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NOTE ON THE PAPER [1] OF S. SĘDZIWY

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0. In the above cited paper the author studies the differential equation (d.e.) (n — positive integer)

$$x^{(n)} + a_1 x^{(n-1)} + \dots + a_{k-1} x^{(n-k+1)} + h_k(x^{(n-k)}) + a_{k+1} x^{(n-k-1)} + \dots + a_n x = e(t)$$
$$(k = 1, 2, \dots, n;\ x^{(0)} = x), \tag{1}$$

where the $a_i (i = 1, 2, \dots, n;\ i \neq k)$ are positive constants and $h_k(x)$, $e(t)$ are continuous functions of their arguments. In the case $h_k(x)$ and $e(t)$ are bounded, two theorems about global boundedness (g.b.) of the solutions of (1) for $k = n$ are proved. In the case of unbounded $h_k(x)$ the validity of four theorems ($n = 3$, $k = 1, 2, 3$ and $n = k = 4$) dealing with the global stability of solutions of the autonomous equation ($e(t) = 0$) is shown. Finally, for $n = k = 3$ a sufficient condition is given for the g.b. of solutions of (1) with unbounded $h_3(x)$. In this note we will show, that theorems concerning the g.b. of solutions can be proved by using the simple method of paper [2] (see also [3] pp. 384—392); moreover it is possible to get some additional results.

1. Let us study, instead of (1), the more general d.e.

$$x^{(n)} + \sum_{i=1}^{n-1} f_i(x^{(n-i)}) + h_n(x) = e(t) \tag{2}$$

with continuous $f_i (i = 1, 2, \dots, n-1)$ and let us pose (with positive a_i)

$$f_i(y) - a_i y = \varphi_i(y) \quad (i = 1, 2, \dots n - 1). \tag{3}$$

In what follows we will suppose that $r^{n-1} + a_1 r^{n-2} + \dots + a_{n-1}$ is a Hurwitz-polynomial.

Theorem 1. Let us consider the equation (2) *and suppose*

$$| h_n(x) | \leqq H \quad \text{for every } x, \tag{i}$$

$$| e(t) | \leqq E \qquad \text{for every } t \geqq 0, \tag{ii}$$

$$\left| \int_0^t e(s)\, ds \right| \leqq E \qquad \text{for every } t \geqq 0, \tag{iii}$$

$$| \varphi_i(y) | \leqq m_i \quad (i = 1, 2, \dots, n-1) \text{ for every } y, \tag{iv}$$

there exist $h > 0$, $\delta > 0$ such that for every $|x| \geqq h$ the inequality $h_n(x) \operatorname{sgn} x \geqq m + \delta$ (where $m = \sum_{i=1}^{n-1} m_i$) holds. (v)

Then the solutions of (2) are g.b.

Proof. The fact that the derivatives $x'(t), x''(t), \ldots, x^{(n-1)}(t)$ are ultimately bounded by a constant independent of the solution $x(t)$ can be proved in a way analogous to that of [2].

We start from the identity

$$y^{(n-1)} + \sum_{k=1}^{n-1} a_k y^{(n-k-1)} = e_1(t) \tag{4}$$

(where $x^{(i+1)}(t) = y^{(i)}(t)$, $i = 0, 1, \ldots, n-1$ and $e_1(t) = e(t) - [h_n(x(t)) + \sum_{i=1}^{n-1} \varphi_i(x^{(n-i)}(t))]$) which is satisfied by each solution $x(t)$ of our equation. We have thus

$$x^{(i+1)} = y^{(i)}(t) + y_0^{(i)}(t) + \int_\tau^t \frac{\partial^i y_1(t-s)}{\partial t^i} e_1(s) \, ds$$

$$(i = 0, 1, \ldots, n-2, \ \tau \text{ stands for a real number}). \tag{5}$$

In this formula $y_0(t)$, $y_1(t)$ are convenient solutions of the d.e. $y^{(n-1)} + \sum_{k=1}^{n-1} a_k y^{(n-k-1)} = 0$. Now, as the function $\frac{\partial^i y_1(t-s)}{\partial t^i}$ admits a majorant of the type $Ae^{-r(t-s)} (r > 0)$ and $e_1(t)$ is bounded by (*i*), (*ii*) and (*iv*), the boundedness of derivatives can be easily proved. In each interval $[\tau, T[$ of existence of $x(t)$ we get the boundedness of derivatives and for this reason the solution $x(t)$ must exist on the whole half-axis $[\tau, +\infty]$. Note that this proof of the boundedness of derivatives may be used for the d.e. with $e(t, x, x', \ldots, x^{(n-1)})$ instead of $e(t)$.

Let us suppose now a chosen solution $x(t)$ satisfies the inequality (I stands for a convenient positive constant)

$$|x^{(n-1)}(t)| + \sum_{k=1}^{n-2} a_k |x^{(n-k-1)}(t)| < I, \qquad |x(t)| \geqq h \tag{6}$$

for every $t \geqq t_0$. From (*iv*), (*v*) we then easily obtain

$$\Phi(t) = \int_{t_0}^{t} [h(x(s)) + \sum_{i=1}^{n-1} \varphi_i(x^{(n-i)}(s))] \operatorname{sgn} x(s) \, ds \geqq \delta(t - t_0) > 0 \tag{7}$$

for every $t \geqq t_0$. Integrating (2) from t_0 to $t \geqq t_0$ and multiplying it with the constant $\operatorname{sgn} x(t)$ we get

$$a_{n-1} |x(t)| \leqq |x^{(n-1)}| + \sum_{k=1}^{n-2} a_k |x^{(n-k-1)}| - \Phi(t) + |\int_{t_0}^{t} e(s) \, ds| + \\ + |x^{(n-1)}(t_0)| + \sum_{k=1}^{n-1} a_k |x^{(n-k-1)}(t_0)| \tag{8}$$

and therefore by (6) (7) and *(iii)*

$$a_{n-1} \mid x(t) \mid \leqq 2(I + E) + a_{n-1} \mid x(t_0) \mid - \delta(t - t_0) \qquad \text{for every } t \geqq t_0. \quad (9)$$

Thus, we have a contradiction from which we finally conclude

$$\liminf_{t \to +\infty} \mid x(t) \mid \leqq h. \quad (10)$$

By (9) and (10) it follows

$$\limsup_{t \to +\infty} \mid x(t) \mid \leqq \frac{2}{a_{n-1}}(I + E) + h$$

and the proof of Theorem 1 is complete.

Remark 1. We see from our proof that asking in (v) the weaker condition $h_n(x) \operatorname{sgn} x \geqq m$ for every $\mid x \mid \geqq h$, we obtain boundedness of solutions. For $f_k(y) = a_k y (k = 1, 2, \ldots, n - 1)$ the conditions in Theorem 1 reduce to the Sędziwy's conditions.

Theorem 2. Let us consider the d.e. (2). If (ii), (iv) and

$$E + m < H_1 \leqq h(x) \operatorname{sgn} x < H_2 \qquad \text{for every } \mid x \mid \geqq h > 0 \quad \text{(vi)}$$

hold, then the solutions of (2) are g.b.

Proof. The g.b. of the derivatives of a solution $x(t)$ as well as its existence on $[t_0, +\infty]$ can be proved in the same way as above. Let us suppose again (6) holds for every $t \geqq t_0$. Instead of (8) we use now the inequality

$$a_{n-1} \mid x(t) \mid \leqq \mid x^{(n-1)} \mid + \sum_{k=1}^{n-2} a_k \mid x^{(n-k-1)} \mid - \varphi(t) + \mid x^{(n-1)}(t_0) \mid + \\ + \sum_{k=1}^{n-1} a_k \mid x^{(n-k-1)}(t_0) \mid$$

with

$$\Psi(t) = \int_{t_0}^{t} [h_n(x(s)) + \sum_{i=1}^{n-1} \varphi_i(x^{(n-i)}(s)) - e(s)] \operatorname{sgn} x(s) \, ds.$$

By (vi) we obtain

$$\Psi(t) \geqq [H_1 - (E + m)] (t - t_0) > 0 \qquad \text{for every } t \geqq t_0,$$

and (6) gives then

$$a_{n-1} \mid x(t) \mid \leqq 2I + a_{n-1} \mid x(t_0) \mid - [H_1 - (E + m)] (t - t_0),$$

for every $t > t_0$. Hence we have (10) again and the rest of the proof is the same as above.

2. *Theorem 3. Let us consider the d.e. (1) for $n = k = 3$. If (ii) holds and constants a_3, K exist with $0 < a_3 < a_1 a_2$ $(a_1 > 0)$ so that*

$$| a_3 - \frac{h(x)}{x} | \leqq K \qquad \text{for every } x \neq 0 \tag{vii}$$

($h_3(0) = 0$), *then the solutions of our d.e. are g.b.*

Remark 2. In the case the condition $0 < \varepsilon < h_3(x)\, x^{-1} < a_1 a_2 - \varepsilon (x \neq 0)$ of Sędziwy is satisfied we can take in (*vii*) i.e. $a_3 = \frac{1}{2} a_1 a_2$, $K = \frac{a_1 a_2 - 2\varepsilon}{2}$.

Proof of Theorem 3. For a chosen solution $x(t)$ of the considered d.e. the identity

$$x''' + a_1 x'' + a_2 x' + a_3 x = e(t) + a_3 x - h_3(x(t)),$$

in the existence interval $[t_0, T[$ $(T > t_0)$ holds and hence

$$x(t) = y_0(t) + \int_{t_0}^{t} y_1(t - s)\, [e(s) + a_3 x(s) - h_3(x(s))]\, \mathrm{d}s, \tag{11}$$

($t \in [t_0, T[$), where $y_0(t)$, $y_1(t)$ are suitable solutions of

$$y''' + a_1 y'' + a_2 y' + a_3 y = 0. \tag{12}$$

From (11) we obtain using (ii), (iii) the inequality

$$| x(t) | \leqq | y_0(t) + E \int_{t_0}^{t} | y_1(t - s) |\, \mathrm{d}s + \int_{t_0}^{t} K | y_1(t - s) | \, | x(s) |\, \mathrm{d}s, \tag{13}$$

($t \in [t_0, T[$). Because the coefficients of (12) satisfy the Hurwitz-condition, the functions $y_0(t)$, $y_1(t)$ have a majorant $Ae^{-rt}(r > 0)$ again and therefore we get from

$$| x(t) | \leqq M + \int_{t_0}^{t} Ne^{-rs} | x(s) |\, \mathrm{d}s \qquad (t \in [t_0, T[),$$

with N not depending on $x(t)$. Hence, by Gronwall's Lemma

$$| x(t) | \leqq M \exp [\int_{t_0}^{+\infty} Ne^{-rs}\, \mathrm{d}s] = MP \qquad (t \in [t_0, T[). \tag{14}$$

We obtain so the boundedness of $x(t)$ on $[t_0, T[$; if we denote $H =$ l.u.b. $h_3(x)$ on $[-MP, MP]$ it becomes clear that the boundedness of derivatives can be shown as in the proofs above. From this we conclude $T = +\infty$ and thus M must not depend on $x(t)$. Theorem 3 is proved.

3. Under assumptions of Theorems 1, 2, 3 and if $xh_3(h) > 0$ $(x \neq 0)$ it is possible to prove the boundedness of $\int_{t_1}^{t} h(x(s))\, \mathrm{d}s$. Thus, in the same way as in [2], we see that under the above assumptions each solution of the considered d.e. is oscillatory or $\to 0$ for $t \to +\infty$. From this it follows again that the periodic solution, whose existence can be asserted if $e(t)$ is periodic and an uniqueness condition holds, oscillates in this cases. If we pose stronger conditions on $e(t)$ we obtain in all the

considered cases simple oscillation—theorems, as Theorems 8, 9 in [4]. It is possible also to prove theorems about divergent solutions. We have i.e.

Theorem 4. Let us consider the d.e. (*I*) *for* $k = n$. *If* $r^{n-1} + \sum_{k=1}^{n-1} a_k r^{n-k-1}$ *is a Hurwitz—polynomial*, (*i*), (*ii*), (*iii*) *and*

$$\limsup_{x \to +\infty} x h_n(x) < -a_{n-1} H[X_{n-1} + a_1 X_{n-2} + \ldots + a_{n-3} X_2 + E] \qquad (viii)$$

(*where* $X_j =$ l.u.b. $x^{(j)}(t)$ *on* $[t_0, +\infty[$, $j = 2, 3, \ldots, n-1$) *hold, then there exist divergent solutions of the considered d.e.* (*with bounded derivatives*).

The proof of this Theorem can be carried out by using the function

$$2V = \frac{2a_{n-2}}{a_{n-1}} \int_0^x h(s)\,\mathrm{d}s + \frac{1}{a_{n-1}} \left(x^{(n-1)} + \sum_{k=1}^{n-1} a_k x^{(n-k-1)} - \int_0^t e(s)\,\mathrm{d}s \right)^2$$

in the same manner as the proof of Theorem 7 [4] (see also [5]).

REFERENCES

[1] *Sędziwy S.:* Asymptotic properties of solutions of nonlinear differential equations of the higher order. Zeszyty naukowe Universitetu Jagiellonskiego CXXXI, 1966, pp. 69—80.

[2] *Voráček J.:* Einige Bemerkungen über eine nichtlineare Differentialgleichung dritter Ordnung. Abhandlungen der Deutschen Akademie der Wissenschaften, Jg. 1965, Nr. 1., pp. 372—378.

[3] *Reissig R., Sansone G., Conti R.:* Nichtlineare Differentialgleichungen höherer Ordnung. Edizioni Cremonese, Roma 1969.

[4] *Voráček J.:* Einige Bemerkungen über eine nichtlineare Differentialgleichung dritter Ordnung. Archivum mathematicum, Brno, T. 2., 1966, pp. 19—26.

[5] *Voráček J.:* Sur une application de la méthode des fonctions de Liapounoff. This issue, pp.

Shrnutí

POZNÁMKA K PRÁCI S. SĘDZIWEHO [1]

JAN VORÁČEK

Ukazuje se, že k důkazu některých vět z práce [1] je možno užít metody publikované autorem (např. [2]). Touto metodou je možno získat výsledky poněkud obecnější a podrobněji studovat asymptotické vlastnosti řešení uvažovaných diferenciálních rovnic.