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OD-CHARACTERIZATION OF ALMOST SIMPLE GROUPS RELATED TO $L_2(49)$

Liangcai Zhang and Wujie Shi

Abstract. In the present paper, we classify groups with the same order and degree pattern as an almost simple group related to the projective special linear simple group $L_2(49)$. As a consequence of this result we can give a positive answer to a conjecture of W. J. Shi and J. X. Bi, for all almost simple groups related to $L_2(49)$ except $L_2(49)\cdot 2^2$. Also, we prove that if M is an almost simple group related to $L_2(49)$ except $L_2(49)\cdot 2^2$ and G is a finite group such that $|G|=|M|$ and $\Gamma(G)=\Gamma(M)$, then $G\cong M$.

1. Introduction

Throughout this paper, groups under consideration are finite. For any group G, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a simple graph called prime graph of G, denoted by $\Gamma(G)$. The vertex set of this graph is $\pi(G)$, and two distinct vertices p, q are joined by an edge if and only if $pq\in\pi_e(G)$. In this case, we write $p\sim q$. Denote by $t(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i=\pi_i(G)$ $(i=1,2,\dots,t(G))$ the connected components of $\Gamma(G)$. When $|G|$ is even, then by our convention $2\in\pi_1(G)$. We also denote by $\pi(n)$ the set of all primes dividing n, where n is a natural number. Then $|G|$ can be expressed as a product of $m_1,m_2,\dots,m_{t(G)}$, where m_i's are positive integers with $\pi(m_i)=\pi_i$. These m_i's are called the order components of G. In particular, if m_i is an odd number, then we call it an odd component of G. Let $OC(G)=\{m_1,m_2,\dots,m_{t(G)}\}$ be the set of order components of G, and $T(G)=\{\pi_i(G)\mid i=1,2,\dots,t(G)\}$.

Let G be a group and $p\in\pi(G)$. We denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all of its Sylow p-subgroups, respectively. We also denote by $\mathrm{Soc}(G)$ the socle of G which is the subgroup generated by the set of all minimal normal subgroups of G. We denote by $A\colon B$ (or $A\cdot B$) a split (or non-split) extension of A by B. Also, $\mathbb{N}$ and $\mathbb{P}$ denote the set of natural numbers and the set of primes, respectively.

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In particular, this paper itself is accessible only with the basic knowledge of group theory. All further unexplained notations are standard and can be found in [4].

Definition 1.1. Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_i \in \mathbb{P}$ and $\alpha_i \in \mathbb{N}$ for $i = 1, 2, \dots, k$. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) \mid p \sim q\}|$, called the degree of p. We also define $D(G) := \big(\deg(p_1), \deg(p_2), \dots, \deg(p_k)\big)$, where $p_1 < p_2 < \cdots < p_k$. We call it the degree pattern of G.

Definition 1.2. A group M is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an *OD-characterizable group.*

Definition 1.3. A group G is said to be an almost simple related to S if and only if $S \trianglelefteq G \leq \mathrm{Aut}\,(S)$ for some non-abelian simple group S.

Definition 1.4. Let p be a prime. A group G is called a $C_{p,p}$-group if and only if $p \in \pi(G)$ and the centralizers of its elements of order p in G are p-groups.

The significance of the prime graphs of finite groups can be found in many articles, for example [6], [18]–[21]. Therefore, the characterizations of finite groups by their orders and degree patterns may help us to know certain properties of the almost simple groups more clearly. In a series of articles (see [10, 11, 22, 23]), it was shown that many finite almost simple groups are OD-characterizable. We point out some of these results.

Result 1 ([10, 11])**.** All sporadic simple groups and their automorphism groups except $\mathrm{Aut}\,(J_2)$ and $\mathrm{Aut}\,(M^cL)$ are OD-characterizable.

Result 2 ([10])**.** The alternating groups A_p, A_{p+1}, A_{p+2} and the symmetric groups S_p and S_{p+1}, where p is a prime, are OD-characterizable.

Result 3 ([10, 11])**.** The simple groups of Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ are OD-characterizable for certain $q \in \mathbb{N}$.

Result 4 ([10])**.** All finite simple $C_{2,2}$-groups are OD-characterizable.

Result 5 ([23])**.** All finite simple groups with exactly four prime divisors except A_{10} are OD-characterizable.

2. Lemmas

Lemma 2.1 ([9, Table 1])**.** *Let G be an almost simple group related to $L := L_2(49)$. Then G is isomorphic to one of the following groups: L, $L\colon 2_1\big(\cong PGL(2,49)\big)$, $L\colon 2_2$, $L \cdot 2_3$, $L \cdot 2^2\big(\cong \mathrm{Aut}\,(L_2(49))\big)$. Moreover, $\pi_e(L) = \{25, 24, 7\}$, $\pi_e(L\colon 2_1) = \{50, 48, 7\}$, $\pi_e(L\colon 2_2) = \{25, 24, 14\}$, $\pi_e(L \cdot 2_3) = \{25, 24, 16, 7\}$, and $\pi_e(L \cdot 2^2) = \{50, 48, 14\}$. More information about the algorithm can be obtained in* [8].

Lemma 2.2 ([5, Theorem 1])**.** *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 2.3 ([9, Table 1]). *If S is a finite non-abelian simple groups such that $\pi(S) \subseteq \{2,3,5,7\}$, then S is isomorphic to one of the following simple groups in Table 1. In particular, $\{2,3\} \subset \pi(S)$ and $\pi(\mathrm{Out}(S)) \subseteq \{2,3\}$ if $S \neq S_6(2)$.*

Table 1. Finite non-abelian simple groups S such that $\pi(S) \subseteq \{2,3,5,7\}$

S	Order of S	Out(S)	S	Order of S	Out(S)
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	S_3
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	D_8
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	D_{12}	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	S_3

Now we quote two lemmas on Frobenius groups.

Lemma 2.4 ([1, Theorem 1]). *Let G be a Frobenius group of even order with H and K its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G)$= $\{\pi(K), \pi(H)\}$.*

Lemma 2.5 ([4, 12]). *Let G be a Frobenius group with kernel F and complement C. Then the following assertions are true.*

(a) *F is a nilpotent group.*

(b) $|F| \equiv 1 (\mathrm{mod} \ \ |C|)$.

(c) *Every subgroup of C of order $p \cdot q$, with p, q (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of C of odd order is cyclic and Sylow 2-subgroup of C is either cyclic or generalized quaternion group. If C is a non-solvable group, then C has a subgroup of index at most 2 isomorphic to $\mathrm{SL}(2,5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|, 30) = 1$; in particular, 15, 20 $\notin \pi_e(C)$. If C is solvable and $O(C) = 1$, then either C is a 2-group or C has a subgroup of index at most 2 isomorphic to $\mathrm{SL}(2,3)$.*

A group G is a 2-Frobenius group if there exists a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Now we quote a lemma on 2-Frobenius groups.

Lemma 2.6 ([1, Theorem 2]). *Let G be a 2-Frobenius group of even order, which has a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then*

(a) $t(G) = 2$ *and* $T(G)= \big\{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\big\}$.

(b) G/K *and* K/H *are cyclic,* $|G/K| \mid |\operatorname{Aut}(K/H)|$, *and* $\big(|G/K|, |K/H|\big) = 1$.

(c) H *is a nilpotent group and* G *is a solvable group.*

The structure of a finite group with disconnected prime graph is described in the following lemma. Though this lemma is a useful tool for the groups with disconnected prime graphs, we should not use it if a finite group has only one connected component.

Lemma 2.7 ([7, 17, Theorem A]). *Let G be a finite group with $t(G) \geq 2$, then G is one of the following groups:*

(a) *G is a Frobenius or 2-Frobenius group;*

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a finite non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/H| \mid |\operatorname{Aut}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.*

Lemma 2.8 ([2, Theorem]). *Let G be a finite non-abelian simple $C_{p,p}$-group, where $p \in \mathbb{P}$.*

(a) *If $p = 5$, then G is isomorphic to one of the following simple groups: A_5, A_6, A_7, M_{11}, M_{22}, $L_3(4)$, $S_4(3)$, $S_4(7)$, $U_4(3)$, $Sz(8)$, $Sz(32)$, $L_2(49)$, $L_2(5^m)$, $L_2(2 \cdot 5^m \pm 1)$, where $m \in \mathbb{N}$ and $2 \cdot 5^m \pm 1 \in \mathbb{P}$.*

(b) *If $p = 7$, then G is isomorphic to one of the following simple groups: A_7, A_8, A_9, J_1, M_{22}, J_2, HS, $L_3(4)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $G_2(13)$, $U_3(3)$, $U_3(5)$, $U_3(19)$, $U_4(3)$, $U_6(2)$, $Sz(8)$, $L_2(8)$, $L_2(7^m)$, $L_2(2 \cdot 7^m - 1)$, where $m \in \mathbb{N}$ and $2 \cdot 7^m - 1 \in \mathbb{P}$.*

Lemma 2.9 ([22, Theorem]). *If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where $M = U_4(3) : 2_2$ or $U_4(3) \cdot 2_3$, then $G \cong U_4(3) : 2_2$ or $U_4(3) \cdot 2_3$.*

3. OD-characterization of almost simple groups related to $L_2(49)$

Theorem 3.1. *If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where M is an almost simple group related to $L := L_2(49)$, then the following assertions are true:*

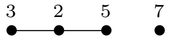
(a) *If $M = L, L : 2_1, L : 2_2$ or $L \cdot 2_3$, then $G \cong M$.*

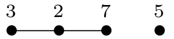
(b) *If $M = L \cdot 2^2$, then $G \cong L \cdot 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L \cdot 2_3)$, $\mathbb{Z}_2 \cdot (L : 2_1)$, $\mathbb{Z}_2 \cdot (L : 2_2)$, $\mathbb{Z}_2 \cdot (L \cdot 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.*

In particular, L, L: 2_1, L: 2_2 and $L \cdot 2_3$ are OD*-characterizable; $L \cdot 2^2$ is 9-fold* OD*-characterizable.*

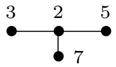
Proof. By Lemma 2.1, first we list the prime graphs of the almost simple groups related to L as follows:

$\Gamma(L)$: 

$\Gamma(L:2_1)$: 

$\Gamma(L:2_2)$: 

$\Gamma(L\cdot 2_3)$: 

$\Gamma(L\cdot 2^2)$: 

Moreover, we break the proof into a number of separate cases.

Case 1. If $M = L$, then $G \cong L$ by Result 5.

Case 2. If $M = L\colon 2_1$, then $G \cong L\colon 2_1$.

If $M = L\colon 2_1$, then $\Gamma(G) = \Gamma(M)$ by our assumptions.

First let G be a solvable group. Then G has a solvable Hall $\{3,5,7\}$-subgroup H. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in H are of prime power order. Hence $t(H) \le 2$ by Lemma 2.2, a contradiction. Thus G is not solvable, which implies that G is not a 2-Frobenius group by Lemma 2.6(c). If G is a non-solvable Frobenius group with H and K being its Frobenius complement and Frobenius kernel, respectively, then, by Lemma 2.5(c), it follows that H has a normal subgroup H_0 with $|H\colon H_0| \le 2$ such that $H_0 = \mathrm{SL}(2,5)\times Z$, where the Sylow subgroups of Z are cyclic and $\big(|Z|,30\big) = 1$. Thus $7 \in \pi(K)$ since $5 \nsim 7$ in $\Gamma(G)$. Since $|G| = |M| = 2^5\cdot 3\cdot 5^2\cdot 7^2$ and $|\,\mathrm{SL}(2,5)| = 2^3\cdot 3\cdot 5$, it follows that $5 \in \pi(K)$ too. Because K is nilpotent by Lemma 2.5(a), it follows that $5 \sim 7$ in $\Gamma(K)$, an obvious contradiction. Hence G is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a finite simple $C_{7,7}$-group and G/G_1 is a solvable π_1-group. By Lemmas 2.3 and 2.8(b), we obtain that G_1/N must be isomorphic to L.

Since $G/N \lesssim \mathrm{Aut}\,(G_1/N)$, it follows that $L \lesssim G/N \lesssim \mathrm{Aut}\,(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lesssim \mathrm{Aut}\,(N) = 1$, it follows that $N \le Z(G)$. Suppose $G_7 \in \mathrm{Syl}_7(G)$. Then NG_7 is a subgroup of G, which implies that $2 \sim 7$ in $\Gamma(NG_7)$, an obvious contradiction. Therefore $G/N \cong L\colon 2_1$, $L\colon 2_2$ or $L\cdot 2_3$ since $|G| = 2|L|$. It follows that $G \cong L\colon 2_1$, $L\colon 2_2$ or $L\cdot 2_3$ by Lemma 2.1. Obviously, $G \cong L\colon 2_1$ since $2 \nsim 5$ in $\Gamma(L\colon 2_2)$ and $\Gamma(L\cdot 2_3)$.

Case 3. If $M = L\colon 2_2$, then $G \cong L\colon 2_2$.

If $M = L\colon 2_2$, then $\Gamma(G) = \Gamma(M)$.

First let G be a solvable group. Then G has a solvable Hall $\{3,5,7\}$-subgroup H. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in H are of prime power order. Hence $t(H) \le 2$ by Lemma 2.2, a contradiction. Thus G is not solvable, which implies that G is not a 2-Frobenius group by Lemma

2.6(c). If G is a non-solvable Frobenius group with H and K being its Frobenius complement and Frobenius kernel, respectively, then, by Lemma 2.5(c), it follows that H has a normal subgroup H_0 with $|H: H_0| \leq 2$ such that $H_0 = \mathrm{SL}(2,5) \times Z$, where the Sylow subgroups of Z are cyclic and $\big(|Z|, 30\big) = 1$. Thus $7 \in \pi(K)$ since $5 \nsim 7$ in $\Gamma(G)$. Since $|G| = |M| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^2$ and $|\,\mathrm{SL}(2,5)| = 2^3 \cdot 3 \cdot 5$, it follows that $5 \in \pi(K)$ too. Because K is nilpotent by Lemma 2.5(a), it follows that $5 \sim 7$ in $\Gamma(K)$, an obvious contradiction. Hence G is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a finite simple $C_{5,5}$-group and G/G_1 is a solvable π_1-group. By Lemmas 2.3 and 2.8(a), we obtain that G_1/N must be isomorphic to L.

Since $G/N \lesssim \mathrm{Aut}\,(G_1/N)$, it follows that $L \lesssim G/N \lesssim \mathrm{Aut}\,(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lesssim \mathrm{Aut}\,(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_5 \in \mathrm{Syl}_5(G)$. Then NG_5 is a subgroup of G, which implies that $2 \sim 5$ in $\Gamma(NG_5)$, an obvious contradiction. Therefore $G/N \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ since $|G| = 2|L|$. It follows that $G \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ by Lemma 2.1. Obviously, $G \cong L\colon 2_2$ since $2 \nsim 7$ in $\Gamma(L\colon 2_1)$ and $\Gamma(L \cdot 2_3)$.

Case 4. If $M = L \cdot 2_3$, then $G \cong L \cdot 2_3$.

If $M = L \cdot 2_3$, then $\Gamma(G) = \Gamma(M)$. Thus $t(G) = t(M) = 3$. By Lemmas 2.4 and 2.6(a), G is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a finite simple $C_{5,5}$- and $C_{7,7}$-group, and G/G_1 is a solvable π_1-group. By Lemmas 2.3 and 2.8, we obtain that G_1/N must be isomorphic to L.

Since $G/N \lesssim Aut(G_1/N)$, it follows that $L \lesssim G/N \lesssim \mathrm{Aut}\,(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lesssim \mathrm{Aut}\,(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_5 \in \mathrm{Syl}_5(G)$. Then NG_5 is a subgroup of G, which implies that $2 \sim 5$ in $\Gamma(NG_5)$, an obvious contradiction. Therefore $G/N \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ since $|G| = 2|L|$. It follows that $G \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ by Lemma 2.1. Obviously, $G \cong L \cdot 2_3$ since $2 \sim 5$ in $\Gamma(L\colon 2_1)$ and $2 \sim 7$ in $\Gamma(L\colon 2_2)$, respectively.

Case 5. If $M = L \cdot 2^2$, then $G \cong L \cdot 2^2$, $\mathbb{Z}_2 \times (L\colon 2_1)$, $\mathbb{Z}_2 \times (L\colon 2_2)$, $\mathbb{Z}_2 \times (L \cdot 2_3)$, $\mathbb{Z}_2 \cdot (L\colon 2_1)$, $\mathbb{Z}_2 \cdot (L\colon 2_2)$, $\mathbb{Z}_2 \cdot (L \cdot 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-subgroup. In particular, G is non-solvable.

If $M = L \cdot 2^2$, then $\Gamma(G) = \Gamma(M)$.

First assume that $\{5,7\} \subseteq \pi(K)$. Let T be a Hall $\{5,7\}$-subgroup of K. It is easy to see that T is an abelian subgroup of order $5^i \cdot 7^j$, where $i, j = 1$ or 2. Thus $5 \cdot 7 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Next, we assume that $5 \in \pi(K)$ and $7 \notin \pi(K)$. Then K is a $\{2,3,5\}$-group. Let $R \in \mathrm{Syl}_5(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order 7, say x. Now $< x > R$ is a subgroup of G of order $5^i \cdot 7$, where $i = 1$ or 2. Hence $\langle x \rangle R$ is an abelian group. Thus $5 \cdot 7 \in \pi_e(\langle x \rangle R) \subseteq \pi_e(G)$, a contradiction. Finally, we assume $7 \in \pi(K)$ and $5 \notin \pi(K)$. In this case, K is a $\{2,3,7\}$-subgroup and we consider the Sylow 7-subgroup P of K. As before, we see that $G = KN_G(P)$ and by a similar argument we get $5 \cdot 7 \in \pi_e(G)$, which is a contradiction. Thus K is a $\{2,3\}$-subgroup.

Let G be a solvable group. Then G has a solvable Hall $\{3,5,7\}$-subgroup H. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in H are of prime power order. Hence $t(H) \leq 2$ by Lemma 2.2, a contradiction. Thus G is not solvable.

Step 2. The quotient G/K is an almost simple group. In fact, $S \lesssim G/K \lesssim \mathrm{Aut}\,(S)$ where S is a finite non-abelian simple group isomorphic to A_5, $L_2(7)$ or L.

Let $\overline{G} := G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \cdots \times P_m$, where P_i's are finite non-abelian simple groups. It is obvious that $\{2,3\} \subseteq \pi(P_i) \subseteq \{2,3,5,7\}$ by Lemma 2.3, where $i = 1, 2, \ldots, m$. Now we assert that $C_{\overline{G}}(S)$ is solvable. In fact, if $C_{\overline{G}}(S)$ is non-solvable, then it can not be a $\{2,3\}$-group by Burnside's Theorem. It follows that $5 \in \pi(C_{\overline{G}}(S))$ or $7 \in \pi(C_{\overline{G}}(S))$, which shows that $3 \cdot 5 \in \pi_e(\overline{G})$ or $3 \cdot 7 \in \pi_e(\overline{G})$ since $\{2,3\} \subseteq \pi(P_i) \subseteq \pi(S)$. It follows that $3 \cdot 5 \in \pi_e(G)$ or $3 \cdot 7 \in \pi_e(G)$, which is a contradiction since $3 \nsim 5$ and $3 \nsim 7$ in $\Gamma(G)$. Suppose $1 \neq T/K =: C_{\overline{G}}(S)$, which is solvable. Then $T/K \neq G/K$ since G/K is non-solvable. Thus $K \trianglelefteq T \trianglelefteq G$, where T is solvable. This is a contradiction by the choice of K. Hence $C_{\overline{G}}(S) = 1$. It follows that $S \lesssim G/K \cong G/K/C_{\overline{G}}(S) \lesssim \mathrm{Aut}\,(S)$.

By Lemma 2.3, it is clear that $m = 1$ since $|G|_3 = 3$, where $|G|_3$ is the 3-part of $|G|$. Using Table 1, S is isomorphic to one of the following simple groups: A_5, $L_2(7)$ or L.

Step 3. $G \cong L\cdot 2^2$, $\mathbb{Z}_2 \times (L\colon 2_1)$, $\mathbb{Z}_2 \times (L\colon 2_2)$, $\mathbb{Z}_2 \times (L \cdot 2_3)$, $\mathbb{Z}_2 \cdot (L\colon 2_1)$, $\mathbb{Z}_2 \cdot (L\colon 2_2)$, $\mathbb{Z}_2 \cdot (L \cdot 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

By Step 2, $S \lesssim G/K \lesssim \mathrm{Aut}\,(S)$ where S is a finite non-abelian simple group isomorphic to A_5, $L_2(7)$ or L.

If $S \cong A_5$, then $A_5 \lesssim \overline{G} \lesssim \mathrm{Aut}\,(A_5)$. It follows that $|K| = 2^4 \cdot 5 \cdot 7^2$ or $2^3 \cdot 5 \cdot 7^2$ by Lemma 2.3. Obviously, this is a contradiction since K is a $\{2,3\}$-group by Step 1.

If $S \cong L_2(7)$, then $L_2(7) \lesssim \overline{G} \lesssim \mathrm{Aut}\,(L_2(7))$. It follows that $|K| = 2^3 \cdot 5^2 \cdot 7$ or $2^2 \cdot 5^2 \cdot 7$ by Lemma 2.3. Obviously, this is a contradiction since K is a $\{2,3\}$-group by Step 1.

Therefore, $S \cong L$. Thus $L \lesssim \overline{G} \lesssim \mathrm{Aut}\,(L)$. Hence $|K| = 1$, 2 or 2^2.

If $|K| = 1$, then $G \cong L \cdot 2^2$ by Lemma 2.1.

If $|K| = 2$, then $K \leq Z(G)$, i.e., G is a central extension of K by $L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$. If G splits over K, we obtain $G \cong \mathbb{Z}_2 \times (L\colon 2_1)$, $\mathbb{Z}_2 \times (L\colon 2_2)$ or $\mathbb{Z}_2 \times (L \cdot 2_3)$. Otherwise we have $G \cong \mathbb{Z}_2 \cdot (L\colon 2_1)$, $\mathbb{Z}_2 \cdot (L\colon 2_1)$ or $\mathbb{Z}_2 \cdot (L \cdot 2_3)$.

If $|K| = 2^2$, then $G/K \cong L$. In this case, we have $G/C_G(K) \lesssim \mathrm{Aut}\,(K) \cong \mathbb{Z}_2$ or S_3. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $\big|G/C_G(K)\big| = 1$, then $K \leq Z(G)$, i.e., G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 2 (see [3]). But this is a contradiction. So we obtain that G splits over K. Hence $G \cong K \times L$. Thus $G \cong \mathbb{Z}_4 \times L$ or

$(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ since $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $\left|G/C_G(K)\right| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$. Since L is simple, we obtain that $G = C_G(K)$, a contradiction. $\square$

Remark 1. W. J. Shi and J. X. Bi in [16] put forward the following conjecture:

Conjecture. Let G be a finite group and M a finite simple group. Then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for the sporadic simple groups (see [14]), alternating groups and some simple groups of Lie type (see [13, 15, 16]). As a consequence of Theorem 3.1, we verify the validity of this conjecture for the groups under discussion.

Theorem 3.2. *If G is a finite group such that $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, where M is an almost simple group related to $L_2(49)$ except $L_2(49) \cdot 2^2$, then $G \cong M$.*

Proof. Since $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, we obtain $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$. It follows that $|G| = |M|$ and $D(G) = D(M)$. By Theorem 3.1, we have $G \cong M$. $\square$

Note that if G is a finite group such that $|G| = |M|$ and $D(G) = D(M)$, where M is a given finite group, then $\pi_e(G)$ is not equal to $\pi_e(M)$ necessarily. Now, we give a counterexample as follows. Let $L := U_4(3)$, then $L\colon 2_2$ is 2-fold OD-characterizable by Lemma 2.9. However, in this case, $\pi_e(L\colon 2_2) = \{18, 12, 10, 8, 7\}$ is not equal to $\pi_e(L \cdot 2_3) = \{24, 10, 9, 7\}$ (see [9]).

Theorem 3.3. *If G is a finite group such that $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, where M is an almost simple group related to $L_2(49)$ except $L_2(49) \cdot 2^2$, then $G \cong M$.*

Proof. Since $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, we obtain that $|G| = |M|$ and $D(G) = D(M)$. By Theorem 3.1, we have $G \cong M$. $\square$

Question. Let G be a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where M is an almost simple group. Is G non-solvable, too?

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