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A NOTE ON THE DEFINITENESS OF THE DIFFERENTIAL EQUATIONS $y'' = q(t)y$

JAROSLAV KRBILA

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To Professor Miroslav Laitoch on the occasion of his 50[th] birthday

In this work we are going first to deduce the necessary and sufficient condition for the definiteness of the 2nd order linear differential equation of Jacobi's type:

$$y'' = q(t)y \tag{q}$$

by means of the first phases. Suppose that the carrier of this differential equation $q(t) \in C_0(j)$.

The differential equation (q) is said to be *definite on the interval* $i = [a, b] \subset j$, $a < b$, $a, b \in E_1^* = E_1 \cup (-\infty) \cup (\infty)$, if any non-trivial solution of (q) has not a zero point and a zero point of its derivative simultaneously in the interval i.

We remark that the notation $i = [a, b]$ signifies an arbitrary interval. If we want to stress the kind of the interval, we shall write for example $(a, b\rangle$ for the open interval from the left and closed from the right.

In book [1], page 31—101, is constructed the theory of phases of the differential equation (q) by O. BORŮVKA. We mention that the first phase of (q) is a function $\alpha(t)$, which has the following property: $\alpha(t) \in C_3(j)$, $\alpha'(t) \neq 0$ for each $t \in j$ that fulfils the nonlinear differential equation

$$-\{\alpha, t\} - \alpha'^2 = q(t) \quad (t \in j), \tag{1}$$

where $\{\alpha, t\} = (\alpha''/2\alpha')' - (\alpha''/2\alpha')^2$.

By means of a phase $\alpha(t)$ we can express the universal solution of (q) with the carrier $q(t)$ given by relation (1) as follows:

$$y(t) = |\alpha'(t)|^{-1/2} (c_1 \sin \alpha(t) + c_2 \cos \alpha(t)), \tag{2}$$

where $c_1, c_2 \in E_1$.

By oscillation of the phase $\alpha(t)$ in the interval i we understand the expression $o(\alpha/i) = |c - d|$, where

$$c = \lim_{t \to a+} \alpha(t), \qquad d = \lim_{t \to b-} \alpha(t).$$

The differential equation (q) is said to be *pure disconjugate* in the interval i if and only if $o(\alpha/i) \leq \pi$ (see [2]).

There is evident the following assertion:

The necessary condition for the definiteness of (q) in the interval i is the pure disconjugacy of (q) in the interval i.

Therefore we shall further suppose that the differential equation (q) is pure disconjugate in the interval i.

Theorem 1. *The differential equation (q) that is pure disconjugate in the interval i, is definite in the interval i if, and only if, for each $t_1, t_2 \in i$, $t_1 \neq t_2$ the following inequality holds:*

$$\operatorname{cotg}(\alpha(t_1) - \alpha(t_2)) \neq (1/2\alpha')'_{t=t_2} \tag{3}$$

where $\alpha(t)$ is the first phase of the differential equation (q).

Proof. 1. If the differential equation (q) is definite in the interval i, then there cannot exist a non-trivial solution $y(t)$ of the differential equation (q) with the property:

$$y(t_1) = 0, \qquad y'(t_2) = 0, \tag{4}$$

where $t_1, t_2 \in i$, $t_1 \neq t_2$. If we put a solution of (q) expressed in the form (2) into the condition (4), we obtain a system of linear algebraic equations with regard to c_1, c_2. This system has just a trivial solution and therefore the determinant of this system must be different from zero. We easily see that it is equivalent with the inequality:

$$\cos(\alpha(t_1) - \alpha(t_2)) \neq (1/2\alpha')'_{t=t_2} \sin(\alpha(t_1) - \alpha(t_2)),$$

which is equivalent with the inequality (3) with regard to the assumption: $0 < o(\alpha/[t_1, t_2]) < \pi$.

2. If the pure disconjugate differential equation (q) in the interval i is not definite in the interval i, then the determinant of the system mentioned in the 1st part of this proof must be equal to zero, which is equivalent with the condition:

$$\operatorname{cotg}(\alpha(t_1) - \alpha(t_2)) = (1/2\alpha')'_{t=t_2}.$$

Thus the theorem is proved.

In the next part of this paper we introduce by means of the first phases the necessary and sufficient conditions and the sufficient conditions for the differential equation (q) not to have conjugate points of the 3rd and the 4th kind in the interval $i \subset j$.

For the sake of completeness we mention (see [1] — I — §3 — page 15) the definitions of conjugate points of the 3rd and 4th kind.

Let $t_1 \in j$ and let $y_1(t)$ [$y_2(t)$] be an arbitrary solution of (q) such that $y_1(t_1) = 0$ [$y'_2(t_1) = 0$]. Number $t_2 \in j$, $t_2 \neq t_1$ is called the *conjugate point* with number t_1 with regard to (q) of the *3rd* [4th] kind if $y'_1(t_2) = 0$ [$y_2(t_2) = 0$].

A conjugate point t_2 is called *conjugate towards t_1 from the right* [*from the left*] if $t_1 < t_2$ [$t_2 < t_1$].

We shall be interested only in the first conjugate points of the 3rd [4th] kind, i. e. that for each $t \in (t_1, t_2)$ $y'_1(t) \neq 0$ [$y_2(t) \neq 0$] holds.

From the conjugate points of view we can interpret the definiteness of the interval i so that it is such a differential equation, which has conjugate points neither of the 3rd nor of the 4th kind.

Analogous to Theorem 1 we can prove the following theorems:

Theorem 2. *The necessary and sufficient condition for the pure disconjugate differential equation (q) in the interval i not to have conjugate points of the 3rd resp. 4th kind from the right [from the left] in the interval i is that for each $t_1, t_2 \in i$, $t_1 < t_2$ [$t_2 < t_1$] the following relation holds:*

$$\operatorname{cotg}(\alpha(t_1) - \alpha(t_2)) \neq (1/2\alpha')'_{t=t_2}, \quad \text{resp.}$$

$$\operatorname{cotg}(\alpha(t_2) - \alpha(t_1)) \neq (1/2\alpha')'_{t=t_2}, \tag{4}$$

where $\alpha(t)$ is the first phase of the differential equation (q).

Theorem 3. *If the differential equation* (q) *has its first phase* $\alpha(t)$ *with the property:* $o(\alpha/i) < \Pi/2$ *and* $sgn\ \alpha' \neq sgn\ \alpha''$, *resp.* $sgn\ \alpha' = sgn\ \alpha''$, *so* (q) *has not conjugate points of the* 3^{rd} $[4^{th}]$ *kind resp. of the* 4^{th} $[3^{rd}]$ *kind from the right* [*from the left*] *in the interval i.*

Proof. As all these four facilities are proved analogically, we shall prove only one of them. Let $t_1, t_2 \in i$, $t_2 < t_1$ have the meaning as in Theorem 2. Let $\alpha'(t) > 0$ $[< 0]$ for any $t \in i$. From this and from the property of function cotgx with regard to the inequality $o(\alpha/i) < \pi/2$ we obtain: $\text{cotg}\,(\alpha(t_2) - \alpha(t_1)) < 0$ $[> 0]$ and from the assumption $\alpha'' < 0$ $[> 0]$ the inequality $(1/2\,\alpha')' > 0$ $[< 0]$ and thus the relation (4) is fulfilled.

By using the preceding results we shall yet prove the sufficient condition for the definiteness of the differential equation (q) in half-open intervals.

Theorem 4. *If the pure disconjugate differential equation* (q) *in the interval* $i = [a, b]$ *has its first phase with the property:*

$$\alpha''(b)/\alpha'^2(b) + \alpha''(a)/\alpha'^2(a) = 0, \tag{5}$$

so (q) is definite in the intervals $\langle a, b,)$ $(a, b\rangle$.

Proof. From the necessary and sufficient condition for the point b to be conjugate of the 3^{rd} resp. of the 4^{th} kind towards the point, a, i. e. from the formula:

$$\text{cotg}\,(\alpha(a) - \alpha(b)) = (1/2\,\alpha')'_{t=b}, \quad \text{resp.}$$
$$\text{cotg}\,(\alpha(b) - \alpha(a)) = (1/2\,\alpha')'_{t=a}$$

we obtain the relation (5), which is the necessary and sufficient condition for the point b to be conjugate simultaneously from the right both of the 3^{rd} and of the 4^{th} kind towards the point a where from we immediately get the statement of this theorem.

SHRNUTÍ

POZNÁMKA O DEFINITNOSTI DIFERENCIÁLNYCH ROVNÍC $y''b = (t)y$

JAROSLAV KRBIĽA

Diferenciálna rovnica (q) : $y'' = q(t)y$ sa nazýva definitnou na intervale i, ak každé jej netriviálne riešenie nemá na intervale i súčasne nulový bod aj nulový bod derivácie. V práci sa odvádza pomocou prvých fáz (veta 1) nutná a postačujúca podmienka pre definitnosť rovnice (q) na intervale i. Ďalej sa uvádza nutná a postačujúca podmienka pre to, aby rovnica (q) nemala na intervale i zprava [zľava] konjugované body 3. resp. 4. druhu (veta 2). Na základe uvedeného sa dokazujú postačujúce podmienky pre to, aby rovnica q nemala na intervale i konjugované body 3. resp. 4. druhu zprava [zľava] (veta 3) a postačujúce podmienky pre definitnosť rovnice (q) na poloотvorených intervaloch (veta 4).

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