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Abstract. In Kepka T., Kinyon M.K., Phillips J.D., *The structure of F-quasigroups*, J. Algebra **317** (2007), 435–461, we showed that every F-quasigroup is linear over a special kind of Moufang loop called an NK-loop. Here we extend this relationship by showing an equivalence between the class of (pointed) F-quasigroups and the class corresponding to a certain notion of generalized module (with noncommutative, nonassociative addition) for an associative ring.

Keywords: F-quasigroup, Moufang loop, generalized modules
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1. Introduction

A *quasigroup* $(Q, \cdot)$ is a set Q with a binary operation $\cdot : Q \times Q \to Q$, denoted by juxtaposition, such that for each $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$. In a quasigroup $(Q, \cdot)$, there exist transformations $\alpha, \beta : Q \to Q$ such that $x\alpha(x) = x = \beta(x)x$ for all $x \in Q$. Now $(Q, \cdot)$ is called an *F-quasigroup* if it satisfies the equations

$$x \cdot yz = y \cdot \alpha(x)z \qquad \text{and} \qquad zy \cdot x = z\beta(x) \cdot yx$$

for all $x, y, z \in Q$.

If $(Q, \cdot)$ is a quasigroup, we set $M(Q) = \{a \in Q : xa \cdot yx = xy \cdot ax,\ \forall\, x, y \in Q\}$. If $(Q, \cdot)$ is an F-quasigroup, then $M(Q)$ is a normal subquasigroup of Q and $Q/M(Q)$ is a group [3, Lemma 7.5].

We denote by $\mathcal{F}_p$ the category of pointed F-quasigroups. That is, $\mathcal{F}_p$ consists of ordered pairs (Q, a), where Q is an F-quasigroup and $a \in Q$. We put $\mathcal{F}_m = \{(Q, a) \in \mathcal{F}_p : a \in M(Q)\}$.

A quasigroup with a neutral element is called a *loop*. Throughout this paper, we adopt an additive notation convention $(Q, +)$ (with neutral 0) for loops, although we do not assume that $+$ is commutative. The *nucleus* of a loop $(Q, +)$ is the set

$$N(Q,+) = \{a \in Q : \left\{ \begin{matrix} (a+x)+y = a+(x+y) \\ (x+a)+y = x+(a+y) \\ (x+y)+a = x+(y+a) \end{matrix} \right\}, \forall\, x, y \in Q\}.$$

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The *Moufang center* is the set

$$K(Q,+) = \{a \in Q : (a+a)+(x+y) = (a+x)+(a+y),\ \forall\, x,y \in Q\}.$$

The intersection of the nucleus and Moufang center of a loop is the *center* $Z(Q,+) = N(Q,+) \cap K(Q,+)$. Each of the nucleus, the Moufang center, and the center is a subloop, and the center is, in fact, a normal subloop [1], [5].

A $(Q,+)$ will be called an *NK-loop* if for each $x \in Q$, there exist $u \in N(Q,+)$ and $v \in K(Q,+)$ such that $x = u+v$ $(= v+u)$. In other words, Q can be decomposed as a central product $Q = N(Q,+)\ K(Q,+)$. It was shown in [3] that every NK-loop is a Moufang A-loop. A *Moufang loop* is a loop satisfying the identity $((x+y)+x)+z = x+(y+(x+z))$ or any of its known equivalents [1], [5]. Every Moufang loop is *diassociative*, that is, the subloop generated by any given pair of elements is a group [4]. For a loop $(Q,+)$, the *inner mapping group* is the stabilizer of 0 in the group of permutations of Q generated by all left and right translations $L_x y = x+y = R_y x$. An *A-loop* is a loop such that every inner mapping is an automorphism [2].

In any Moufang A-loop $(Q,+)$, such as an NK-loop, the nucleus $N(Q,+)$ is normal (in fact, this is true in any Moufang loop), and $Q/N(Q,+)$ is a commutative Moufang loop of exponent 3. In particular, for each $x \in Q$, $3x \in N(Q,+)$, where $3x = x+x+x$. The Moufang center $K(Q,+)$ is also normal in Q (but this is not necessarily the case in arbitrary Moufang loops), and $Q/K(Q,+)$ is a group [3, Lemma 4.3]. In an NK-loop $(Q,+)$, we also have $Z(Q,+) = Z(N(Q,+)) = K(N(Q,+)) = Z(K(Q,+)) = N(K(Q,+))$. In addition, $K(Q,+) = \{a \in Q : a+x = x+a\ \ \forall\, x \in Q\}$.

The connection between F-quasigroups and NK-loops was established in [3].

Proposition. *For a quasigroup $(Q,\cdot)$, the following are equivalent.*

1. *$(Q,\cdot)$ is an F-quasigroup.*
2. *There exist an NK-loop $(Q,+)$, $f,g \in \mathcal{A}ut(Q,+)$, and $e \in N(Q,+)$ such that $x\cdot y = f(x)+e+g(y)$ for all $x,y \in Q$, $fg = gf$, and $x+f(x), x+g(x) \in N(Q,+)$, $-x+f(x), -x+g(x) \in K(Q,+)$ for all $x \in Q$.*

We refer to the data $(Q,+,f,g,e)$ of the proposition as being an *arithmetic form* of the F-quasigroup $(Q,\cdot)$. If (Q,a) is a pointed F-quasigroup in $\mathcal{F}_p$, then there is an arithmetic form such that $a = 0$ is the neutral element of $(Q,+)$.

The purpose of this paper is to extend the connection between (pointed) F-quasigroups and NK-loops further by showing an equivalence of classes between $\mathcal{F}_p$ and a certain notion of generalized module for an associative ring. Thus the study of (pointed) F-quasigroups effectively becomes a part of ring theory. The generalization we require weakens the additive abelian group structure of a module to an NK-loop structure.

Definition. Let R be an associative ring, possibly without unity. A *generalized* (*left*) *R-module* is an NK-loop $(Q,+)$ supplied with an R-scalar multiplication $R \times Q \to Q$ such that the following conditions are satisfied: for all $a, b \in R$, $x, y \in Q$, $z \in N(Q,+)$, and $w \in K(Q,+)$,

1. $a(x+y) = ax + ay$,
2. $(a+b)x = ax + bx$,
3. $a(bx) = (ab)x$,
4. $ax \in K(Q,+)$,
5. $az \in N(Q,+)$, and
6. there exists an integer m such that $mw + aw \in Z(Q,+)$.

Here $mw = w + \cdots + w$ (m terms) is unambiguous by diassociativity.

If Q is a generalized R-module, then define the *annihilator* of Q to be $\mathrm{Ann}(Q) = \{a \in R : aQ = 0\}$. Clearly, $\mathrm{Ann}(Q)$ is an ideal of the ring R.

In order to state our main result, we need to describe a particular ring. Let $\mathbf{S} = \mathbb{Z}[\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}]$ be the polynomial ring in four commuting indeterminates $\mathbf{x}, \mathbf{y}, \mathbf{u}$, and $\mathbf{v}$ over the ring $\mathbb{Z}$ of integers. Put $\mathbf{R} = \mathbf{Sx} + \mathbf{Sy} + \mathbf{Su} + \mathbf{Sv}$, so that $\mathbf{R}$ is the ideal generated by the indeterminates. Clearly, $\mathbf{R}$ is a free commutative and associative ring (without unity) freely generated by the indeterminates.

Let $\mathcal{M}$ denote the category of generalized $\mathbf{R}$-modules Q such that:

1. $2z + \mathbf{x}z \in N(Q,+)$, $2z + \mathbf{y}z \in N(Q,+)$ for all $z \in Q$,
2. $\mathbf{x} + \mathbf{u} + \mathbf{xu} \in \mathrm{Ann}(Q)$, and
3. $\mathbf{y} + \mathbf{v} + \mathbf{yv} \in \mathrm{Ann}(Q)$.

Further, let $\mathcal{M}_p$ be the category of pointed objects from $\mathcal{M}$. That is, $\mathcal{M}_p$ consists of ordered pairs (Q, e), where $Q \in \mathcal{M}$ and $e \in Q$. Put $\mathcal{M}_n = \{(Q,e) \in \mathcal{F}_p : e \in N(Q,+)\}$, the category of *nuclearly* pointed objects from $\mathcal{M}$, and put $\mathcal{M}_c = \{(Q,e) \in \mathcal{F}_p : e \in Z(Q,+)\}$, the category of *centrally* pointed objects from $\mathcal{M}$.

Our main result is the following equivalence between pointed F-quasigroups and generalized $\mathbf{R}$-modules.

Main Theorem. *The classes $\mathcal{F}_p$ and $\mathcal{M}_n$ are equivalent. The equivalence restricts to an equivalence between $\mathcal{F}_m$ and $\mathcal{M}_c$.*

2. Quasicentral endomorphisms

In this section, let $(Q,+)$ denote a (possibly non-commutative) diassociative loop. We endow the set $\mathcal{E}nd(Q,+)$ of all endomorphisms of $(Q,+)$ with the standard operations of addition, negation, and composition, *viz.*, for $f, g \in \mathcal{E}nd(Q,+)$, $f+g$ is defined by $(f+g)(x) = f(x) + g(x)$, $-f$ is defined by $(-f)(x) = -f(x) = f(-x)$, and fg is defined by $fg(x) = f(g(x))$ for all $x \in Q$.

An endomorphism f of $(Q,+)$ is called *central* if $f(Q) \subset Z(Q,+)$. We denote the set of all central endomorphisms of $(Q,+)$ by $\mathcal{ZE}nd(Q,+)$. The verification of the following result is easy and omitted.

Lemma 2.1. *Let $f, g, h \in \mathcal{ZE}nd(Q,+)$ be given. Then:*

1. $f + g \in \mathcal{ZE}nd(Q,+)$,
2. $f + (g + h) = (f + g) + h$ *and* $f + g = g + f$,
3. *the zero endomorphism of $(Q,+)$ is central,*
4. $-f \in \mathcal{ZE}nd(Q,+)$,
5. $f + (-f) = 0$ *and* $f + 0 = f$,
6. $fg \in \mathcal{ZE}nd(Q,+)$.

Corollary 2.2. *$\mathcal{ZE}nd(Q,+)$ is an associative ring (possibly without unity) with respect to the standard operations.*

Let m be an integer. An endomorphism f of $(Q,+)$ is called *m-quasicentral* if $mx + f(x) \in Z(Q,+)$ for all $x \in Q$ (in which case $mx + f(x) = f(x) + mx$). An endomorphism is called *quasicentral* if it is m-quasicentral for at least one integer m. We denote by $\mathcal{QE}nd(Q,+)$ the set of all quasicentral endomorphisms of $(Q,+)$. The following is an obvious consequence of these definitions.

Lemma 2.3. 1. *An endomorphism is 0-quasicentral if and only if it is central,*
2. $\mathcal{ZE}nd(Q,+) \subset \mathcal{QE}nd(Q,+)$, *and*
3. *the identity automorphism, Id_Q, of $(Q,+)$ is (-1)-quasicentral.*

Lemma 2.4. *Let $f, g \in \mathcal{E}nd(Q,+)$.*

1. *If f is m-quasicentral and g is n-quasicentral, then fg is $(-mn)$-quasicentral.*
2. *If $f, g \in \mathcal{QE}nd(Q,+)$, then $fg \in \mathcal{QE}nd(Q,+)$.*

Proof: For (1): Fix $x \in Q$. Since f is m-quasicentral, $g(mx) + fg(x) = mg(x) + fg(x) \in Z(Q,+)$. Since g is n-quasicentral, $-mnx - mg(x) = -(g(mx) + nmx) \in Z(Q,+)$. Consequently,

$$\begin{aligned} -mnx + fg(x) &= ([-mnx - mg(x)] + mg(x)) + fg(x) \\ &= [-mnx - mg(x)] + [mg(x) + fg(x)] \in Z(Q,+). \end{aligned}$$

Thus, fg is $(-mn)$-quasicentral, as claimed.

(2) follows immediately from (1). □

Lemma 2.5. *Assume that $(Q,+)$ is commutative, let $f, g \in \mathcal{E}nd(Q,+)$ be m-quasicentral and n-quasicentral, respectively. Then*

1. *$-f$ is $(-m)$-quasicentral,*
2. *$f + g$ is an $(m+n)$-quasicentral endomorphism.*

In particular, for $f, g \in \mathcal{QE}nd(Q,+)$, $-f, f + g \in \mathcal{QE}nd(Q,+)$.

Proof: (1) is clear. For (2), set $z = (-mx - f(x)) + (-my - f(y)) + (-mx - g(x)) + (-my - g(y))$. Then $z \in Z(Q,+)$. It follows that

$$\begin{aligned} z + (f+g)(x+y) &= z + ([f(x)+f(y)] + [g(x)+g(y)]) = -mx - my - nx - ny \\ &= -mx - nx - my - ny = z + ([f(x)+g(x)] + [f(y)+g(y)]) \\ &= z + ((f+g)(x) + (f+g)(y)). \end{aligned}$$

Thus $f + g \in \mathcal{E}nd(Q,+)$. Similarly, $(m+n)x + (f+g)(x) = [mx + f(x)] + [nx + g(x)] \in Z(Q,+)$. That is, (2) holds. □

Lemma 2.6. *Assume that* $(Q,+)$ *is commutative and let* $f, g, h \in \mathcal{QE}nd(Q,+)$. *Then*

1. $f + g = g + f$,
2. $f + (g + h) = (f + g) + h$,
3. $f + (-f) = 0$, *and*
4. $f + 0 = f$.

Proof: (1), (3), and (4) are obvious. For (2): There exist $m, n, k \in \mathbb{Z}$ such that $mx+f(x), nx+g(x), kx+h(x) \in Z(Q,+)$. Set $y = (-f(x)-mx)+(-g(x)-nx)+(-h(x)-kx)$. Then $y \in Z(Q,+)$ and $y+(f(x)+(g(x)+h(x))) = -(m+n+k)x = y + ((f(x) + g(x)) + h(x))$ for all $x \in Q$. □

Corollary 2.7. *If* $(Q,+)$ *is commutative, then* $\mathcal{QE}nd(Q,+)$ *is an associative ring with unity.*

We conclude this section with a straightforward observation.

Lemma 2.8. *Assume that for* $k \in \{1,2,3\}$, $kx \in Z(Q,+)$ *for all* $x \in Q$. *Then*

1. *every quasicentral endomorphism is* m*-central for some* $m \in \{0, 1, -1\}$,
2. *if* $f \in \mathcal{QE}nd(Q,+) \cap \mathcal{A}ut(Q,+)$, *then* $f^{-1} \in \mathcal{QE}nd(Q,+)$.

3. Special endomorphisms of NK-loops

In this section, let $(Q,+)$ be an NK-loop. We denote by N, K, and Z the underlying sets of $N(Q,+)$, $K(Q,+)$, and $Z(Q,+)$, respectively. As noted in §1, $Z(Q,+) = Z(N,+) = Z(K,+)$ and $Z = N \cap K$.

An endomorphism f of $(Q,+)$ will be called *special* if $f(Q) \subset K$, $f|_K$ is a quasicentral endomorphism of $(K,+)$, and $f(N) \subset N$. Then $f|_N$ is a central endomorphism of $(N,+)$ and $f(N) \subset Z$. We denote by $\mathcal{SE}nd(Q,+)$ the set of special endomorphisms of $(Q,+)$.

Lemma 3.1. *Let* $f, g, h \in \mathcal{SE}nd(Q,+)$. *Then*

1. $fg \in \mathcal{SE}nd(Q,+)$,
2. $f + g \in \mathcal{SE}nd(Q,+)$, *and* $f + g = g + f$,
3. $f + (g + h) = (f + g) + h$,
4. $-f \in \mathcal{SE}nd(Q,+)$, $f + (-f) = 0$, *and* $f + 0 = f$.

Proof: For (1), use Lemma 2.4.

For (2): Take $x, y \in Q$. Then $x = a+b$ and $y = c+d$ for some $a, c \in N, b, d \in K$ so that

$$\begin{aligned} u &= (f+g)(x+y) = f(x+y) + g(x+y) = [f(x)+f(y)] + [g(x)+g(y)] \\ &= [(f(a)+f(b)) + (f(c)+f(d))] + [(g(a)+g(b)) + (g(c)+g(d))] \end{aligned}$$

and

$$\begin{aligned} v &= (f+g)(x) + (f+g)(y) = [f(x)+g(x)] + [f(y)+g(y)] \\ &= [(f(a)+f(b)) + (g(a)+g(b))] + [(f(c)+f(d)) + (g(c)+g(d))]. \end{aligned}$$

The restrictions $f|_N$ and $g|_N$ are central endomorphisms of $(N,+)$, and it follows that $f(N) \cup g(N) \subset Z(N,+) = Z(Q,+)$. Thus, $f(a), f(c), g(a), g(c) \in Z$ and in order to check that $u = v$ it is sufficient to show that $(f(b)+f(d)) + (g(b)+g(d)) = (f(b)+g(b)) + (f(d)+g(d))$. However, the latter equality holds, since the restrictions $f|_K$ and $g|_K$ are quasicentral endomorphisms of the commutative loop $(K,+)$ and Corollary 2.7 applies.

We have shown that $f+g \in \mathcal{E}nd(Q,+)$. The facts that $f+g$ is special and $f+g = g+f$ are easily seen, using Lemma 2.6 applied to the loop $(K,+)$.

For (3): Using the facts that $(Q,+)$ is an NK-loop and $f(N) \cup g(N) \cup h(N) \subset Z$, it is enough to show that $f(u)+(g(u)+h(u)) = (f(u)+g(u))+h(u)$ for all $u \in K$. Now we proceed similarly as in the proof of Lemma 2.6.

Finally, (4) is easy. □

Corollary 3.2. *$\mathcal{SE}nd(Q,+)$ is an associative ring (possibly without unity).*

An endomorphism f of $(Q,+)$ will be said to satisfy *condition* (F) if

$$-x + f(x) \in K \qquad \text{and} \qquad x + f(x) \in N$$

for all $x \in Q$. Then $f(K) \subset K$ and $f(N) \subset N$.

Lemma 3.3. *Let $f \in \mathcal{E}nd(Q,+)$ satisfy (F). Define $h : Q \to Q$ by $h(x) = -x + f(x)$ for all $x \in Q$. Then $h \in \mathcal{SE}nd(Q,+)$.*

Proof: First we check that $h \in \mathcal{E}nd(Q,+)$. Fix $x, y \in Q$ with $x = a+b$, $y = c+d$, $a, c \in N$, $b, d \in K$. Set $u = h(x+y)$, $v = h(x)+h(y)$, and $w = (a - f(a)) + (c - f(c)) + (-b - f(b)) + (-d - f(d))$. Then $w \in Z$,

$$u = (-y-x) + f(x+y) = ((-d-c)+(-b-a)) + ((f(a)+f(b))) + (f(c)+f(d))$$

and

$$v = (-x+f(x)) + (-y+f(y)) = (-b-a) + (f(a)+f(b)) + ((-d-c)+(f(c)+f(d))).$$

On the other hand,

$$\begin{aligned} u+w &= [(-d-c)+(-b-a)]+[(a-b)+(c-d)] \\ &= [(-d-c)-b]+[-a+(a-b)]+(c-d) \\ &= [(-d-c)-b]+[(c-d)-b] \\ &= [(-d-c)+(c-d)]-2b \\ &= -2d-2b \\ &= -2(b+d) \\ &= [(-b-a)+(a-b)]+[(-d-c)+(c-d)] \\ &= v+w. \end{aligned}$$

Consequently, $u=v$, so that $h \in \mathcal{E}nd(Q,+)$, as claimed. Further, it follows immediately from the definition of h that $h(Q) \subset K$ and $h(N) \subset N$ (then $h(N) \subset Z$). Finally, $2a+h(a)=a+f(a) \in Z$ for all $a \in K$, and therefore $h|_K$ is a 2-quasicentral endomorphism of $(K,+)$. Thus $h \in \mathcal{SE}nd(Q,+)$. □

Lemma 3.4. *Let $f,g \in \mathcal{E}nd(Q,+)$ satisfy (F). Define $h,k: Q \to Q$ by $h(x) = -x+f(x)$ and $k(x)=-x+g(x)$ for all $x \in Q$. Then $hk=kh$ if and only if $fg=gf$.*

PROOF: By Lemma 3.3, $h \in \mathcal{E}nd(Q,+)$, and hence

$$hk(x)=h(-x+g(x))=-h(x)+hg(x)=(-f(x)+x)+(-g(x)+fg(x)).$$

On the other hand,

$$kh(x)=-h(x)+gh(x)=(-f(x)+x)+(-g(x)+gf(x))$$

by the definition of h and k. The result is now clear. □

Lemma 3.5. *Let $f,g \in \mathcal{A}ut(Q,+)$ satisfy (F). Define $h,k,p,q: Q \to Q$ by $h(x)=-x+f(x)$, $k(x)=-x+g(x)$, $p(x)=-x+f^{-1}(x)$, and $q(x)=-x+g^{-1}(x)$ for all $x \in Q$. Then*

1. *$h,k,p,q \in \mathcal{SE}nd(Q,+)$,*
2. *$hp=ph$ and $h+p+hp=0$,*
3. *$kq=qk$ and $k+q+kq=0$, and*
4. *if $fg=gf$, then the endomorphisms h,k,p,q commute pairwise.*

PROOF: (1) follows from Lemma 3.3.

For (2): We have $ff^{-1}=f^{-1}f$ and hence $hp=ph$ by Lemma 3.4. Now, put $A=h+p+hp$. Then A is a (special) endomorphism of $(Q,+)$ and $A(x)=[-x+f(x)]+[-x+f^{-1}(x)]+[(-f^{-1}(x)+x)+(-f(x)+x)]$. Clearly, $N \subset \ker(A)(=\{u \in$

$Q : A(u) = 0\}$). On the other hand, if $x \in K$, then $-x + f(x), -x + f^{-1}(x) \in Z$ and the equality $A(x) = 0$ is clear, too. Thus, $N \cup K \subset \ker(A)$. But $(Q, +)$ is an NK-loop and $\ker(A)$ is a subloop of $(Q, +)$. It follows $\ker(A) = Q$ and $A = 0$.

(3) is proven similarly to (2).

For (4), combine (2), (3), and Lemma 3.4. □

4. The equivalence

We now turn to the proof of the Main Theorem. First, recall the definition of generalized module over a ring R, and observe that the conditions (1), (4), (5), and (6) of the definition imply that for each $a \in R$, the transformation $Q \to Q; x \mapsto ax$ is a special endomorphism of $(Q, +)$. Recall also the ring $\mathbf{R}$, which is the ideal of $\mathbf{S} = \mathbb{Z}[\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}]$ freely generated by the commuting indeterminates $\mathbf{x}, \mathbf{y}, \mathbf{u}$, and $\mathbf{v}$.

First, take $(Q, a) \in \mathcal{F}_p$. As described in §1, let $(Q, +, f, g, e)$ be the arithmetic form of the F-quasigroup $(Q, \cdot)$ such that $a = 0$ in $(Q, +)$. Then $f, g \in \mathcal{A}ut(Q, +)$ satisfy condition (F). Further, define $\varphi, \mu, \psi, \nu : Q \to Q$ by $\varphi(x) = -x + f(x)$, $\mu(x) = -x + f^{-1}(x)$, $\psi(x) = -x + g(x)$, and $\nu(x) = -x + g^{-1}(x)$ for all $x \in Q$. By Lemma 3.5, the special endomorphisms φ, ψ, μ, and ν of the loop $(Q, +)$ commute pairwise, and $\varphi + \mu + \varphi\mu = 0 = \psi + \nu + \psi\nu$. Consequently, these endomorphisms generate a commutative subring of the ring $\mathcal{SE}nd(Q, +)$ (see Corollary 3.2) and there exists a (uniquely determined) homomorphism $\lambda : \mathbf{R} \to \mathcal{SE}nd(Q, +)$ such that $\lambda(\mathbf{x}) = \varphi$, $\lambda(\mathbf{y}) = \psi$, $\lambda(\mathbf{u}) = \mu$, and $\lambda(\mathbf{v}) = \nu$. The homomorphism λ induces an $\mathbf{R}$-scalar multiplication on the loop $(Q, +)$, and the resulting generalized $\mathbf{R}$-module will be denoted by $\overline{Q}$. By Lemma 3.5, $\lambda(\mathbf{x} + \mathbf{u} + \mathbf{xu}) = 0 = \lambda(\mathbf{y} + \mathbf{v} + \mathbf{yv})$, and so $\mathbf{x} + \mathbf{u} + \mathbf{xu}, \mathbf{y} + \mathbf{v} + \mathbf{yv} \in \mathrm{Ann}(Q)$. Also, since f, g satisfy (F), we have $2z + \lambda(\mathbf{x})z = 2z + \varphi(z) = z + f(z) \in N(Q, +)$ and similarly $2z + \lambda(\mathbf{y})z \in N(Q, +)$ for all $z \in Q$. It follows that $\overline{Q} \in \mathcal{M}$. Now define $\rho : \mathcal{F}_p \to \mathcal{M}_n$ by $\rho(Q, a) = (\overline{Q}, e)$, and observe that $(\overline{Q}, e) \in \mathcal{M}_c$ if and only if $e \in Z(Q, +)$.

Next, take $(\overline{Q}, e) \in \mathcal{M}_n$ and define $f, g : Q \to Q$ by $f(z) = z + \mathbf{x}z$ and $g(z) = z + \mathbf{y}z$ for all $z \in Q$. We have $f(x+y) = (x+y) + (\mathbf{x}x + \mathbf{x}y)$ and $f(x) + f(y) = (x + \mathbf{x}x) + (y + \mathbf{x}y)$. Further, $x = u_1 + v_1$, $y = u_2 + v_2$ for some $u_1, u_2 \in N(Q, +)$, $v_1, v_2 \in K(Q, +)$, and hence, $f(x+y) = (u_1 + u_2 + v_1 + v_2) + (\mathbf{x}u_1 + \mathbf{x}u_2 + \mathbf{x}v_1 + \mathbf{x}v_2)$, and $f(x) + f(y) = (u_1 + \mathbf{x}u_1 + v_1 + \mathbf{x}v_1) + (u_2 + \mathbf{x}u_2 + v_1 + \mathbf{x}v_2)$. But $\mathbf{x}u_1, \mathbf{x}u_2 \in Z(Q, +)$, and so in order to show $f(x+y) = f(x) + f(y)$, it is enough to check that $(v_1 + v_2) + (\mathbf{x}v_1 + \mathbf{x}v_2) = (v_1 + \mathbf{x}v_1) + (v_2 + \mathbf{x}v_2)$. However, $-2v_1 - \mathbf{x}v_1 \in Z(Q, +)$ and $-2v_2 - \mathbf{x}v_2 \in Z(Q, +)$, and so the latter equality is clear.

We have proven that $f \in \mathcal{E}nd(Q, +)$, and the proof that $g \in \mathcal{E}nd(Q, +)$ is similar. Now by definition of generalized module, $-x + f(x) = \mathbf{x}x \in K(Q, +)$ and $-x + g(x) = \mathbf{y}x \in K(Q, +)$ for all $x \in Q$. By definition of $\mathcal{M}$, $x + f(x) = 2x + \mathbf{x}x \in N(Q, +)$ and $x + g(x) = 2x + \mathbf{y}x \in N(Q, +)$ for all $x \in Q$. This means that both f and g satisfy (F) and it follows from Lemma 3.4 that $fg = gf$.

Define $h : Q \to Q$ by $h(x) = x + \mathbf{u}x$ for $x \in Q$. We have $\mathbf{u}x + \mathbf{x}x + \mathbf{xu}x = 0$, and so $\mathbf{x}x + \mathbf{xu}x = -\mathbf{u}x$. Now, $fh(x) = h(x) + \mathbf{x}h(x) = (x + \mathbf{u}x) + (\mathbf{x}x + \mathbf{xu}x) = (x + \mathbf{u}x) - \mathbf{u}x = x$ and $fh = \mathrm{Id}_Q$. Similarly, $hf = \mathrm{Id}_Q$ and we see that $f \in \mathcal{A}ut(Q, +)$. Similarly, $g \in \mathcal{A}ut(Q, +)$.

We have that $f, g \in \mathcal{A}ut(Q, +)$, and $e \in Q$ satisfy the conditions of the Proposition of §1, and so defining a multiplication on Q by $xy = f(x) + e + g(y)$ for all $x, y \in Q$ gives an F-quasigroup. Define $\sigma : \mathcal{M}_n \to \mathcal{F}_p$ by $\sigma(\overline{Q}, e) = (Q, 0)$.

It is easy to check that the operators ρ and σ represent an equivalence between $\mathcal{F}_p$ and $\mathcal{M}_n$. Further, $0 \in \mathcal{M}(Q)$ if and only if $e \in Z(Q, +)$, so that ρ and σ restrict to an equivalence between $\mathcal{F}_m$ and $\mathcal{M}_c$. This completes the proof of the Main Theorem.

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