.D.

To prove the remaining part (iii) let be $\varepsilon > 0$. Let's take $K = | a_\alpha |_\sigma + \varepsilon$. Then by (ii) there exists the unique square root $y \in A_\alpha$ of a_α such that $\sigma(y) > 0$ and $| y |_\sigma < < K^{1/2} = (| a_\alpha |_\sigma + \varepsilon)^{1/2}$. Applying the standard technique we get that y is the unique square root of a_α, $y \in B(a_\alpha)$ such, that for each integer n holds

$$| y |_\sigma \leqq (| a_\alpha |_\sigma + 1/n)^{1/2}.$$

The last inequality implies $| y |_\sigma \leqq (| a_\alpha |_\sigma)^{1/2}$.

Q.E.D

3.7. Proposition: Let be $a \in A$. Let's suppose the square root $x \in A$ of a exists. Then the following holds:

(i) If $| a |_\sigma < 1$ then $| x |_\sigma < 1$ too.

(ii) If for some $\alpha \in \Sigma$ holds $| a_\alpha |_\sigma < 1$ then $| x_\alpha |_\sigma < 1$ holds too.

Proof: We prove only the first statement the second having an analogous proof. From the equality $x^2 = a$ follows that $\sigma(x^2) = (\sigma(x))^2 = \sigma(a)$ and this implies for the spectral radius

$$| a |_\sigma = \sup \{| \lambda | : \lambda \in \sigma(a)\} = \sup \{| \xi |^2 : \xi \in \sigma(x)\} < 1.$$

The required result follows be the elementary properties of multiplication of complex numbers.

Q.E.D.

3.8. Corollary: Let be $a \in A$ and $\alpha \in \Sigma$. Let's suppose that $\sigma(a_\alpha) > 0$ and $| a_\alpha |_\sigma < 1$. Then there exists the unique square root $x \in A_\alpha$ of a_α with positive spectrum and, moreover, is $x \in B(a_\alpha)$.

Proof: Immediately follows from 3.7. and 3.6.

Q.E.D.

Now we are able to state first of the main results.

3.9. Theorem: Let be $a \in A$ and $|a|_\sigma < 1$. Then there exists the unique quasi-square root $q \in A$ of a such that $|q|_\sigma < 1$. If, moreover, the algebra A is a *-algebra and a is selfadjoint then q is selfadjoint too.

Proof: Because the index set Σ is of arbitrary cardinality and we don't make additional requirements on existence of countable cofinal subset in Σ we cannot use the Mittag–Leffler theorem to prove that the required set of quasi-square roots being nonempty.

Obviously there exists a positive η so that for each $\alpha \in \Sigma$ holds $|a_\alpha|_\sigma < \eta < 1$. By 3.3. and 3.4. there exists for each $\alpha \in \Sigma$ the unique quasi-square root $q_\alpha \in A_\alpha$ of a_α so that $|q_\alpha|_\sigma < 1$, $q_\alpha \in B(a_\alpha)$. By the definition of $B(a_\alpha)$ there exists a sequence $\{P_n^\alpha(a)_\alpha\}_{n=1}^\infty$ from $B(a_\alpha)$, such that for each integer n is $|P_n^\alpha(a_\alpha)|_\sigma < 1$ and

$$q_\alpha = \lim_{n\to\infty} P_n^\alpha(a_\alpha).$$

Now we set $q = (q_\alpha)_{\alpha\in\Sigma}$ and we have to prove that q is the required quasi-square root of a. At first we have to show that q belongs to A. Because of the fact that $A = \varprojlim A_\alpha$ we must prove that for each pair $\alpha, \beta \in \Sigma$ satisfying $\alpha < \beta$ holds $\pi_{\alpha\beta}(q_\beta) = q_\alpha$. Let be $\alpha < \beta$. By the definition there exists a positive K so that for each $x \in A$ is $p_\alpha(x) \leqq Kp_\beta(x)$. This easily implies $N_\beta \subset N_\alpha$ and

$$\sigma(a_\alpha) \subset \sigma(a_\beta). \tag{1}$$

As $\pi_{\alpha\beta}$ is a homomorphism for each integer n holds

$$\pi_{\alpha\beta}(P_n^\beta(a_\beta)) = P_n^\beta(a_\alpha) \subset B(a_\alpha) \subset A_\alpha,$$

$\{P_n^\beta(a_\beta)\}_{n=1}^\infty$ being a Cauchy, sequence in A_β so is $\{P_\beta^n(a_\alpha)\}_{n=1}^\infty$ in A_α. The last fact together with that of $B(a_\alpha)$ being closed implies the existence of an element $q'_\alpha \in A_\alpha$ such that $q'_\alpha = \lim_{n\to\infty} P_n^\beta(a_\alpha) \in B(a_\alpha)$. By the continuity of the quasi-product follows immediately

$$\lim_{n\to\infty} (P_n^\beta(a_\beta) \,.\, P_n^\beta(a_\beta)) = a_\beta$$

and again by the continuity of $\pi_{\alpha\beta}$ we get

$$\lim_{n\to\infty} (P_n^\beta(a_\alpha) \,.\, P_n^\beta(a_\alpha)) = a_\alpha.$$

Now by (1) we easily see that for each integer n holds the inequality

$$|P_n^\beta(a_\alpha)|_\sigma < 1\,.$$

So we got $q_\alpha, q'_\alpha \in B(a_\alpha)$, both commuting quasi-square roots of a_α, satisfying the condition of spectral radius being less one. (The last fact follows by using the Gelfand representation theory for $B(a_\alpha)$). By 3.4. we conclude that $q_\alpha = q'_\alpha$. Now, again using the continuity of $\pi_{\alpha\beta}$ we get

$$\pi_{\alpha\beta}(q_\beta) = \pi_{\alpha\beta} \lim_{n\to\infty} P_n^\beta(a_\beta)) = \lim_{n\to\infty} \pi_{\alpha\beta}(P_n^\beta(a_\beta)) = \lim_{n\to\infty} P_n^\beta(a_\alpha) = q_\alpha .$$

So we see that $q \in A$ and a simple application of Gelfand's representation on the algebra $C(a)$ showes that $| q |_\sigma < 1$. (The converse assumption leads to the existence of a suitable sequence of continuous multiplicative functionals on $C(a)$, denoted by $\{f_n\}_{n=1}^\infty$ such that $| f_n(q) | \to 1$, but on the other hand at the same time must hold for each integer n

$$| 2 | f_n(q) | - | f_n(q) |^2 | = | 2 | f_n(q) | - | f_n(q^2) | | \leqq$$
$$\leqq | 2f_n(q) - f_n(q)^2 | = | f_n(a) | < \eta < 1.$$

So we get a contradiction that $1 < \eta < 1$.

To prove the rest of the theorem let's suppose that A is a $*$-algebra and a is a selfadjoint element of A. We get immediately

$$q \,.\, q = a = a^* = q^* \,.\, q^*.$$

From the fact that in each $*$-algebra holds $| q |_\sigma = | q^* |_\sigma$ and by 3.4. follows that $q = q^*$.

Q.E.D.

3.10. Theorem: The following holds for $a \in A$ such that $\sigma(a) > 0$:

(i) Let be $| a |_\sigma < 1$. Then there exists the unique square root $s \in A$ of a such that $\sigma(s) > 0$. The square root s commutes with a. If, moreover, a is selfadjoint, so is s.

(ii) Let K positive be given so that $| a |_\sigma < K$. Then there exists the unique square root $s \in A$ of a such that it's spectrum is positive and $| s |_\sigma < K^{1/2}$, s commuting with a. If, again, a is selfadjoint, so is s.

Proof: It's obviously enough to prove (i) because (ii) follows from (i) if applied on a/K. To prove (i) we apply 3.9. on the element $(e - a)$. Thus we get the unique quasi-square root $q \in A$ of $(e - a)$ and by 3.4. we easily see that the required square root is the element $s = (e - q)$.

Q.E.D.

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Souhrn

O ODMOCNINÁCH A QUASI-ODMOCNINÁCH V LOKÁLNĚ MULTIPLIKATIVNĚ KONVEXNÍCH TOPOLOGICKÝCH ALGEBRÁCH

DINA ŠTĚRBOVÁ

V práci se dokazuje existence quasi-odmocniny prvku se spektrem obsaženým ve vnitřku jednotkové koule komplexních čísel. V případě, že spektrum prvku je kladné, je ukázán jednoduchý vztah mezi odmocninou a quasi-odmocninou, který umožňuje najít jedinou odmocninu s kladným spektrem pro každý prvek lokálně multiplikativně konvexní úplné algebry, jenž má kladné a omezené spektrum.

Резюме

О КВАДРАТНЫХ КОРНЯХ И КВАЗИ-КОРНЯХ В ПОЛУНОРМИРОВАННЫХ КОЛЬЦАХ

ДИНА ШТЕРБОВА

В настоящей статье показывается существование квадратных квази-корней для элементов полунормированных колец, спектры которых лежат во внутренности единичного круга. Когда эти спектры положительны, можно найти простую связь между корнями и квази-корнями, с помощью которой можно легко показать существование и единственность квадратных корней тех элементов, спектры которых положительны и ограничены.