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ON BANACH *-ALGEBRAS WITHOUT THE UNIT

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The present paper is to show that a Banach *-algebra A without the unit element is hermitian if and only if the fundamental V. Pták's inequality $| x |_\sigma \leqq p(x)$ for each $x \in$ A holds.

1. Introduction

V. Pták in [3] characterized fully the hermitian Banach algebras containing the unit element endowed by a not necessarily continuous involution, by spectral properties of its elements. He showed at the same time the important role of the function $p(x) = \sqrt{| x^*x |_\sigma}$, the square root of the spectral radius of x^*x. In [3] are listed fourteen conditions, all equivalent to the fact of algebra being hermitian. The most interesting of these is the "fundamental inequality"

$$| x |_\sigma \leqq p(x) \tag{1}$$

playing the basic role in the theory of hermitian algebras and their connections with the theory of C*-algebras.

The aim of the present paper is to show that (1) characterizes hermitian Banach algebras also in such a case, when the existence of the unit element is not assumed.

2. Preliminaries

Let A be an algebra over the complex field. A is said to be a topological algebra if it is at the same time a topological space with respect to which the algebraic operations in A are continuous. An involution on A is a map $x \to x^*$ of A onto itself such that for each complex number λ and each $x, y \in$ A the following holds:

(i) $(x^*)^* = x$,

(ii) $(x + y)^* = x^* + y^*$,

(iii) $(\lambda x)^* = \bar{\lambda} x^*$,

(iv) $(xy)^* = y^*x^*$.

A *-algebra (or algebra with involution) is an algebra endowed by an involution. A star algebra which is also normed (respectively complete with respect to the norm) is called normed (respectively Banach) *-algebra. Let A be a *-algebra. An element $u \in A$ is said to be unitary if $u^*u = uu^* = e$, where e denotes the unit element of A. An element $a \in A$ is said to be normal if $a^*a = aa^*$. An element $h \in A$ is said to be selfadjoint if $h^* = h$. The sets of all unitary, selfadjoint, normal elements of A will be denoted respectively by U(A), H(A), N(A). It holds obviously $U(A) \subset N(A)$ and $H(A) \subset N(A)$. For any set $S \subset A$ let's denote $S^* = \{x^*, x \in S\}$. If $S = S^*$, we say S is selfadjoint. If the elements of $S \cup S^*$ are pairwise commutative, we say S is normal. The set of all regular elements of A will be denoted by R(A). The spectrum of an element $x \in A$ will be denoted by $\sigma(x)$. If it is necessary to specify the algebra with respect to which the spectrum is taken, we use the notation $\sigma(x, A)$. The spectral radius of the element $x \in A$ is denoted by $| x |_\sigma$ and we recall its definition.

$$| x |_\sigma = \sup \{| \lambda | : \lambda \in \sigma(x)\}$$

In the *-algebra A we define the spectral norm of each element $x \in A$ as follows: $p(x) = (| x^*x |_\sigma)^{1/2}$. The involution is called hermitian if the spectrum $\sigma(x)$ is real for each $x \in H(A)$. The *-algebra A is called hermitian if its involution is hermitian.

We suppose the reader to be familiar with elementary properties of all notions introduced. Further there are supposed knowledges on Gelfand representation theory of commutative Banach algebras. These can be found e.g. in [4]. The set of all multiplicative functionals of A is denoted by $\mathcal{M}(A)$. Now, we recall some known but necessary facts.

2.1. Proposition: The following conditions are equivalent:

(i) A is a commutative hermitian Banach algebra.

(ii) For each $f \in \mathcal{M}(A)$ and for each $x \in A$ it holds $f(x^*) = \overline{f(x)}$.

Proof: See [2].

For the possibility of using the Gelfand representation theory in some reduced extent, of course, also in the non-commutative case the following is useful:

2.2. Proposition: Let A be a Banach *-algebra possessing the unit element e. Then every normal set $N \subset A$ is contained in a closed maximal commutative *-subalgebra C of A so that for each $x \in C$ it holds: $\sigma(x, A) = \sigma(x, C)$.

Proof: See [3].

2.3. Note: Recall now that throughout this paper the continuity of the studied involution is not assumed.

Let's now concentrate our attention on algebras not necessarily containing the unit element.

3. On Banach *-algebras not necessarily possessing the unit

Some results valid for algebras with the unit element can be extended to algebras without it by using the concept of quasi-inverse or by the adjunction of a unit.

3.1. Definition: The unitization of a normed algebra A over the complex field $\boldsymbol{C}$, denoted by A^1, is the normed algebra consisting of the set $A \times \boldsymbol{C}$ with addition, scalar multiplication and product defined for all $x, y \in A$, $\alpha, \beta \in \boldsymbol{C}$ by

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$
$$\beta(x, \alpha) = (\beta x, \alpha\beta)$$
$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

and with the norm defined by

$$\| (x, \alpha) \| = \| x \| + | \alpha |$$

It is easy to verify that A^1 is a normed algebra with the unit element (0,1) that $\| (0,1) \| = 1$ and that the mapping $a \to (a, 0)$ is an isometric isomorfism of A onto a subalgebra of A^1. It is also obvious that A^1 is complete whenever A is complete.

3.2. Definition: If the normed algebra A is endowed by the involution *, we define a new involution on A^1 as follows:

$$(x, \alpha)^{\#} = (x^*, \bar{\alpha})$$

3.3. Definition: Given elements $x, y \in A$, the quasi-product of x, y is the element $x \circ y \in A$ defined by

$$x \circ y = x + y - xy$$

3.4. Definition: Let x be an element of the algebra A. Elements $y, z \in A$ are respectively left and right quasi-inverses if

$$y \circ x = 0, \qquad x \circ z = 0$$

A quasi-inverse of an element $x \in A$ that is both a left quasi-inverse and a right quasi-inverse of x. An element that has a quasi-inverse is said to be quasi-invertible (or quasi-regular), all other elements are said to be quasi-singular. The set of all quasi-invertible elements of A is denoted by q-Inv(A) and the set of all quasi-singular elements of A is denoted by q-Sing(A).

3.5. Proposition: (i) An element $x \in A$ has the quasi-inverse y if and only if $(0, 1) - (x, 0)$ has the inverse $(0, 1) - (y, 0)$ in A^1.

(ii) If A has a unit element e, an element $x \in A$ has the quasi-inverse y if and only if $e - x$ has the inverse $e - y$.

Proof: See [1].

Let's recall now the definition of spectra in the case when the algebra A is not endowed by the unit element.

3.6. Definition: If the algebra A does not possess the unit, we define the spectrum of the element $x \in \mathrm{A}$ as follows

$$\sigma(x, \mathrm{A}) = \{0\} \cup \{\lambda \in C\backslash\{0\} : (1/\lambda) \,.\, x \in q\text{-Sing}(\mathrm{A})\}$$

3.6. Proposition: Let A be without the unit element, and let A^1 be the unitization of A. Then $\sigma(x, \mathrm{A}) = \sigma((x, 0), \mathrm{A}^1)$ for each $x \in \mathrm{A}$.

Proof: The proof is a straightforward consequence of 3.5. and can be also found in [1].

3.7. Note: Let A be an algebra without the unit. Then it is easily seen for each $x \in \mathrm{A}$ that $| x |_\sigma = | (x, 0) |_\sigma$, where the first is taken in A and the second term is taken with respect to A^1.

3.8. Note: By polynomial identity for spectra it immediately follows for each element a from the algebra A and for each complex λ:

$$\sigma(a, \lambda) = \sigma(a) + \lambda$$

3.9. Lemma: Let A be an algebra with involution *, which does not possess the unit element. Then the following holds:

(i) An element $x \in \mathrm{A}$ is selfadjoint if and only if (x, λ) is selfadjoint in A^1 for each real number λ.

(ii) An element $x \in \mathrm{A}$ is normal if and only if (x, λ) is normal in A^1 for each complex λ.

Proof: The statements are obvious.

3.10. Proposition: Let A be an algebra with involution *, which does not possess the unit element. Then the following conditions are equivalent for each element $u \in \mathrm{A}$:

(i) There exists a complex λ such that $| \lambda | = 1$ and $\lambda u \circ \bar{\lambda} u^* = 0 = \bar{\lambda} u^* \circ \lambda u$.

(ii) There exists a complex number λ such that $| \lambda | = 1$ and it holds that $(u, -\lambda)$ is a unitary element of the unitization A^1.

Proof: The condition from (i) is equivalent by the definition to the following one

$$\lambda\bar{\lambda} u u^* - \lambda u - \bar{\lambda} u^* = 0 = \lambda\bar{\lambda} u^* u - \lambda u - \bar{\lambda} u^* \tag{1}$$

It is easily seen the equivalence of (1) and the following

$$(u, -\bar{\lambda})\,(u^*, -\lambda) = (uu^* - \lambda u - \bar{\lambda} u^*, \lambda\bar{\lambda}) = (0, 1) = (u^*, -\lambda)\,(u, -\bar{\lambda})$$

Q.E.D.

3.11. Definition: Let A be a *-algebra which does not possess the unit element. The element $u \in A$ is said to be quasi-unitary if there exists a complex number λ such that $|\lambda| = 1$ and it holds $\lambda u \circ \bar{\lambda} u^* = \bar{\lambda} u^* \circ \lambda u = 0$. The set of all quasi-unitary elements of A is denoted by $U_q(A)$.

3.12. Proposition: Let A be a *-algebra which does not possess the unit element. Let A^1 be the unitization of A. Then the following conditions are equivalent:

(i) There exists a positive number K such that for each quasi-unitary element $u \in U_q(A)$ holds $|\sigma(u, A)| < K$.

(ii) There exists a positive number K′ such that for each unitary element $(u, \lambda) \in U(A^1)$ holds $|\sigma((u, \lambda), A^1)| < K'$.

Proof: (i) → (ii).

Let $(u, \lambda) \in U(A^1)$. By 3.10. it follows that $|\lambda| = 1$ and $u(-\lambda) \circ u^*(-\bar{\lambda}) = 0$. This yields $u \in U_q(A)$ and we have the inequality

$$|\sigma(u, A)| < K \tag{1}$$

Now, 3.8. and (1) yield

$$|\sigma((u, \lambda), A^1)| K + 1 = K' \tag{2}$$

(ii) → (i).

Conversely, for each $u \in U_q(A)$ there exists a complex λ, $|\lambda| = 1$ so that $\lambda u \circ \bar{\lambda} u^* = 0$. It immediately follows that $(u, -\lambda)$ is a unitary element from A^1 which means that

$$|\sigma((u, -\lambda), A^1)| < K' \tag{3}$$

Again, by 3.8. and (3), it follows that $|\sigma(u, A)| < K' + 1$. Q.E.D.

4. The fundamental inequality in Banach *-algebras without the unit.

Now, we are able to formulate the main result of this paper.

4.1. Theorem: Let A be a Banach *-algebra without the unit element. The following conditions are equivalent:

(i) A is hermitian.
(ii) Spectra of all quasi-unitary elements of A are contained on the unit circle of $\boldsymbol{C}$.
(iii) The unitization A^1 is hermitian.
(iv) It holds $|x|_\sigma = p(x)$ for each normal element $x \in N(A)$.
(v) It holds $|x|_\sigma \leqq p(x)$ for each $x \in A$.

Proof: The proof is divided in several steps as follows:

(i) ↔ (iii)

(i) → (v) → (iv) → (ii) → (i)

(i) ↔ (iii):

Let's suppose that A is hermitian; and let $(a, \lambda) \in H(A^1)$. By definition we obtain $a = a^*$ and $\lambda = \bar{\lambda}$. From the fact of A being hermitian it follows that $\sigma(a, A)$ is real. By note 3.8. we have

$$\sigma((a, \lambda), A^1) = \sigma((a, 0), A^1) + \sigma((0, \lambda), A^1) = \sigma(a, A) + \lambda$$

and so we proved that the spectrum of (a, λ) is real.

Suppose conversely that the unitization A^1 is hermitian. By definition 3.1. A is identified by a subalgebra of A^1. If the element $h \in A$ is selfadjoint, then, by definition 3.2. also $(h, 0)$ is a selfadjoint element of A^1. By proposition 3.6. we get

$$\sigma(h, A) = \sigma((h, 0), A^1)$$

and it immediately follows that the spectrum of h is real. Q.E.D.

(i) → (v):

Let A be supposed to be hermitian. By the preceding part of our proof we see that also A^1 is hermitian and by theorem 5.1. of V. Pták's paper [3] it follows that for each element $(a, \lambda) \in A^1$ the fundamental inequality $|(a, \lambda)|_\sigma \leqq p(a, \lambda)$ holds. By proposition 3.6. we get the following inequality:

$$|a|_\sigma = |(a, 0)|_\sigma \leqq p(a, 0) = p(a)$$

and so we proved the desired implication.

Q.E.D.

(v) → (iv):

For an arbitrary given normal element $a \in N(A)$ the equality $p(a) = |a|_\sigma$ follows by simple use of Gelfand representation for the maximal commutative *-subalgebra C of A^1, containing the normal set $\{(a, 0), (0, 1)\}$. See 2.2. and recall the wellknown fact of spectral radius being submultiplicative on commutative Banach algebras with the unit. The last can be found in [4].

Q.E.D.

(iv) → (ii):

Now, let $u \in U_q(A)$. Then there exists a complex λ, $|\lambda| = 1$ so that $\lambda u \circ \bar{\lambda} u^* = 0 = \bar{\lambda} u^* \circ \lambda u$. By the definition we see that

$$uu^* - \lambda u - \bar{\lambda} u^* = 0 = u^*u - \lambda u - \bar{\lambda} u^* \tag{1}$$

(1) immediately implies

$$u^*u = \lambda u + \bar{\lambda} u^* = uu^* \tag{2}$$

By (2) we see that u is a normal element of A. Further we obtain by (iv) and by (2)

$$|uu^*|_\sigma = |\lambda\bar{\lambda} uu^*|_\sigma = |\lambda u + \bar{\lambda} u^*|_\sigma = |u|_\sigma^2 \leqq 2|u|_\sigma \tag{3}$$

where the inequality follows by seeing the subadditivity of spectral radius on commutative Banach algebra; in our case on maximal commutative *-subalgebra of A^1 which contains the normal element $(u, 0)$. (See proposition 2.2.) Now, by (3), it is obvious that

$$| u |_\sigma^2 \leqq 2 | u |_\sigma$$

and so

$$| u |_\sigma \leqq 2$$

Q.E.D.

(ii) → (i):

Now, let $U_q(A)$ be supposed equibounded which means that there exists a positive K such that

$$| \sigma U(A^1) | < K \qquad \text{and} \qquad | \sigma U_q(A) | < K \tag{4}$$

Let $h \in \dot{H}(A)$. We can suppose without any loss of generality that $| \sigma(h) | \leqq 1$. Now we take the maximal closed commutative *-subalgebra C of A^1 containing the set $\{(h, 0), (0, 1)\}$. By 5.1. in V. Pták's paper [3] we see that $| \sigma(U(A^1)) | = 1$. By Ford's lemma, (see [1]), we get the existence of the selfadjoint element $k' \in C$ such that $h^2 = k' \circ k'$ and $| \sigma(k', A^1) | \leqq 1$. It immediately follows by definition 3.3. that there exists a selfadjoint $k \in C$ so that $e = h^2 + k^2$. (Here we do not distinguish between h and $(h, 0)$ and e denotes the unit element $(0, 1)$ of A^1.) Now, we put $u = h + ik$ which means $u^* = h - ik$ and we obtain the equation

$$uu^* = h^2 + k^2 = u^*u = e \tag{5}$$

(5) says that there is $u \in U(A^1)$ which implies that the spectrum $| \sigma(u) | = 1$. Again, we use the Gelfand reprezentation of C and for each multiplicative functional $f \in \mathcal{M}(C)$ get:

$$\begin{aligned} f(u) &= f(h) + if(k) \\ f(u^*) &= f(h) - if(k) \end{aligned} \tag{6}$$

Since $f(u) \in \sigma(u)$ and $f(u^*) \in \sigma(u^*)$ we see that

$$| f(u) | = | f(u^*) | = 1 \tag{7}$$

Because of f being multiplicative, it follows from (6), (7)

$$f(u^*u) = f(h^2 + k^2) = 1$$

Now, by elementary properties of complex numbers it follows that $f(h)$ is real. As f is an arbitrary multiplicative functional from $\mathcal{M}(C)$ we see that $\sigma(h)$ is real, and A si he hermitian.

Q.E.D.

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Souhrn

O BANACHOVÝCH *-ALGEBRÁCH BEZ JEDNOTKY

DINA ŠTĚRBOVÁ

V práci se dokazuje, že Banachova algebra s involucí, která nemá jednotkový prvek, je hermiteovská tehdy a jen tehdy, jestliže je pro každý její prvek x splněna nerovnost

$$| x |_\sigma \leqq p(x) = (| xx |_\sigma)^{1/2} \qquad (1)$$

kde $| \, |_\sigma$ značí spektrální poloměr v uvažované algebře. V provedených úvahách se nepředpokládá spojitost involuce. Dále je ukázáno, že splnění podmínky (1) pro každý prvek x z uvažované algebry je ekvivalentní s podmínkou stejné omezenosti spekter všech quasiunitárních prvků algebry. Pojem quasiunitárnosti vhodně nahrazuje unitárnost v algebře bez jednotkového prvku. Dosažené výsledky představují zobecnění výsledků V. Ptáka [3], který dokázal, že (1) charakterizuje hermiteovské Banachovy algebry s jednotkovým prvkem.

Резюме

О БАНАХОВЫХ *-АЛГЕБРАХ БЕЗ ЕДИНИЦЫ

ДИНА ШТЕРБОВА

В настоящей работе доказывается следующее предложение: Польное нормированное кольцо с инволюцией без единичного элемента является вполне симметрическим тогда и только тогда если для каждого элемента x из кольца выполняется фундаментальное неравенство В. Птака:

$$| x |_\sigma \leqq p(x) \qquad (1)$$

При этом не предполагается непрерывность инволюции. Этот результат является обобщением результата В. Птака (3), где было автором доказано то же самое для колец, у которых существование единицы предполагается.