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Gaps and dualities in Heyting categories

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Abstract. We present an algebraic treatment of the correspondence of gaps and dualities in partial ordered classes induced by the morphism structures of certain categories which we call Heyting (such are for instance all cartesian closed categories, but there are other important examples). This allows to extend the results of [14] to a wide range of more general structures. Also, we introduce a notion of combined dualities and discuss the relation of their structure to that of the plain ones.

Keywords: Heyting algebras, dualities and gaps, Heyting categories

Classification: 06D20, 18D15, 05C99

Introduction

The object of the theory of homomorphism duality is to characterise a family $\mathcal{C}$ of obstructions to the existence of a homomorphism into a given structure B. In a large sense, such a class $\mathcal{C}$ always exists; for instance, the class of all the structures not admitting a homomorphism to B has this property. However, it is desirable to seek a more tractable family of obstructions to make this characterisation meaningful.

When the family $\mathcal{C}$ of obstructions is finite (or algorithmically "well behaved"), then such theorems clearly provide an example of *good characterisations* (in the sense of Edmonds [4]). Any instance of such good characterisation is called a *homomorphism duality*. This concept was introduced by Nešetřil and Pultr [12] and applied to various graph-theoretical good characterisations (see [11], [5], and references there). The simplest homomorphism dualities are those where the family of obstructions consists from just one structure. In other words, such homomorphism dualities are described by a pair A, B of structures as follows.

> (*Singleton*) *Homomorphism Duality Scheme:*
> C admits a homomorphism into B if and only if A does not admit a homomorphism into C.

The (singleton) homomorphism duality may capture general theorems such as Farkas Lemma (see [6]) and Menger-type theorems ([7]). For undirected graphs

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there is only one singleton duality (Nešetřil and Pultr [12]), but there are many in the directed case (Komárek [9], and Nešetřil and Tardif [13] present a complete list). In [14] the problem is solved in a surprising generality for all finite relational structures. In view of the scarcity of examples that arise in the category of undirected graphs, and in view of the difficulty of the dichotomy problem even for directed graphs it seemed unlikely that the framework for such a generalisation would be found in this context. Yet paradoxically, this is precisely what happened. The absence of good characterisations for undirected graphs is explained by an apparently unrelated result, that is, the density theorem of Welzl ([15]), which states that the class of undirected non-bipartite graphs is dense with respect to the homomorphism order.

The arguments in [14], formulated in terms of finite relational structures, have a much more general range. One considers the partially ordered class (in fact, lattice) obtained from the preorder

$$A \leq B \ \text{ iff there exists a morphism } \ f : A \to B$$

on the class of objects of the category in question; then, all the reasoning is based on the lattice structure and its two special properties, namely that

(1) there is a Heyting operation (that is, an operation adjoint to the meet), and
(2) the elements are suprema of systems of connected ones.

In categories we are usually concerned with (they are typically such that the coproducts are disjoint unions), the second property is ubiquitous. The first ("Heyting") condition is somewhat more special; still, it is being satisfied quite frequently. For instance it holds true whenever the original category is cartesian closed; we point out several such categories of a combinatorial nature. However, cartesian closedness is not necessary. One of the aims of this article is to present examples of non-cartesian categories with the properties (one such is, e.g., the category of classical symmetric graphs without loops).

The paper is divided into five sections. In Preliminaries we recall the necessary (very simple) facts about the Heyting operations and about categories, and introduce the notion of Heyting category. Section 2 is devoted to proving the facts from [14] in a more general lattice context. In Section 3 we define combined dualities (as an extension of finitary dualities) and prove an analogy of the correspondence from Section 2. Section 4 contains a number of cartesian closed categories of combinatorial character. In Section 5 we discuss some Heyting categories that are not cartesian closed; after trivial examples we present more involved natural ones, like the above mentioned category of classical graphs, or that of partial unary algebras.

1. Preliminaries

1.1. The facts about partial order and lattices we will need will be explained below. The reader wishing for more can consult, e.g., [2].

We will deal with partially ordered classes (typically proper ones). We keep, however, the terminology and notation standardly used for partially ordered *sets*: we speak of a *lattice* L if any two elements $a, b \in L$ have a supremum (join) $a \vee b$ and an infimum (meet) $a \wedge b$ in L; the least element (bottom) resp. the largest element (top), if it exists, will be denoted by

$$\bot \quad \text{resp.} \quad \top.$$

In particular we speak of a lattice with $\bot$ and $\top$ as of a *Heyting algebra* if there is an additional operation $\Rightarrow$ satisfying

$$\text{(Hey)} \qquad a \wedge b \leq c \quad \text{iff} \quad a \leq (b \Rightarrow c).$$

1.1.1 Notes. 1. If L is non-empty, the existence of top follows from (Hey): take any $a \in L$; then $a \Rightarrow a = \top$ since $x \wedge a \leq a$.

2. Recall that a (*Galois*) *adjunction* between monotone (non-decreasing) maps $f : X \to Y$ and $g : Y \to X$ is the situation

$$\text{(adj)} \qquad \forall x \in X, y \in Y, \quad f(x) \leq y \quad \text{iff} \quad x \leq g(y)$$

and that the *left adjoints* (the maps standing to the left in (adj)) preserve all the existing suprema (and the right adjoints preserve the existing infima). In particular, (Hey) is an adjunction $f_b(x) \leq y$ iff $x \leq g_b(y)$ with $f_b(x) = x \wedge b$ to the left and $g_b(y) = b \Rightarrow y$ to the right. Consequently,

> *each $f_b = (-) \wedge b$ preserves all the existing suprema so that in particular a Heyting algebra is always a distributive lattice* (*that is,* $(a_1 \vee a_2) \wedge b = (a_1 \wedge b) \vee (a_2 \wedge b)$).

In finite lattices, the existence of a Heyting operation is equivalent to distributivity.

1.1.2 Some simple Heyting rules. Since $b \wedge a \leq b$ we have

$$\text{(1.1.1)} \qquad b \leq (a \Rightarrow b)$$

and since $a \Rightarrow b \leq a \Rightarrow b$ we have

$$\text{(1.1.2)} \qquad a \wedge (a \Rightarrow b) \leq b \quad \text{(modus ponens)},$$

and combining these two formulas we obtain

$$\text{(1.1.3)} \qquad a \wedge (a \Rightarrow b) = a \wedge b.$$

We have $1 \wedge a \leq b$ iff $1 \leq (a \Rightarrow b)$ and hence

$$\text{(1.1.4)} \qquad a \Rightarrow b = 1 \quad \text{iff} \quad a \leq b.$$

1.2. An element a of a lattice is *connected* if

$$a = a_1 \vee a_2 \quad \text{implies} \quad a = a_i \quad \text{for some of the} \quad i.$$

We will say that a Heyting algebra L has *connected decompositions* if

(CD) every $x \in L$ is a supremum of a *set* of connected elements.

1.3. We shall use only basic notions and facts from category theory (such as can be found in the introductory chapters of [10] or [1]): category, objects, morphisms, functor, (natural) transformation, natural equivalence (this last will be indicated by $\cong$, and the reader will mostly need only to know that it is a one-one correspondence). Once, in a note, we mention reflectivity.

If $\mathcal{C}$ is a category, the symbol $\mathcal{C}(A, B)$ will be used for the set of all morphisms $f : A \to B$ in $\mathcal{C}$.

Recall that a product of two objects X_1, X_2 consists of an object X (often indicated as $X_1 \times X_2$) and morphisms $p_i : X \to X_i$ such that for every couple of morphisms $f_i : Y \to X_i$ there is a unique $f : Y \to X$ such that $p_i f = f_i$. Dually, a coproduct (or sum) of two objects X_1, X_2 consists of an object X (often indicated as $X_1 + X_2$) and morphisms $\iota_i : X_i \to X$ such that for every couple of morphisms $f_i : X_i \to Y$ there is a unique $f : X \to Y$ such that $f\iota_i = f_i$.

In the last section and in the notes on cartesian closedness 1.5 and 1.6 we will recall the *adjunction of functors* and the fact that a left adjoint preserves colimits.

1.4. For a category $\mathcal{C}$ define a partially ordered class $\widehat{\mathcal{C}}$ as the class of objects of $\mathcal{C}$ ordered by

$$A \leq B \quad \text{iff} \quad \exists f : A \to B$$

factorized by the relation $A \sim B$ iff $A \leq B \leq A$. If there is no danger of confusion, the class containing A is denoted by the same symbol.

If the category $\mathcal{C}$ has products then $A_1 \times A_2$ is the meet (infimum) of A_1 and A_2 in $\widehat{\mathcal{C}}$, and if $\mathcal{C}$ has coproducts, $A_1 + A_2$ is the join (supremum) of A_1, A_2. Thus, if $\mathcal{C}$ has products and coproducts, $\widehat{\mathcal{C}}$ is a lattice.

Note. In the categories relevant for our purposes (graphs, special graphs, relational systems, hypergraphs, unary algebras, etc.) the coproducts are typically disjoint unions and we easily see that $\widehat{\mathcal{C}}$ has connected decompositions.

1.5. A category with products is said to be *cartesian closed* if there is a functor $[-,-] : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$ (the superscript “op” indicates that $[-,-]$ is contravariant in the first variable) such that

$$\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, [B, C]).$$

The functor $[-,-]$ will be called the *exponentiation* in $\mathcal{C}$.

The formula constitutes an *adjunction* with $F_B : \mathcal{C} \to \mathcal{C}$ defined by $F_B(X) = X \times B$, $F_B(f) = f \times \mathrm{id}_B$, to the left. Thus, F_B preserves all colimits (in particular, coproducts, but also others).

In Section 4 we will present several examples of cartesian closed categories. Here, let us just illustrate the notion by the simple example of the category of sets with the exponentiation $[B, C]$ constituted by the set C^B of all maps $B \to C$ and the natural equivalence associating the map $f : A \times B \to C$ with $\widetilde{f} : A \to C^B$ defined by $\widetilde{f}(x)(y) = f(x, y)$.

1.6. A category $\mathcal{C}$ is said to be *Heyting* if the partially ordered class $\widehat{\mathcal{C}}$ is a Heyting algebra.

Note that a cartesian closed category is Heyting, with the Heyting operation $A \Rightarrow B = [A, B]$. Not every Heyting category is cartesian closed, though. This will be discussed in Section 5 where we present several counterexamples, among others the categories of classical graphs (without loops, symmetric or oriented), where in the oriented case the structure of dualities is particularly rich (see [9] and [13]).

2. Gaps and dualities

Throughout this section, L is a Heyting algebra (possibly carried by proper class) with connected decompositions.

2.1. A *gap* (or *cover*, here we prefer the more suggestive of the two synonymous terms) in L is a couple (c, d) such that $c < d$, and $c \leq x \leq d$ implies that either $x = c$ or $x = d$. We will usually say "$c < d$ is a gap" instead of "(c, d) is a gap".

2.1.1 Lemma. *Let $a < b$ be a gap in a distributive lattice, and let $a \leq c < c \vee a$. Then $c < c \vee b$ is a gap.*

Proof: Let $c \leq x \leq c \vee b$. Then $a \leq x \wedge b \leq b$ and hence either $a = x \wedge b$ and $x = x \wedge (c \vee b) = (x \wedge c) \vee (x \wedge b) = c \vee a = c$, or $x \wedge b = b$, hence $x \geq b$ and since also $x \geq c$ we conclude $x \geq c \vee b$. □

2.1.2 Lemma. *Let $c < d$ be a gap in a distributive lattice with connected decompositions. Then there is precisely one connected a such that $d = c \vee a$.*

Proof: Let $d = \bigvee_{i \in J} d_i$ with d_i connected and let $a = d_i$ be some of the summands such that $a \not\leq c$. Then $c < c \vee a \leq d$ and hence $c \vee a = d$. Now let also $c \vee b = d$ with b connected. Then $a = a \wedge (c \vee b) = (a \wedge c) \vee (a \wedge b)$ and since $a \neq a \wedge c$ we have $a = a \wedge b$ and $a \leq b$. Similarly $b \leq a$. □

2.2. A *duality* in L is a couple (a, b) such that

$$a \leq x \quad \text{iff} \quad x \not\leq b.$$

(Thus, a duality is a pair (a, b) such that L is the disjoint union of the principal filter $\uparrow a = \{x \mid x \geq a\}$ and the principal ideal $\downarrow b = \{x \mid x \leq b\}$.)

2.2.1 Proposition. *If (a, b) is a duality then a is connected.*

PROOF: Suppose $a = a_1 \vee a_2 \neq a_i$, $i = 1, 2$. Hence $a \not\leq a_i$ and, by duality, $a_1, a_2 \leq b$. Then $a = a_1 \vee a_2 \leq b$, and we have the contradiction $a \not\leq a$. $\square$

2.3.1 Lemma. *Let $c < d$ be a gap and let a be the unique connected element such that $d = c \vee a$. Then $(a, a \Rightarrow c)$ is a duality. Consequently, if d is connected, $(d, d \Rightarrow c)$ is a duality.*

PROOF: If $a \leq x$ we cannot have $x \leq (a \Rightarrow c)$ since else $a = x \wedge a \leq c$. On the other hand, if $x \not\leq (a \Rightarrow b)$ then $x \wedge a \not\leq c$, and since $c < (x \wedge a) \vee c \leq d = a \vee c$ we have $(x \wedge a) \vee c = a \vee c$. Thus, $(x \wedge a) \vee (c \wedge a) = (a \vee c) \wedge a = a$ and since a is connected either $a = c \wedge a$ or $a \wedge x = a$. The first is impossible since it would yield $d = c \vee a = c$. Hence $a = x \wedge a$, that is, $a \leq x$. $\square$

2.3.2 Lemma. *Let (a, b) be a duality. Then $(a \wedge b, a)$ is a gap.*

PROOF: Let $a \wedge b \leq x \leq a$. If $a \not\leq x$ we have $x \leq b$ and $x \leq a \wedge b \leq x$. $\square$

2.4 Proposition. *The formulas*

$$\alpha(a, b) = (a \wedge b, a), \quad \beta(c, d) = (d, d \Rightarrow c)$$

constitute a one-one correspondence between dualities and gaps with connected d.

PROOF: By 2.3.2 and 2.2.1, $(a \wedge b, a)$ is a gap with connected a, and by 2.3.1, $(d, d \Rightarrow c)$ is a duality. Further, $\beta\alpha(a, b) = (a, a \Rightarrow (a \wedge b))$. Trivially, $b \leq a \Rightarrow (a \wedge b)$; if $(a \Rightarrow (a \wedge b)) \not\leq b$ we have $a \leq (a \Rightarrow (a \wedge b))$ and $a \leq a \wedge b$, that is, $a \leq b$ and $a \not\leq a$. Thus, $\beta\alpha(a, b) = (a, b)$. Finally, $\alpha\beta(c, d) = (d \wedge (d \Rightarrow c), d) = (d \wedge c, d) = (c, d)$ by (1.1.3). $\square$

2.4.1 Note. The existence of the Heyting operation (in the finite case, the same as distributivity) is essential. It is not just that lacking $\Rightarrow$ we would not have the simple formula: for instance, in the Chinese lantern $\{\perp < 1, 2, \ldots, n < \top\}$ with $n \geq 3$ there is no duality while we have n gaps $i < \top$ with connected i. The relation of distributivity to the links between (connected) gaps and dualities in finite lattices is not quite clear and may be of some interest.

2.5 Proposition. *Let $c < d$ be a gap and let a be the unique connected element such that $d = c \vee a$. Then $e = c \wedge a$ forms a gap $e < a$ such that $e \leq c \leq (a \Rightarrow e)$.*

PROOF: By 2.3.1, $(a, a \Rightarrow c)$ is a duality and hence, by 2.3.2, $a \wedge (a \Rightarrow c) < a$ is a gap. By (1.1.3), $a \wedge (a \Rightarrow c) = a \wedge c$. $\square$

2.6 Proposition. *The gaps in L are exactly the couples $c < d$ such that for some duality (a, b),*

$$a \wedge b \leq c \leq b \quad \text{and} \quad d = a \vee c.$$

PROOF: If $c < d$ is a gap consider the duality $(a, a \Rightarrow c)$ as in 2.5. Then $a \wedge (a \Rightarrow c) \leq c \leq (a \Rightarrow c)$. On the other hand, let for a duality (a, b), $a \wedge b \leq c \leq b$ and $d = a \vee c$. Let $c \leq x \leq d = a \vee c$. If $a \vee c \not\leq x$ then $a \not\leq x$ and hence $x \leq b$; consequently $x = x \wedge (c \vee a) \leq b \wedge (c \vee a) = c \vee (a \wedge b) = c$. □

3. Combined dualities

3.1. A *combined duality* is a couple $((a_i)_{i \in J}, b)$ such that

(1) if $i \neq j$ then $a_i \not\leq a_j$,
(2) $(\forall i,\ a_i \not\leq x)$ iff $x \leq b$.

Note. Suppose we have a system satisfying just the condition (2). If J is finite, it can be easily modified, omitting some of the a_i, to a combined duality with equivalent (2). If J is infinite, however, the first assumption is essential.

3.2 Lemma. *Let $((a_i)_{i \in J}, b)$ be a combined duality. Then all the a_i are connected.*

PROOF: Let $a_{i_0} = c \vee d$ with $a_{i_0} \not\leq c, d$. Then for all i, $a_i \not\leq c, d$, hence $c, d \leq b$ and finally $a_{i_0} = c \vee d \leq b$ contradicting $a_{i_0} \leq a_{i_0}$. □

3.3 Proposition. *I. Let $((a_i)_{i \in J}, b)$ be a combined duality. Let either J be finite or L admit infima of sets of the size of the J. Then there are dualities (a_i, b_i), $i \in J$, such that $b = \bigwedge_{i \in J} b_i$.*

II. On the other hand, if (a_i, b_i), $i \in J$, are dualities and $b = \bigwedge_{i \in J} b_i$ and $a_i \not\leq a_j$ for $i \neq j$ then $((a_i)_{i \in J}, b)$ is a combined duality.

PROOF: I. By 3.2, all the a_i are connected. We have $a_i \not\leq b$ (else $a_i \not\leq a_i$) and hence $a_i < b \vee a_i$. Now each of these $a_i < b \vee a_i$ is a gap. Indeed, let $b \leq x \leq b \vee a_i$. If $x \not\leq b$ there is a j such that $a_j \leq x$. If $j \neq i$ we had a non-trivial decomposition $a_j = (b \wedge a_i) \vee (a_i \wedge a_j)$ so that necessarily $i = j$. Thus, $a_i \leq x$ and $b \leq x$ and we have $b \vee a_i \leq x$. Hence by 2.5 there are gaps $c_i < a_i$ such that $c_i \leq b \leq (a_i \Rightarrow c_i)$. By 2.3, $(a_i, a_i \Rightarrow c_i)$ are dualities and hence

$$\forall i,\ a_i \not\leq x \quad \text{iff} \quad \forall i,\ x \leq (a_i \Rightarrow c_i) \quad \text{iff} \quad x \leq \bigwedge (a_i \Rightarrow c_i).$$

II. The second statement is obvious. □

4. Some cartesian closed categories

This section contains several examples of cartesian closed categories relevant for combinatorics. Some of these cartesian structures are known but seldom explicitly presented. We describe in detail the exponentiation mechanisms and leave the checking to the reader.

4.1 Systems of binary relations. In $\mathbf{Rel}(n)$, the category of sets (X, R), $R = (R_1, \ldots, R_n)$, with n binary relations, and the (relations preserving) homomorphisms $f : (X, R) \to (Y, S)$ (that is, maps $f : X \to Y$ such that xR_iy implies $f(x)S_if(y)$), one has the exponentiation

$$[(X, R), (Y, S)] = (\{\varphi \mid \varphi : X \to Y \textit{ all } \text{maps}\}, T)$$

with $(\varphi, \psi) \in T_i$ iff $(x, y) \in R_i \Rightarrow (\varphi(x), \psi(y)) \in S_i$.

For homomorphisms $f : (X', R') \to (X, R)$ and $g : (Y, S) \to (Y', S')$ define a homomorphism $[f, g] : [(X, R), (Y, S)] \to [(X', R'), (Y', S')]$ by setting $[f, g](\alpha) = g\alpha f$. We have the same formula also in most of the following examples; it will not be unnecessarily repeated.

4.1.1 Symmetric relations. The subcategory $\mathbf{SymRel}(n)$ generated by the (X, R), $R = (R_1, \ldots, R_n)$, where the R_i are symmetric is obviously closed in $\mathbf{Rel}(n)$ under product and the exponentiation, and hence inherits the cartesian structure.

4.2. Let A be an arbitrary set. The category

$$A\textbf{-Graph}$$

of A-*graphs* is defined as follows:

the objects X are couples $(\mathsf{V}(X), \mathsf{E}(X))$ where $\mathsf{V}(X)$ is a set, and $\mathsf{E}(X)$ is a subset of $\mathsf{V}(X)^A$;

the morphisms $f : X \to Y$ are maps $f : \mathsf{V}(X) \to \mathsf{V}(Y)$ such that for every $\alpha \in \mathsf{E}(X)$ the composition $f \cdot \alpha$ is in $\mathsf{E}(Y)$.

We obviously have the product given by the formula

$$\mathsf{V}(X_1 \times X_2) = \mathsf{V}(X_1) \times \mathsf{V}(X_2),$$
$$\mathsf{E}(X_1 \times X_2) = \{(\alpha_1 \times \alpha_2) \cdot \Delta \mid \alpha_i \in \mathsf{E}(X_i),\ \Delta \text{ the diagonal } A \to A \times A\}.$$

For A-graphs Y, Z define an A-graph $[Y, Z]$ by setting

$$\mathsf{V}([Y, Z]) = \{\varphi : \mathsf{V}(Y) \to \mathsf{V}(Z) \mid (\textit{all } \text{maps})\},$$
$$\varphi \in \mathsf{E}([Y, Z]) \quad \text{iff for each } \beta \in \mathsf{E}(Y) \quad (a \mapsto \varphi(a)(\beta(a))) : A \to Z \text{ is in } \mathsf{E}(Z).$$

This cartesian closedness structure restricts to the category

$$\mathbf{Sym}A\textbf{-Graph},$$

the full subcategory of A-**Graph** generated by the X such that for every $\alpha \in \mathsf{E}(X)$ and every permutation π of A, $\alpha\pi$ is in $\mathsf{E}(X)$.

Note. If $A = \{1, 2, \dots, n\}$, A-**Graph** is the category of sets with $n-$ary relations. This example can be generalized for systems of relations of various arities.

4.3 Hypergraphs. The category **HGraph** of *hypergraphs* is defined as follows:
the objects (*hypergraphs*) X are couples $(\mathsf{V}(X), \mathsf{E}(X))$ where $\mathsf{V}(X)$ is a set (the set of *vertices* of X) and $\mathsf{E}(X)$, the set of *hyperedges* of X is any subset of $\mathfrak{P}(\mathsf{V}(X))$;
the morphisms $f : X \to Y$ are maps $f : \mathsf{V}(X) \to \mathsf{V}(Y)$ such that for every $U \in \mathsf{E}(X)$ the image $f[U]$ is in $\mathsf{E}(Y)$.

The product in **HGraph** is given by the formula

$$\mathsf{V}(X_1 \times X_2) = \mathsf{V}(X_1) \times \mathsf{V}(X_2),$$
$$\mathsf{E}(X_1 \times X_2) = \{U \mid p_i[U] \in \mathsf{E}(X_i)\}.$$

The exponentiation $[Y, Z]$ is defined by setting

$\mathsf{V}([Y, Z]) = \{\varphi : \mathsf{V}(Y) \to \mathsf{V}(Z) \mid (\textit{all} \text{ maps})\}$,
$\Phi \in \mathsf{E}([Y, Z])$ iff

for any $B \in \mathsf{E}(Y)$, any set M, and any two onto maps $\alpha : M \to \Phi$, $\beta : M \to B \in \mathsf{E}(Y), \{\alpha(m)(\beta(m)) \mid m \in M\} \in \mathsf{E}(Z)$.

4.3.1 Hypergraphs with bounded hyperedges. Let α be an infinite cardinal. Denote by $\mathbf{HGraph}_\alpha$ the full subcategory of **HGraph** generated by the hypergraphs X such that for each $A \in \mathsf{E}(X)$, $|A| < \alpha$. Thus,

$$\mathbf{HGraph}_{\omega_0} = \mathbf{HGraph}_{\mathrm{fin}}$$

is the category of hypergraphs with finite hyperedges.

Obviously,

(1) if X, Y are in $\mathbf{HGraph}_\alpha$ then so is the product $X \times Y$, and
(2) if we set, for a general hypergraph X,

$$X_{<\alpha} = (\mathsf{V}(X), \{A \in \mathsf{E}(X) \mid |A| < \alpha\})$$

then for any $Y \in \mathbf{HGraph}_\alpha$ and any X, the morphisms $Y \to X$ coincide with the morphisms $Y \to X_{<\alpha}$.

Consequently

the category $\mathbf{HGraph}_\alpha$ *is cartesian closed with the exponentiation* $[X, Y]_{<\alpha}$.

Note. This is, of course, a special case of the general categorical fact that if $\mathcal{A}$ is cartesian closed and $\mathcal{B}$ a coreflective subcategory closed under finite products, then $\mathcal{B}$ is cartesian closed.

Here we have such a coreflection given by the system $(X_{<\alpha} \to X)_X$ carried by the identities.

4.4 The categories of functors $\mathbf{Set}^{\mathsf{A}}$. Actually, the categories in this example have a considerably stronger property than just the cartesian closedness: they are topoi (see, e.g., [J]), which is a well known and widely used fact. But this is not relevant for our purposes and need not be of interest for the reader. We present here explicitly the exponentiation since the mechanism differs from the previous cases.

Let A be a small category (that is a category that is a set). For functors $F, G : \mathsf{A} \to \mathbf{Set}$ denote by

$$\langle F, G\rangle$$

the set of all transformations $F \to G$. For objects $a \in A$ define $A(a)(c) = \mathsf{A}(a, c)$, the set of all morphisms $a \to c$ in A, and for a morphism $\varphi : c \to d$ define $A(a)(\varphi) : A(a)(c) \to A(a)(d)$ by setting $A(a)(\varphi)(\alpha) = \varphi\alpha$. Obviously each $A(a)$ is a functor $\mathsf{A} \to \mathbf{Set}$.

Further, if $f : b \to a$ is a morphism in A, define a transformation $A(f) : A(a) \to A(b)$ by setting $A(f)_a(\alpha) = \alpha f$.

For functors $G, H : \mathsf{A} \to \mathbf{Set}$ define

$$\begin{aligned}&[G, H](a) = \langle A(a) \times G, H\rangle \text{ for objects } a \in \mathsf{A}, \text{ and}\\ &[G, H](f)(\tau) = \tau \cdot (A(f) \times \mathrm{id}) \text{ for morphisms } f : a \to b.\end{aligned}$$

If we define, for a transformation $\tau : F \times G \to H$, a transformation

$$\widetilde{\tau} : F \to \langle A(-) \times G, H\rangle = [G, H]$$

by setting, for $x \in F(a)$, $(\widetilde{\tau}_a(x))_b(\alpha, y) = \tau_b(F(\alpha)(x), y)$ (one has to prove, of course, that each individual $\widetilde{\tau}_a(x)$ is a transformation $A(a)\times G \to H$, and that $\widetilde{\tau}$ is a transformation as a whole), and if we define for $\theta : F \to [G, H]$ a transformation $\overline{\theta} : F \times G \to H$ by setting $\overline{\theta}_a(x, y) = (\theta_a(x))_a(\mathrm{id}, y)$ we find that $\overline{\widetilde{\tau}} = \tau$ and $\widetilde{\overline{\theta}} = \theta$ and that the correspondences $\tau \mapsto \widetilde{\tau}$ and $\theta \mapsto \overline{\theta}$ constitute a natural equivalence. Thus,

the functors $[G, H]$ constitute a cartesian exponentiation in $\mathbf{Set}^{\mathsf{A}}$.

4.4.1. A particular case is for instance *the category of multigraphs* with A constituted by two objects a, b and non-identical morphisms $\alpha, \beta : a \to b$.

Other examples are *arbitrary varieties of unary algebras* obtained from suitable monoids A. For instance, in the simplest case of sets with unique unary operations the exponentiation is very transparent: we have ($\mathbb{N}$ is the set of natural numbers endowed with the successor operation $(i \mapsto i + 1)$)

$$[(Y, \beta), (Z, \gamma)] = (\{\varphi \mid \varphi : \mathbb{N} \times (Y, \beta) \to (Z, \gamma)\}, \nu)$$

with $\nu(\varphi)(i, y) = \varphi(i + 1, y)$.

4.5 Note. The reader may have observed that none of our examples had the exponentiation $[A, B]$ given as the set of *the morphisms* $A \to B$ (endowed by suitable structures). Cartesian categories with such exponentiations are not rare (for instance, the categories of reflexive relations have the property). They are not very interesting in our context, though. We have

Fact. *Let $\mathcal{C}$ be a cartesian closed category and let there be a forgetful functor $U : \mathcal{C} \to$ **Set** such that $G \in \mathcal{C}$ for some object G, $U(X) \cong \mathcal{C}(G, X)$ and that we have $U([A, B]) = \mathcal{C}(A, B)$. Then the partially ordered class $\widehat{\mathcal{C}}$ has at most two elements.*

PROOF: We have $\mathcal{C}(A, B) \cong U([A, B]) \cong \mathcal{C}(G, [A, B]) \cong \mathcal{C}(G \times A, B)$. Thus, there is always a morphism $A \to G \times A$ (take $B = G \times A$ and the morphism corresponding to the identity $G \times A \to G \times A$), and consequently also $A \to G \times A \to G$. On the other hand, unless $U(A)$ is void (which, by the faithfulness, can happen at most for one isomorphism type), we have also $G \to A$. $\square$

5. Some more Heyting categories

5.1. A Heyting category is not necessarily cartesian closed: for instance every connected $\mathcal{C}$ (that is, a $\mathcal{C}$ with trivial $\widehat{\mathcal{C}}$) is Heyting. To obtain a less trivial (but still very primitive) example consider the obvious fact that a product of Heyting categories is Heyting; take a cartesian closed $\mathcal{C}_1$ and a connected $\mathcal{C}_2$, and form $\mathcal{C}_1 \times \mathcal{C}_2$.

5.2 Classical symmetric and oriented graphs. Consider the category $\mathbf{Graph}_0$ of classical graphs, the full subcategory of **SymRel**(1) generated by the objects without loops, and by the one-vertex graph with loop $P = (\{0\}, \{(0, 0)\})$ (P has to be added for technical reasons: it is the product of the void system which the category would otherwise lack). Consider the graphs

$$A = (\{0, 1\}, \emptyset),\ B = (\{0, 1, 2\}, \{(0, 1), (1, 0), (1, 2), (2, 1)\}),\ D = (\{0\}, \emptyset)$$

and the homomorphisms

$$f = (i \mapsto i) : A \to B, \quad g = (0 \mapsto 0, 1 \mapsto 2) : A \to B.$$

There is only one homomorphism h such that $hf = hg$, namely the constant $q : B \to P$ (since $h(1) = h(2)$, $1, 2$ have to be sent to a loop and there is no other loop in the whole of the category); thus, this constant is the coequalizer of f, g. On the other hand, $A \times D$ and $B \times D$ are discrete graphs, and the target of the coequalizer of $f \times \mathrm{id}_D, g \times \mathrm{id}_C$ is a two-vertex graph. Thus, $- \times D$ does not preserve coequalizers, and $\mathbf{Graph}_0$ cannot be cartesian closed.

$\mathbf{Graph}_0$ is a Heyting category, though. We can take the Heyting structure induced by that of **SymRel**(1): indeed, $[A, B]$ from 4.1 contains a loop (ϕ, ϕ)

only if ϕ is a homomorphism $A \to B$, that is, if $A \le B$, in which case $A \Rightarrow B$ is the top anyway (recall (1.1.4)). Thus, we can set

$$A \Rightarrow B = \begin{cases} [A, B] & \text{if } A \not\le B, \\ P & \text{if } A \le B. \end{cases}$$

Similarly for $\mathbf{OrGraph}_0$ of classical oriented graphs without loops, the full subcategory of $\mathbf{Rel}(1)$.

5.3 Another example. Consider the category $\mathcal{C}$ of transitive relations with (strictly) monotone maps. It is obvious that $\widehat{\mathcal{C}}$ is isomorphic to the ordinal $\omega + 1$ (also it is a special case of the simplicial sets, see [10]). The poset $\omega + 1$ is Heyting, but we will show that $\mathcal{C}$ is not cartesian closed.

Suppose there is an exponentiation $[Y, Z]$ with a natural equivalence $\varepsilon : \mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, [Y, Z])$. Consider

$$\begin{aligned} &P = (\{0\}, \emptyset), \quad A = (\{0, 1\}, \{(0, 1)\}), \\ &B = (\{0, 1, 2\} \times \{0, 1\}, \{(0, 0), (1, 1)), ((1, 0), (2, 1))\}, \end{aligned}$$

and the morphisms

$$\xi_i : P \to A,\ \xi_i(0) = i; \quad \varphi_i : A \times A \to B,\ \varphi_i(j, k) = (j + i, k).$$

Then $\varphi_0(\xi_1 \times \mathrm{id}) = \varphi_1(\xi_0 \times \mathrm{id})$ and hence we can define, for $i = 0, 1, 2$,

$$\alpha_i = \varphi_j(\xi_k \times \mathrm{id}) \quad \text{with} \quad i = j + k.$$

Set $x_i = \varepsilon(\alpha_i(0))$. Then

$$\begin{aligned} \varepsilon(\varphi_i)(j) &= (\varepsilon(\varphi_i \cdot \xi_j))(0) = \mathcal{C}(\xi_j, \mathrm{id})(\varepsilon(\varphi_i))(0) \\ &= \varepsilon(\mathcal{C}(\xi_i \times \mathrm{id}, \mathrm{id})(\varphi_i))(0) = \varepsilon(\varphi_i \cdot (\xi_j \times \mathrm{id}))(0) = x_{i+j} \end{aligned}$$

so that $x_0 < x_1 < x_2$. Hence $x_0 < x_2$ and there is a $\psi : A \to [A, B]$ such that $\psi(0) = x_0$, that is, $\psi \cdot \xi_0 = \varepsilon(\alpha_0)$, and $\psi(1) = x_2$, that is, $\psi \cdot \xi_1 = \varepsilon(\alpha_2)$. Then

$$\varepsilon^{-1}(\psi)(0, 0) = (\varepsilon^{-1}(\xi_0 \times \mathrm{id}))(0, 0) = \varepsilon^{-1}(\psi \cdot \xi_0)(0) = \alpha_0(0, 0) = (0, 0)$$

and

$$\varepsilon^{-1}(\psi)(1, 1) = (\varepsilon^{-1}(\xi_1 \times \mathrm{id}))(0, 1) = \varepsilon^{-1}(\psi \cdot \xi_1)(0) = \alpha_2(0, 1) = (2, 1),$$

while $(0, 0) \not< (2, 1)$.

5.4. Another class of examples of Heyting categories that are not cartesian closed is provided by the following trivial fact.

Proposition. *Let $\mathcal{A}$ be a cartesian closed category with the exponentiation $[X,Y]$ and let $\mathcal{B}$ be a full subcategory closed under products. Let there be a functor $F : \mathcal{A} \to \mathcal{B}$ such that*

(a) *for each A in $\mathcal{A}$ there is a morphism $F(A) \to A$, and*
(b) *for each B in $\mathcal{B}$ there is a morphism $B \to F(B)$.*

Then $\mathcal{B}$ inherits the Heyting structure by way of $F([X,Y])$.

Proof: If B is in $\mathcal{B}$ and A is general we have $B \to A$ iff $B \to F(A)$ ($\Rightarrow$ because of

$$B \longrightarrow F(B) \xrightarrow{F(f)} F(A)$$

using (b), and $\Leftarrow$ immediately from (a).) Thus, $B_1 \times B_2 \to B_3$ iff $B_1 \to [B_2, B_3]$ iff $B_1 \to F([B_2, B_3])$. $\square$

5.5 Partial unary algebras. Denote by

$$\mathbf{PA}(n \times 1)$$

the full subcategory of $\mathbf{Rel}(n)$ generated by the partial unary algebras (X,R) (that is, the (X,R) such that for any i, if xR_iy and xR_iy' then $y = y'$; we then write $y = R_i(x)$).

Proposition. *The category $\mathbf{PA}(n \times 1)$ is Heyting but not cartesian closed.*

Proof: Denote by $\mathbb{T}_n$ the system of all finite words in $1, 2, \ldots, n$, including the empty word $\emptyset$. For an object (X,R) of $\mathbf{Rel}(n)$ set

$$\widetilde{X_R} = \{t : \mathbb{T}_n \rightharpoonup X \mid \text{partial maps satisfying (1) and (2)}\},$$

where

(0) $t(\emptyset)$ is defined,
(1) if t is defined for vw then it is defined for v, and
(2) $t(i_1 \cdots i_k) R_i t(i i_1 \cdots i_k)$.

On $\widetilde{X_n}$ define the relational system $\widetilde{R} = (\widetilde{R}_1, \ldots, \widetilde{R}_n)$ by setting

$$t\widetilde{R}_i\tau \quad \text{iff} \quad \tau(i_1 \cdots i_k) = t(i_1 \cdots i_k i)$$

(meaning: both the values are defined and equal).
Obviously, $(\widetilde{X_R}, \widetilde{R})$ is in $\mathbf{PA}(n \times 1)$.

Define $p = p_{(X,R)} : (\widetilde{X_R}, \widetilde{R}) \to (X,R)$ by setting $p(t) = t(\emptyset)$ (if $t\widetilde{R}_i\tau$ we have in particular $\tau(\emptyset) = t(i)$ and hence $p(t) = t(\emptyset) R_i t(i) = p(\tau)$).

For a homomorphism $f : (X,R) \to (Y,S)$ define $\widetilde{f} : (\widetilde{X_R}, \widetilde{R}) \to (\widetilde{Y_S}, \widetilde{S})$ by setting $\widetilde{f}(t) = f \cdot t$. Obviously this is a homomorphism and we see that we have obtained a functor $\mathbf{Rel}(n) \to \mathbf{PA}(n \times 1)$. If (X,R) is in $\mathbf{PA}(n \times 1)$ we can define $q : (X,R) \to (\widetilde{X_R}, \widetilde{R})$ by setting

$$q(\emptyset) = x, \quad q(x)(i_1 i_2 \cdots i_k) = R_{i_1} R_{i_2} \cdots R_{i_k}(x) \text{ whenever defined.}$$

(The condition (1) is obviously satisfied, and $q(x)(iw) = R_i(q(x)(w))$, hence also (2). Now if xR_iy, that is, $y = R_i(x)$, we have $q(y)(i_1 \cdots i_k) = R_{i_1} \cdots R_{i_k} R_i(x) = q(x)(i_1 \cdots i_k i)$ so that q is a homomorphism.)

Thus, by 5.4, $\mathbf{PA}(n \times 1)$ is Heyting.

As for the second statement, consider $A = (\{0\}, \emptyset)$, $B = (\{0,1\} \times \{0,1\}$, $R_j = \{((0,i),(1,i)), i = 0,1\}$, $j = 1,\ldots,n)$ and $C = (\{0,1\}$, $S_j = \{(0,1)\}$, $j = 1,\ldots,,n)$, and the maps $f_i : A \to B$ sending 0 to $(0,i)$. Then the coequalizer of f_1, f_2 is the homomorphism $g : B \to C$ defined by $g(i,j) = i$ while $g \times \mathrm{id}_A$ is not the coequalizer of $f_i \times \mathrm{id}_A$. Thus, the functor $- \times A$ does not preserve coequalizers, and since coequalizers are colimits, our category is not cartesian closed (recall 1.3). $\square$

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