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# $\Sigma$-products of paracompact Čech-scattered spaces

Hidenori Tanaka

*Abstract.* In this paper, we shall discuss $\Sigma$-products of paracompact Čech-scattered spaces and show the following: (1) Let $\Sigma$ be a $\Sigma$-product of paracompact Čech-scattered spaces. If $\Sigma$ has countable tightness, then it is collectionwise normal. (2) If $\Sigma$ is a $\Sigma$-product of first countable, paracompact (subparacompact) Čech-scattered spaces, then it is shrinking (subshrinking).

*Keywords:* $\Sigma$-product, C-scattered, Čech-scattered, paracompact, subparacompact, collectionwise normal, shrinking, subshrinking, countable tightness

*Classification:* Primary 54B10, 54D15, 54D20, 54G12

## 1. Introduction

Since the concept of $\Sigma$-products was introduced by Corson [Co], the normality of $\Sigma$-products has been studied by several authors. In particular, the normality of $\Sigma$-products of metric spaces was proved by Gul'ko [Gu] and Rudin [R1]. Furthermore, Rudin [R2] proved the shrinking property of $\Sigma$-products of metric spaces. So, the shrinking property of $\Sigma$-products has been another interesting subject (Yajima [Y2]).

Telgársky [Te] defined C-scattered spaces, which is a generalization of scattered spaces and locally compact spaces. As the spaces consisting of ordinals (with the usual order topology) are scattered, many important examples using ordinals are scattered. The author and Yajima [TY] showed the following (cf. Hanaoka and the author [HaT]):

(A) Let $\Sigma$ be a $\Sigma$-product of paracompact C-scattered spaces. If $\Sigma$ has countable tightness, then it is collectionwise normal.

(B) If $\Sigma$ is a $\Sigma$-product of first countable, paracompact (subparacompact) C-scattered spaces, then it is shrinking (subshrinking).

On the other hand, Kombarov [1] proved the following.

(C) Let $\Sigma$ be a $\Sigma$-product of paracompact Čech-complete spaces. If $\Sigma$ has countable tightness, then it is collectionwise normal.

Furthermore Kombarov [2] also proved the following.

(D) Let $\Sigma$ be a $\Sigma$-product of paracompact p-spaces. Then the following are equivalent:

(a) $\Sigma$ has countable tightness,
(b) $\Sigma$ is collectionwise normal,
(c) $\Sigma$ is normal.

Hohti and Ziqiu [HZ] introduced the concept of Čech-scattered spaces, which is a generalization of C-scattered spaces. Aoki, Mori and the author [AMT], Higuchi and the author [HiT] proved that if $Y$ is a perfect paracompact (hereditarily Lindelöf, perfect subparacompact) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf, subparacompact) Čech-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindelöf, subparacompact).

It seems to be natural to consider $\Sigma$-products of paracompact Čech-scattered spaces. So, we shall discuss normality and shrinking property of $\Sigma$-products of paracompact Čech-scattered spaces and obtain generalizations of (A), (B) and (C) as follows: (1) Let $\Sigma$ be a $\Sigma$-product of paracompact Čech-scattered spaces. If $\Sigma$ has countable tightness, then it is collectionwise normal. (2) If $\Sigma$ is a $\Sigma$-product of first countable, paracompact (subparacompact) Čech-scattered spaces, then it is shrinking (subshrinking).

All spaces are assumed to be Tychonoff spaces. Let $\omega$ denote the set of natural numbers and $|A|$ denote the cardinality of a set $A$. Undefined terminology can be found in Engelking [E].

## 2. Preliminaries

A space $X$ is said to be *scattered* if every nonempty (closed) subset $A$ of $X$ has an isolated point in $A$. A space $X$ is said to be *C-scattered* if for every nonempty closed subset $A$ of $X$, there is an $x \in A$ which has a compact neighborhood in $A$. Then scattered spaces and locally compact spaces are C-scattered. A space $X$ is said to be *Čech-scattered* if for every nonempty closed subset $A$ of $X$, there is an $x \in A$ which has a Čech-complete neighborhood in $A$. It is well known that the space of irrational numbers $P = \omega^\omega$ is not C-scattered. However, it is Čech-complete and hence, is Čech-scattered.

Let $X$ be a space. For a closed subset $A$ of $X$, let

$$A^* = \{x \in A : x \text{ has no Čech-complete neighborhood in } A\}.$$

Let $A^{(0)} = A, A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal $\alpha$. Note that every $A^{(\alpha)}$ is a closed subset of $X$ and $X$ is Čech-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$.

Let $X$ be a Čech-scattered space. A subset $A$ of $X$ is said to be *topped* if there is an ordinal $\alpha(A)$ such that $A \cap X^{(\alpha(A))}$ is a nonempty Čech-complete subset and $A \cap X^{(\alpha(A)+1)} = \emptyset$. Let $\mathrm{Top}(A) = A \cap X^{(\alpha(A))}$. It is clear that if $X$ and $Y$ are Čech-scattered spaces, then the product $X \times Y$ is Čech-scattered.

**Lemma 1** (Engelking [E]). *A space $X$ is Čech-complete if and only if there is a sequence $(\mathcal{A}_n)$ of open covers of $X$ satisfying that if $\mathcal{F}$ is a collection of closed subsets of $X$, with the finite intersection property, such that for each $n \in \omega$, there are $F_n \in \mathcal{F}$ and $A_n \in \mathcal{A}_n$ with $F_n \subset A_n$, then the intersection $\bigcap \mathcal{F}$ is nonempty.*

$(\mathcal{A}_n)$ is said to be a *complete sequence* of open covers of $X$. It is well known that if $\mathcal{F}$ satisfies the condition, then $\bigcap \mathcal{F}$ is countably compact. So, if $X$ is subparacompact, then $\bigcap \mathcal{F}$ is compact.

Let $n \in \omega$, $\{X_i : i \le n\}$ be a finite collection of spaces and $X = \prod_{i \le n} X_i$. A subset of the form $A = \prod_{i \le n} A_i$ is said to be a *rectangle* in $X$. A rectangle $A = \prod_{i \le n} A_i$ in $X$ is said to be *open* (*closed*) if $A_i$ is open (closed) in $X_i$ for each $i \le n$. An open (closed) rectangle $A = \prod_{i \le n} A_i$ in $X$ is said to be *topped* if for each $i \le n$, $\overline{A_i}$ $(A_i)$ is topped in $X_i$ and let $\mathrm{Top}(A) = \prod_{i \le n} \mathrm{Top}(\overline{A_i})$ $(\mathrm{Top}(A) = \prod_{i \le n} \mathrm{Top}(A_i))$. A cover $\mathcal{A}$ of $X$ is said to be *open* (*closed*) *rectangle* if it consists of open (closed) rectangles.

For Čech-scattered spaces, we have the following, which is essentially proved by [AMT] and [HiT]. So we omit the proofs of them.

**Lemma 2.** (1) *If $\{X_i : i \le n\}$, $n \in \omega$, is a finite collection of paracompact Čech-scattered spaces, then for every open cover $\mathcal{U}$ of the product $X = \prod_{i \le n} X_i$, there is a $\sigma$-locally finite cover $\mathcal{V}$ of $X$, consisting of topped, open rectangles such that for each $V \in \mathcal{V}$, there is an $U \in \mathcal{U}$ with $\overline{V} \subset U$.*

(2) *If $\{X_i : i \le n\}$, $n \in \omega$, is a finite collection of subparacompact Čech-scattered spaces, then every open cover of the product $X = \prod_{i \le n} X_i$ has a $\sigma$-locally finite refinement, consisting of topped, closed rectangles.*

Let $\mathcal{U}$, $\mathcal{V}$ be collections of subsets of a space $X$ and $A \subset X$. Define $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$, $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$ and $\mathcal{U} \,|\, A = \{U \cap A : U \in \mathcal{U}\}$. For a mapping $f : X \to Y$ of $X$ to a space $Y$ and a collection $\mathcal{W}$ of subsets of $Y$, let $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}(\mathcal{W}) = \{f^{-1}(W) : W \in \mathcal{W}\}$.

**Basic constructions. I.** Let $n \in \omega$, $\{X_i : i \le n\}$ be a finite collection of topped, Čech-scattered spaces and $X = \prod_{i \le n} X_i$. For each $R \subset \{0, 1, \cdots, n\}$, let $p_R : X \to X_R = \prod_{i \in R} X_i$ be the projection of $X$ onto $X_R$. Since $p_R(\mathrm{Top}(X))$ is Čech-complete, take a complete sequence $(\mathcal{A}(R)_j)$ of open (in $p_R(\mathrm{Top}(X))$) covers of $p_R(\mathrm{Top}(X))$. For each $j \in \omega$, let $\mathcal{U}_j = \wedge\{p_R^{-1}(\mathcal{A}(R)_j) : R \subset \{0, 1, \cdots, n\}\}$. Then $(\mathcal{U}_j)$ is a sequence of open (in $\mathrm{Top}(X)$) covers of $\mathrm{Top}(X)$ such that for $R \subset \{0, 1, \cdots, n\}$ and $j \in \omega$, $p_R(\mathcal{U}_j)$ refines $\mathcal{A}(R)_j$ and hence, $(p_R(\mathcal{U}_j))$ is a complete sequence of open covers of $p_R(\mathrm{Top}(X))$.

**II.** Furthermore, assume that every $X_i$ is paracompact (subparacompact). For each $U \in \mathcal{U}_0$, there is an open subset $U'$ of $X$ such that $U' \cap \mathrm{Top}(X) = U$. By Lemma 2, there is a $\sigma$-locally finite cover $\mathcal{A}_0$ of $X$, consisting of topped open (closed) rectangles, such that $\overline{\mathcal{A}_0}$ refines $\{U' : U \in \mathcal{U}_0\} \cup \{X - \mathrm{Top}(X)\}$. For each

$A \in \mathcal{A}_0$, $\mathrm{Top}(A)$ is Čech-complete. We say that $\{\mathrm{Top}(A) : A \in \mathcal{A}_0\}$ is a $\sigma$-locally finite collection of Čech-complete subsets of $X$, induced by $\mathcal{U}_0$. Inductively, we have a sequence $(\mathcal{A}_j)$ of $\sigma$-locally finite covers of $X$, consisting of topped open (closed) rectangles, such that for each $j \in \omega$, $\overline{\mathcal{A}_{j+1}}$ refines $\mathcal{A}_j$ and $\overline{\mathcal{A}_j}$ refines $\mathcal{U}_j$. Then, for $R \subset \{0, 1, \cdots, n\}$ and $j \in \omega$, $\overline{p_R(\mathcal{A}_j \mid \mathrm{Top}(X))}$ refines $\mathcal{A}(R)_j$. So, if every $X_i$ is paracompact, then for each $R \subset \{0, 1, \cdots, n\}$, $(p_R(\mathcal{A}_j \mid \mathrm{Top}(X)))$ is a complete sequence of topped, open rectangle covers of $\mathrm{Top}(X_R)$. We say that $(\mathcal{A}_j \mid \mathrm{Top}(X))$ is a *complete sequence of topped, open rectangle covers* of $\mathrm{Top}(X)$.

## 3. Normality of $\Sigma$-products

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a collection of spaces. We may assume that the index set $\Lambda$ is uncountable and every $X_\lambda$ contains at least two points. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ and take a point $x^* = (x^*_\lambda) \in X$. The subspace

$$\Sigma = \{x = (x_\lambda) \in X : |\mathrm{Supp}(x)| \leq \omega\}$$

of $X$ is called a $\Sigma$-*product* of spaces $X_\lambda$, $\lambda \in \Lambda$, where $\mathrm{Supp}(x) = \{\lambda \in \Lambda : x_\lambda \neq x^*_\lambda\}$. The $x^* \in \Sigma$ is called a *base point* of $\Sigma$. The mention of base point $x^*$ is often omitted.

For a set $\Lambda$, we denote $[\Lambda]^{<\omega}$ the set of all finite subsets of $\Lambda$. For each $R \in [\Lambda]^{<\omega}$, we also denote $X_R$ the finite subproduct $\prod_{\lambda \in R} X_\lambda$ of $\Sigma$, and denote by $p_R$ the projection of $\Sigma$ onto $X_R$. In particular, $p_{\{\lambda\}}$ is denoted by $p_\lambda$ for each $\lambda \in \Lambda$. Furthermore, we denote by $p^{R'}_R$ the projection of $X_{R'}$ onto $X_R$ for $R, R' \in [\Lambda]^{<\omega}$ with $R \subset R'$. For each $R \in [\Lambda]^{<\omega}$, let $\Sigma_{\Lambda - R}$ be the $\Sigma$-product of spaces $X_\lambda$, $\lambda \in \Lambda - R$, with the base point $x^* \mid (\Lambda - R) = (x^*_\lambda)_{\lambda \in \Lambda - R}$.

Let $\Theta$ be an index set such that $\theta, \gamma \in \Theta$ assign $R_\theta, R_\gamma \in [\Lambda]^{<\omega}$. Then $X_{R_\gamma}$, $X_{R_\theta}$, $X_{R_\theta - R_\gamma}$, $\Sigma_{\Lambda - R_\gamma}$, $p_{R_\gamma}$, $p_{R_\theta}$ and $p^{R_\theta}_{R_\gamma}$ are abbreviated by $X_\gamma$, $X_\theta$, $X_{\theta - \gamma}$, $\Sigma_{\Lambda - \gamma}$, $p_\gamma$, $p_\theta$ and $p^\theta_\gamma$ respectively.

Let $R \in [\Lambda]^{<\omega}$. A subset $H$ is said to be $R$-*cylindrically open* (*closed*) in $\Sigma$ if $H = \prod_{\lambda \in R} H_\lambda \times \Sigma_{\Lambda - R}$, where $\prod_{\lambda \in R} H_\lambda$ is an open (closed) rectangle in $X_R$ and $H$ is said to be *cylindrically open* (*closed*) in $\Sigma$ if $H$ is $R$-cylindrically open (closed) in $\Sigma$ for some $R \in [\Lambda]^{<\omega}$. The set of all cylindrically open subsets in $\Sigma$ is a base in $\Sigma$. Notice that for every $R$-cylindrically open (closed) set $H$ in $\Sigma$, $H$ is homeomorphic to $p_R^{-1} p_R(H)$. Let $R \in [\Lambda]^{<\omega}$. An $R$-cylindrically open (closed) subset $H = \prod_{\lambda \in R} H_\lambda \times \Sigma_{\Lambda - R}$ in $\Sigma$ is said to be *topped* if $p_R(H) = \prod_{\lambda \in R} H_\lambda$ is topped in $X_R$. Let $\mathrm{Top}(H) = \mathrm{Top}(p_R(H))$. Then $\mathrm{Top}(H)$ is Čech-complete.

Let $X$ be a space and $\mathcal{D}$ be a collection of subset of $X$. We say that $\mathcal{D}$ is *discrete at* $x \in X$ if there is an open neighborhood $U$ of $x$ in $X$ such that $|\{D \in \mathcal{D} : D \cap U \neq \emptyset\}| \leq 1$. $\mathcal{D}$ is said to be *discrete in* $X$ if for each $x$ in $X$, $\mathcal{D}$ is discrete at $x$. A space $X$ is said to be *collectionwise normal* if every discrete collection of closed subsets of $X$ can be separated by disjoint open subsets.

A space $X$ *has countable tightness* if for each $A \subset X$ and $x \in \overline{A}$, there is a countable subset $B \subset A$ such that $x \in \overline{B}$. Every first countable space has countable tightness. Kombarov and Malykhin [KM] proved that a $\Sigma$-product $\Sigma$ has countable tightness if and only if every finite subproduct of $\Sigma$ has countable tightness.

**Lemma 3** (Yajima [Y1]). *Let $X$ be a space which has countable tightness, $\mathcal{B}$ be a collection of subsets of $Y$ and $p : Y \to X$ be a continuous mapping from $Y$ into $X$. If $p(\mathcal{B})$ is not discrete at $x \in X$, then there is a countable subset $M$ of $\bigcup \mathcal{B}$ such that $p(\mathcal{B} \,|\, M)$ is not discrete at $x$.*

Let $\Theta = \bigcup_{n\in\omega} \Theta_n$ be an index set, constructed inductively. If $\theta \in \Theta_n$, $n \geq 1$, is constructed by $\mu \in \Theta_{n-1}$, then we denote $\theta_- = \mu$.

**Theorem 1.** *Let $\Sigma$ be a $\Sigma$-product of paracompact Čech-scattered spaces. If $\Sigma$ has countable tightness, then it is collectionwise normal.*

Proof: Let $\Sigma$ be a $\Sigma$-product of paracompact Čech-scattered spaces $X_\lambda$, $\lambda \in \Lambda$, with a base point $x^* = (x^*_\lambda) \in \Sigma$. For each $x \in \Sigma$, we denote $\mathrm{Supp}(x) = \{\lambda_{x,i} : i \in \omega\}$ and for each $n \in \omega$, let $\langle \mathrm{Supp}(x)\rangle_n = \{\lambda_{x,0}, \lambda_{x,1}, \lambda_{x,2}, \dots, \lambda_{x,n}\}$.

Let $\mathcal{D}$ be a discrete collection of closed subsets in $\Sigma$. A subset $F$ of $\Sigma$ is said to satisfy (*) if there are a finite collection $\{B(i) : i \leq n\}$ of cylindrically open subsets in $\Sigma$ such that $F \subset \bigcup_{i\leq n} B(i)$ and for each $i \leq n$, $\overline{B(i)}$ meets at most one member of $\mathcal{D}$.

To construct sequences of $\sigma$-locally finite collections of cylindrically open subsets in $\Sigma$, define the following: $(H, C_H, (\mathcal{A}(H)_j), x_H) \in \mathcal{B}$ if

(1) (a) $H$ is a topped, $R_H$-cylindrically open subset in $\Sigma$ and $C_H = \mathrm{Top}(H)$,
  (b) $(\mathcal{A}(H)_j)$ is a complete sequence of topped, open (in $C_H$) rectangle covers of $C_H$,
  (c) $x_H \in X_{R_H}$.

Inductively, for each $n \in \omega$, we obtain collections $\mathcal{G}_n$, $\mathcal{H}_n$ of topped, cylindrically open subsets of $\Sigma$ and an index set $\Theta_n = \Theta_n^+ \cup \Theta_n^-$ of $n$-th level of a tree with the height $\omega$ and index sets $\Gamma_\theta = \Gamma_\theta^+ \oplus \Gamma_\theta^-$ such that $\theta \in \Theta_n^-$, $\gamma \in \Gamma_\theta$ assign finite subsets $R_\gamma, R_\theta, P_\theta \in [\Lambda]^{<\omega}$ and an index set $\Xi_\gamma = \Xi_\gamma^+ \oplus \Xi_\gamma^-$, Čech-complete sets $C_\gamma$, $C_\theta$, a point $x_\theta$, a countable subset $Y_\theta \subset \Sigma$ (if possible), satisfying the following conditions (2)–(5): for each $n \in \omega$,

(2) $\mathcal{G}_n = \{G_\theta : \theta \in \Theta_n^+\}$ is $\sigma$-locally finite in $\Sigma$ such that $\overline{G}$ meets at most one member of $\mathcal{D}$ for each $G \in \mathcal{G}_n$,
(3) $\mathcal{H}_n = \{H_\theta : \theta \in \Theta_n^-\}$ is $\sigma$-locally finite in $\Sigma$,
(4) for $n \geq 1$,
  (a) for each $\theta_- \in \Theta_{n-1}, \{C_\gamma : \gamma \in \Gamma_{\theta_-}\}$ is a $\sigma$-locally finite collection of Čech-complete closed subsets of $\overline{H_{\theta_-}}$, induced by $\mathcal{A}(\theta_-)_0$ and $\Gamma_{\theta_-}^+ = \{\gamma \in \Gamma_{\theta_-} : C_\gamma \cap C_{\theta_-} \neq \emptyset$ and $C_\gamma \times \prod_{\lambda\in\Lambda-R_\gamma}\{x^*_\lambda\}$ satisfies (*)$\}$ and $\Gamma_{\theta_-}^- = \Gamma_{\theta_-} - \Gamma_{\theta_-}^+$,

(b) for each $\gamma \in \Gamma_{\theta_-}$, $\Xi_\gamma = \Xi_\gamma^+ \oplus \Xi_\gamma^-$ is defined and $\Theta_n^+ = \{\theta = (\gamma, \xi) : \gamma \in \Gamma_{\theta_-}^+, \xi \in \Xi_\gamma^+, \theta_- \in \Theta_{n-1}^-\}$, $\Theta_n^- = \{\theta = (\gamma, \xi) : \gamma \in \Gamma_{\theta_-}, \xi \in \Xi_\gamma^-, \theta_- \in \Theta_{n-1}^-\}$ and $\Theta_n = \Theta_n^+ \oplus \Theta_n^-$,

(c) $\theta_- < \theta$,

(d) $R_{\theta_-} \subset R_\gamma$.

(5) for $\theta \in \Theta_n^-$ and $\theta_- \in \Theta_{n-1}^-$, $n \geq 1$,

(a) $H_{\theta_-}$ is a topped, $R_{\gamma_-}$-cylindrically open subset in $\Sigma$ such that $(H_{\theta_-}, C_{\theta_-}, (\mathcal{A}(\theta_-)_j), x_{\theta_-}) \in \mathcal{B}$, where $H_{\theta_-} = \prod_{\lambda \in R_{\gamma_-}} H_{\theta_-, \lambda} \times \Sigma_{\Lambda - \gamma_-}$,

(b) $H_{\theta_-} - \bigcup \mathcal{G}_n \subset \bigcup \{H_\mu : \mu \in \Theta_n \text{ with } \theta_- < \mu\} \subset H_{\theta_-}$,

(c) $P_\theta = \{\lambda \in R_{\theta_-} : \alpha(\overline{H_{\theta,\lambda}}) < \alpha(\overline{H_{\theta_-,\lambda}})\}$,

(d) if $\gamma \in \Gamma_{\theta_-}^+$, then

(d-1) $x_\theta = (x_{\theta,\lambda}) \in \overline{H_{\theta_-}}$,

(d-2) $Y_\theta = \{y_{\theta,i} : i \in \omega\}$ is a countable subset of $\bigcup \mathcal{D}$ such that $p_\gamma(\mathcal{D} \,|\, Y_\theta)$ is not discrete at $x_\theta$ in $X_\gamma$,

(d-3) if $\lambda \in R_{\theta_-}$ and $x_{\theta,\lambda} \notin p_\lambda(C_{\theta_-})$, then $\lambda \in P_\theta$,

(d-4) $R_\theta = \bigcup \{\langle \mathrm{Supp}(y_{\mu,j}) \rangle_k : \mu \leq \theta \text{ and } j, k \leq n\} \cup R_\gamma$,

and if $\gamma \in \Gamma_{\theta_-}^-$, then

(d-5) $R_\theta = R_\gamma = R_{\theta_-}$,

(d-6) $x_\theta = x_{\theta_-}$,

(e) $C_\theta = \mathrm{Top}(H_\theta)$ and $(\mathcal{A}(\theta)_j)$ is a complete sequence of topped, open (in $C_\theta$) rectangle covers of $C_\theta$ such that for $R \subset R_{\theta_-}$ with $R \cap P_\theta = \emptyset$, $p_R^\gamma(\mathcal{A}(\theta)j)$ refines $p_R^{\gamma_-}(\mathcal{A}(\theta_-)_{j+1})$ for each $j \in \omega$,

(f) $(H_\theta, C_\theta, (\mathcal{A}(\theta)_j), x_\theta) \in \mathcal{B}$.

Let $\mathcal{G}_0 = \Theta_0^+ = \{\emptyset\}$. Take an arbitrary $\lambda_0 \in \Lambda$ and a $\sigma$-locally finite open cover $\mathcal{H}' = \{H_\theta' : \theta \in \Theta_0^-\}$ of $X_{\lambda_0}$ such that for each $\theta \in \Theta_0^-, \overline{H_\theta'}$ is topped. Let $H_\theta = p_{\lambda_0}^{-1}(H_\theta')$ for each $\theta \in \Theta_0^-$ and $\mathcal{H}_0 = \{H_\theta : \theta \in \Theta_0^-\}$. Then $\mathcal{H}_0$ is a $\sigma$-locally finite collection of topped, cylindrically open subsets in $\Sigma$. For each $\theta \in \Theta_0^-$, let $C_\theta = \mathrm{Top}(H_\theta)$ and take a complete sequence $(\mathcal{A}(\theta)_j)$ of open (in $C_\theta$) covers of $C_\theta$ and a $x_\theta \in C_\theta$. Then, for each $\theta \in \Theta_0^-$, $(H_\theta, C_\theta, (\mathcal{A}(\theta)_j), x_\theta) \in \mathcal{B}$.

Let $n \in \omega$ and assume that for each $k \leq n$, we have already obtained collections $\mathcal{G}_n, \mathcal{H}_n$ and other collections, satisfying the conditions (2)–(5). Take a $\theta_- \in \Theta_n^-$. Let $(H_{\theta_-}, C_{\theta_-}, (\mathcal{A}(\theta_-)_j), x_{\theta_-}) \in \mathcal{B}$ and $H_{\theta_-} = \prod_{\lambda \in \gamma_-} H_\lambda \times \Sigma_{\Lambda - \gamma_-}$. For each $A \in \mathcal{A}(\theta_-)_0$, take an open subset $B_A$ of $\overline{\prod_{\lambda \in R_{\gamma_-}} H_\lambda} = \prod_{\lambda \in R_{\gamma_-}} \overline{H_\lambda}$ such that $B_A \cap C_{\theta_-} = A$. By Lemma 2(1), there is a $\sigma$-locally finite collection $\mathcal{W}'(\theta_-) =$

$\{W'_\gamma : \gamma \in \Gamma_{\theta_-}\}$ of topped, open rectangles of $X_{\gamma_-}$ such that $\prod_{\lambda\in R_{\gamma_-}} H_\lambda = \bigcup \mathcal{W}'(\theta_-)$ and for each element $W' \in \mathcal{W}'(\theta_-)$, $\overline{W'}$ is contained in some member of $\{B_A : A \in \mathcal{A}(\theta_-)_0\} \cup \{\prod_{\lambda\in R_{\gamma_-}} \overline{H_\lambda} - C_{\theta_-}\}$.

For each $\gamma \in \Gamma_{\theta_-}$, let $C_\gamma = \mathrm{Top}(W'_\gamma \times \Sigma_{\Lambda-\gamma_-})$ and $\Gamma^+_{\theta_-} = \{\gamma \in \Gamma_{\theta_-} : C_\gamma \cap C_{\theta_-} \neq \emptyset$ and $C_\gamma \times \prod_{\lambda\in\Lambda-R_{\gamma_-}} \{x^*_\lambda\}$ satisfies (*)$\}$ and $\Gamma^-_{\theta_-} = \Gamma_{\theta_-} - \Gamma^+_{\theta_-}$.

Let $\gamma \in \Gamma^+_{\theta_-}$. Then there are a finite collection $\{B(i) : i \leq n\}$ of $R(i)$-cylindrically open subsets in $\Sigma$, $i \leq n$, such that $C_\gamma \times \prod_{\lambda\in\Lambda-\gamma_-} \{x^*_\lambda\} \subset \bigcup_{i\leq n} B(i)$ and for each $i \leq n$, $\overline{B(i)}$ meets at most one member of $\mathcal{D}$. Put $R_\gamma = R_{\gamma_-} \cup (\bigcup_{i\leq n} R(i))$ and $W_\gamma = W'_\gamma \times X_{\gamma-\theta_-}$. Put

$$\Phi = \{x \in \overline{W_\gamma} : p_\gamma(\mathcal{D}) \text{ is not discrete at } x \text{ in } X_\gamma\}.$$

Then $\Phi$ is a closed subset of $\overline{W_\gamma}$ and $p_\gamma(C_\gamma \times \prod_{\lambda\in\Lambda-R_{\gamma_-}} \{x^*_\lambda\}) \cap \Phi = \emptyset$. For each $x \in \overline{W_\gamma} - \Phi - \bigcup_{i=0}^n p_\gamma(B(i))$, take an open rectangle neighborhood $U(x)$ of $x$ in $X_\gamma$ such that $\overline{U(x)}$ meets at most one member of $\mathcal{D}$ and for each $x = (x_\lambda) \in \Phi$, take an open rectangle neighborhood $U(x) = \prod_{\lambda\in R_\gamma} U(x_\lambda)$ of $x$ in $X_\gamma$, such that if $x_\lambda \notin p_\lambda(C_\gamma \times \prod_{\lambda\in\Lambda-R_{\theta_-}} \{x^*_\lambda\})$, $\lambda \in R_\gamma$, then $\overline{U(x_\lambda)} \cap p_\lambda(C_\gamma \times \prod_{\lambda\in\Lambda-R_{\theta_-}} \{x^*_\lambda\}) = \emptyset$. By Lemma 2(1), there is a $\sigma$-locally finite (in $X_\gamma$) collection $\mathcal{V}_\gamma = \{V_\xi : \xi \in \Xi_\gamma\}, V_\xi = \prod_{\lambda\in R_\gamma} V_{\xi,\lambda}$ for each $\xi \in \Xi_\gamma$, of topped, open rectangles in $W_\gamma$ such that $W_\gamma = \bigcup \mathcal{V}_\gamma = \bigcup\{V_\xi : \xi \in \Xi_\gamma\}$ and $\{\overline{V_\xi} : \xi \in \Xi_\gamma\}$ refines $\{U(x) : x \in \overline{W_\gamma} - \bigcup_{i=0}^n p_\gamma(B(i))\} \cup \{p_\gamma(B(0)), \cdots, p_\gamma(B(n))\}$. Put

$$\Xi^+_\gamma = \{\xi \in \Xi_\gamma : \overline{V_\xi} \text{ meets at most one member of } p_\gamma(\mathcal{D})\}$$

and $\Xi^-_\gamma = \Xi_\gamma - \Xi^+_\gamma$. Let

$$\Theta^+_\gamma = \{(\gamma,\xi) : \xi \in \Xi^+_\gamma\} \text{ and } \Theta^-_\gamma = \{(\gamma,\xi) : \xi \in \Xi^-_\gamma\}.$$

For each $\theta = (\gamma,\xi) \in \Theta^+_\gamma$, let $G_\theta = p_\gamma^{-1}(V_\xi)$. For each $\theta = (\gamma,\xi) \in \Theta^-_\gamma$, $H_\theta = p_\gamma^{-1}(V_\xi) = \prod_{\lambda\in R_\gamma} H_{\theta,\lambda} \times \Sigma_{\Lambda-\gamma}$ is a topped, $R_\gamma$-cylindrically open subset of $\Sigma$. Take an $x_\theta = (x_{\theta,\lambda}) \in \Phi$ such that $V_\xi \subset U(x_\theta)$. Since $X_\gamma$ has countable tightness, it follows from Lemma 3 that there is a countable subset $Y_\theta = \{y_{\theta,i} : i \in \omega\}$ of $\bigcup \mathcal{D}$ such that $p_\gamma(\mathcal{D} \,|\, Y_\theta)$ is not discrete at $x_\theta$ in $X_\gamma$. Let $P_\theta = \{\lambda \in R_{\theta_-} : \overline{p_\lambda(H_\theta)} \cap p_\lambda(C_{\theta_-}) = \overline{H_{\theta,\lambda}} \cap p_\lambda(C_{\theta_-}) = \emptyset\}$. Then it is easy to prove the following.

**Claim 1.** *Let $\theta \in \Theta_\gamma^-$ and $\lambda \in R_{\theta_-}$. Then*
(1) $\alpha(\overline{H_{\theta,\lambda}}) \leq \alpha(\overline{H_{\theta_-,\lambda}})$ *and* $\lambda \in P_\theta$ *if and only if* $\alpha(\overline{H_{\theta,\lambda}}) < \alpha(\overline{H_{\theta_-,\lambda}})$,
(2) *if* $x_{\theta,\lambda} \notin p_\lambda(C_{\theta_-})$, *then* $\lambda \in P_\theta$.

Let $C_\theta = \mathrm{Top}(H_\theta)$. If $\lambda \in R_{\theta_-} - P_\theta$, then $p_\lambda(C_\theta) = \overline{p_\lambda(H_\theta)} \cap p_\lambda(C_{\theta_-})$. Take a complete sequence $(\mathcal{A}(\theta)_j)$ of topped, open (in $C_\theta$) rectangle covers of $C_\theta$ such that if $R \subset R_{\theta_-}$ with $R \cap P_\theta = \emptyset$, $p_R^\gamma(\mathcal{A}(\theta)_j)$ refines $p_R^{\theta_-}(\mathcal{A}(\theta_-)_{j+1})$ for each $j \in \omega$. Then $(H_\theta, C_\theta, (\mathcal{A}(\theta)_j), x_\theta) \in \mathcal{B}$.

Assume that $\gamma \in \Gamma_{\theta_-}^-$. Let $R_\gamma = R_{\theta_-}$, $W_\gamma = W'_\gamma$, $\Xi_\gamma^+ = \Theta_\gamma^+ = \mathcal{G}_\gamma = \{\emptyset\}$, $\Xi_\gamma^- = \{\xi_\gamma\}$, $V_{\xi_\gamma} = W_\gamma$, $\Theta_\gamma^- = \{(\gamma,\xi_\gamma)\}$. For $\theta = (\gamma,\xi_\gamma)$, $H_\theta = p_\gamma^{-1}(W_\gamma)$ and $C_\theta = \mathrm{Top}(H_\theta)$. Define $P_\theta$ as before and let $x_\theta = x_{\theta_-}$. Take a complete sequence $(\mathcal{A}(\theta)_j)$ of open (in $C_\theta$) rectangle covers of $C_\theta$, satisfying the same condition for each $R \subset R_{\theta_-}$ with $R \cap P_\theta = \emptyset$. For $\theta \in \Theta_\gamma^-$, $\gamma \in \Gamma_{\theta_-}$, $(H_\theta, C_\theta, (\mathcal{A}(\theta)_j), x_\theta) \in \mathcal{B}$.

Let $\Theta_{n+1}^+ = \bigcup\{\Theta_\gamma^+ : \gamma \in \Gamma_{\theta_-}^+, \theta_- \in \Theta_n^-\}$, $\Theta_{n+1}^- = \bigcup\{\Theta_\gamma^- : \gamma \in \Gamma_{\theta_-}, \theta_- \in \Theta_n^-\}$ and $\Theta_{n+1} = \Theta_{n+1}^+ \oplus \Theta_{n+1}^-$. For $\theta \in \Theta_{n+1}^-$ and $\mu \in \Theta_n^-$, $\mu < \theta$ if $\theta_- = \mu$ and there are $\gamma \in \Gamma_{\theta_-}$ and $\xi \in \Theta_\gamma^-$ such that $\theta = (\gamma, \xi)$. Let $\mathcal{G}_{n+1} = \{G_\theta : \theta \in \Theta_{n+1}^+\}$ and $\mathcal{H}_{n+1} = \{H_\theta : \theta \in \Theta_{n+1}^-\}$. Then $\mathcal{G}_{n+1}$ and $\mathcal{H}_{n+1}$ satisfy (2) and (3). Other properties are satisfied by the above construction.

Let $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$. By (2), it suffices to prove that $\mathcal{G}$ covers $\Sigma$. Assume that $\mathcal{G}$ does not cover $\Sigma$. Take an $x = (x_\lambda) \in \Sigma - \bigcup \mathcal{G}$. Then, by (5)(b), we can inductively choose a sequence $\{\theta_n : n \in \omega\}$ such that $\theta_n = (\gamma_n, \xi_n) \in \Theta_n^-$, $\gamma_n \in \Gamma_{\theta_{n-1}}$, $\xi_n \in \Xi_{\gamma_n}^-$, $\theta_{n-1} < \theta_n$, $n \geq 1$ and $x \in H_{\theta_n}$ for each $n \in \omega$.

**Claim 2.** $\{n \geq 1 : \gamma_n \in \Gamma_{\theta_{n-1}}^+\}$ *is infinite.*

Proof: Assume that $\{n \geq 1 : \gamma_n \in \Gamma_{\theta_{n-1}}^+\}$ is finite. Then there is an $n_0 \geq 1$ such that if $n \geq n_0$, then $\gamma_n \notin \Gamma_{\theta_{n-1}}^+$, that is, $\gamma_n \in \Gamma_{\theta_{n-1}}^-$. Then for each $n \geq n_0$, $R_{\gamma_n} = R_{\theta_n} = R_{\gamma_{n_0}} = R_{\theta_{n_0}}$. Let $\lambda \in R_{\theta_{n_0}}$ and $n > n_0$. If $\lambda \in P_{\theta_n}$, then by (5)(c), we have $\alpha(p_\lambda(C_{\theta_{n+1}})) < \alpha(p_\lambda(C_{\theta_n}))$. So, there is an $n_\lambda \geq n_0$ such that if $n > n_\lambda$, then $\lambda \notin P_{\theta_n}$ and hence, $\alpha(p_\lambda(C_{\theta_n})) = \alpha(p_\lambda(C_{\theta_{n_\lambda}}))$ and $p_\lambda(C_{\theta_n}) \subset p_\lambda(C_{\theta_{n_\lambda}})$. Take an $\tilde{n} \in \omega$ such that $\tilde{n} > n_\lambda$ for $\lambda \in R_{\theta_{n_0}}$. Thus, if $n > \tilde{n}$, then $R_{\theta_{n_0}} \cap P_{\theta_n} = \emptyset$ and hence, $P_{\theta_n} = \emptyset$. Then $\{C_{\theta_n} : n > \tilde{n}\}$ is a decreasing sequence of nonempty closed subsets of $C_{\theta_{\tilde{n}}}$. For each $n > \tilde{n}$, there is an $A_n \in \mathcal{A}(\theta_n)_0$ such that $C_{\theta_n} \subset A_n$. By (5)(e), there is an $A'_n \in \mathcal{A}(\theta_{\tilde{n}})_{n-\tilde{n}}$ such that $A_n \subset A'_n$. Then $C' = \bigcap_{n \geq \tilde{n}} C_{\theta_n}$ is nonempty and compact. Let $C = C' \times \prod_{\lambda \in \Lambda - R_{\theta_{\tilde{n}}}} \{x_\lambda^*\}$. Then $C$ is compact. There is a finite collection $\{B(i) : i \leq k\}$ of cylindrically open subsets in $\Sigma$ such that $C \subset \bigcup_{i \leq k} B(i)$ and for each $i \leq k$, $\overline{B(i)}$ meets at most one member of $\mathcal{D}$. Then, by Lemma 1, there is an $m > \tilde{n}$ such that $C_{\gamma_m} \times \prod_{\lambda \in \Lambda - R_{\gamma_m}} \{x_\lambda^*\} \subset \bigcup_{i \leq k} B(i)$. Thus $\gamma_m \in \Gamma_{\theta_{m-1}}^+$, which is a contradiction. $\square$

By Claim 2, $\{n \geq 1 : \gamma_n \in \Gamma^+_{\theta_{n-1}}\}$ is infinite. So there is a mapping $\phi : \omega \to \omega$ such that $1 \leq \phi(0)$ and for each $n \in \omega$, $\phi(n) < \phi(n+1)$, $\gamma_{\phi(n)} \in \Gamma^+_{\theta_{\phi(n)-1}}$, $p_{\gamma_{\phi(n)}}(\mathcal{D}\,|\,Y_{\theta_{\phi(n)}})$ is not discrete at $x_{\theta_{\phi(n)}}$ and for each $k$ $(\phi(n) < k < \phi(n+1))$, $\gamma_k \in \Gamma^-_{\theta_{k-1}}$. Let $Q = \bigcup_{n\in\omega} R_{\theta_n} = \bigcup_{n\in\omega} R_{\theta_{\phi(n)}}$. Then

**Claim 3.** (1) $R_{\theta_{\phi(n)}} \subset R_{\gamma_{\phi(n+1)}} \subset R_{\theta_{\phi(n+1)}}$ *for each* $n \in \omega$ *and hence, for each finite subset* $F \subset Q$, *there is an* $n \geq 1$ *such that* $F \subset R_{\gamma_{\phi(n)}}$,
(2) $\bigcup\{\mathrm{Supp}(y) : y \in Y_{\theta_{\phi(n)}}, n \in \omega\} \subset Q$.

After this in proof, we omit the index letter $\theta$, $\gamma$ and $\phi$ for simplicity. That is, $x_{\theta_{\phi(n)}}$, $H_{\theta_{\phi(n)}}$, $R_{\gamma_{\phi(n)}}$, $p^{\gamma_{\phi(n)}}_{\gamma_{\phi(k)}}$, $p_{\gamma_{\phi(k)}}$, $P_{\theta_{\phi(n)}}$, $C_{\theta_{\phi(n)}}$ and $\mathcal{A}(\theta_{\phi(n)})_j$ are abbreviated by $x_n$, $H_n$, $R_n$, $p^n_k$, $p_k$, $P_n$, $C_n$ and $\mathcal{A}(n)_j$ respectively.

**Claim 4.** *For each* $\lambda \in Q$, *there is an* $m_\lambda \in \omega$ *such that for* $n > m_\lambda$, $\lambda \in R_{n-1} - P_n$ *and* $x_{n,\lambda} \in p_\lambda(C_n) \subset p_\lambda(C_{n-1})$.

Proof: For each $\lambda \in Q$, by Claim 3(1), take an $m \in \omega$ such that if $n > m$, then $\lambda \in R_{n-1}$. By the similar proof of Claim 2, there is an $m_\lambda \in \omega$ with $m_\lambda \geq m$ such that if $n > m_\lambda$, then $\lambda \notin P_n$. If there is an $n > m_\lambda$ such that $x_{n,\lambda} \notin p_\lambda(C_n)$, then by (5)(d-3), $\lambda \in P_n$, which is a contradiction. Thus for each $n > m_\lambda$, $x_{n,\lambda} \in p_\lambda(C_n) \subset p_\lambda(C_{n-1})$. □

For $m, k \geq 1$ with $m > k$, let $E^m_k = \{p^n_k(x_n) : n \geq m\}$. Then, for each $m > k$, $E^{m+1}_k \subset E^m_k$. Notice that for each $n \geq m$, $p^n_k : X_{\gamma_{\phi(n)}} \to X_{\gamma_{\phi(k)}}$. Choose an $m_k > k$ with $m_k > \max\{m_\lambda : \lambda \in R_k\}$. It follows from Claim 4 that for each $n > m_k$, $R_k \cap P_n = \emptyset$. Thus $p^n_k(x_n)_\lambda = x_{n,\lambda} \in p_\lambda(C_{m_k})$ for $\lambda \in R_k$ and $n \geq m_k$. Thus $\{\overline{E^m_k} : m \geq m_k\}$ is a decreasing sequence of closed subsets of $p^{m_k}_k(C_{m_k})$. Let $m > m_k$. Then we obtain $A_m \in \mathcal{A}(H_m)_0$ and $A'_m \in p^{m_k}_k(\mathcal{A}(H_{m_k})_{m-m_k})$ such that $\overline{E^m_k} \subset p^m_k(C_m) \subset p^m_k(A_m) \subset A'_m$. Thus $K_k = \bigcap_{m\geq k} \overline{E^m_k}$ $(= \bigcap_{m\geq m_k} \overline{E^m_k})$ is nonempty and compact. Since $p^{k+1}_k(K_{k+1}) \subset K_k$ for each $k \in \omega$, $\{K_k, p^{k+1}_k\}$ is an inverse sequence of nonempty compact spaces. Hence there is a point $(z_k) \in \varprojlim\{K_k, p^{k+1}_k\}$. Define a point $z = (z_\lambda) \in \Sigma$ such that $p_k(z) = z_k$ for each $k \geq 1$ and $z_\lambda = x^*_\lambda$ otherwise.

We can show that $\mathcal{D}$ is not discrete at $z$. However, by Claim 4 and (5)(d-2), this is verified in the same manner as the proof of [K1, Theorem 1]. □

## 4. Shrinking property of Σ-products

A space $X$ is said to be *shrinking* if for every open cover $\{U_\lambda : \lambda \in \Lambda\}$ of $X$, there is a closed cover $\{F_\lambda : \lambda \in \Lambda\}$ of $X$ such that $F_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$. A space $X$ is said to be *subshrinking* if for every open cover $\{U_\lambda : \lambda \in \Lambda\}$ of $X$, there is a closed cover $\{F_{\lambda,n} : \lambda \in \Lambda$ and $n \in \omega\}$ of $X$ such that $F_{\lambda,n} \subset U_\lambda$ for each $\lambda \in \Lambda$ and $n \in \omega$. It is well known that a space $X$ is shrinking if and only if it is normal and subshrinking.

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a collection of spaces and $\Sigma$ be a $\Sigma$-product of them. We may take a point $y_\lambda \in X_\lambda$ different from $x^*_\lambda$ for each $\lambda \in \Lambda$. For each $s \in [\Lambda - R]^{<\omega}$, an open neighborhood $W_s$ of $x^* \,|\, (\Lambda - R)$ in $\Sigma_{\Lambda - R}$ is said to be *s-basic* if $W_s = \prod_{\lambda \in s} W_\lambda \times \Sigma_{\Lambda - (R \cup s)}$, where $W_\lambda$ is an open neighborhood of $x^*_\lambda$ in $X_\lambda$ with $y_\lambda \notin W_\lambda$ for each $\lambda \in s$.

Let $\mathcal{G} = \{G_\upsilon : \upsilon \in \Upsilon\}$ be an open cover of $\Sigma$. For $R \in [\Lambda]^{<\omega}$ and a subset $F$ in $X_R$, let

$$\begin{aligned} M(F) = \{s \in [\Lambda - R]^{<\omega} :\ & \text{there is an s-basic open neighborhood} \\ W_s \ \text{ of } \ x^* \,|\, (\Lambda - R) \ \text{ such that } \ & \overline{F} \times \overline{W_s} \subset G_\upsilon \ \text{ for some } \ \upsilon \in \Upsilon\}. \end{aligned}$$

**Lemma 4** (Tanaka and Yajima [TY]). *Let $\Sigma$ be a $\Sigma$-product of spaces $X_\lambda$, $\lambda \in \Lambda$, and let $\mathcal{G} = \{G_\upsilon : \upsilon \in \Upsilon\}$ be an open cover of $\Sigma$. If there is a $\sigma$-locally finite closed cover $\{E_\theta : \theta \in \Theta^+\}$ of $\Sigma$ such that each $E_\theta$, $\theta = (\gamma, \xi) \in \Theta^+$, $\gamma \in \Gamma_{\theta_-}$, $\xi \in \Xi^-_\gamma$, $\theta_- \in \Theta$, is an $R_\gamma$-cylindrically closed set in $\Sigma$, satisfying*

$$E_\theta \subset \bigcup \{p^{-1}_{\Lambda - \gamma}(W_s) : s \in M(p_\gamma(E_\theta))\},$$

*where $R_\gamma \in [\Lambda]^{<\omega}$ and $p_{\Lambda-\gamma}$ denotes the projection of $\Sigma$ onto $\Sigma_{\Lambda-\gamma}$, then there is a closed cover $\{F_{\upsilon,n} : \upsilon \in \Upsilon$ and $n \in \omega\}$ of $\Sigma$ such that $F_{\upsilon,n} \subset G_\upsilon$ for each $\upsilon \in \Upsilon$ and $n \in \omega$.*

**Theorem 2.** *If $\Sigma$ is a $\Sigma$-product of first countable, subparacompact Čech-scattered spaces, then it is subshrinking.*

Proof: Let $\Sigma$ be a $\Sigma$-product of first countable, subparacompact $\mathcal{DC}$-like spaces $X_\lambda$, $\lambda \in \Lambda$, with a base point $x^* = (x^*_\lambda) \in \Sigma$. Let $\mathcal{G} = \{G_\upsilon : \upsilon \in \Upsilon\}$ be an open cover of $\Sigma$. A subset $A$ of $\Sigma$ is said to satisfy (**) if there is a finite collection $\mathcal{B}$ of cylindrically open subsets in $\Sigma$ such that $A \subset \bigcup \mathcal{B}$ and for each $B \in \mathcal{B}$, there is a $G \in \mathcal{G}$ such that $\overline{B} \subset G$.

Define a collection $\mathcal{B}$ similarly: $(E, C_E, (\mathcal{A}(E)_j), x_E) \in \mathcal{B}$ if

(1) (a) $E = \prod_{\lambda \in R_E} E_\lambda \times \Sigma_{\Lambda - R_E}$ is a topped, $R_E$-cylindrically closed subset in $\Sigma$ and $C_E = \mathrm{Top}(E)$,
   (b) $(\mathcal{A}(E)_j)$ is a sequence of open (in $C_E$) covers of $C_E$ such that for each $R \subset R_E$, $(p^{R_E}_R(\mathcal{A}(E)_j))$ is a complete sequence of open covers of $p^{R_E}_R(C_E)$,
   (c) $x_E \in X_{R_E}$.

Inductively, for each $n \in \omega$, we shall obtain an index set $\Theta_n = \Theta^+_n \oplus \Theta^-_n$ of $n$-th level of a tree with height $\omega$ such that $\theta \in \Theta_n$ and $\gamma \in \Gamma_\theta$ assign cylindrically closed subset $E_\theta$, index sets $\Gamma_\theta$ and $\Xi_\gamma = \Xi^+_\gamma \oplus \Xi^-_\gamma$, finite subsets $R_\gamma, R_\theta, P_\theta \in [\Lambda]^{<\omega}$, Čech-complete subsets $C_\gamma$, $C_\theta$, points $x_\theta \in X_\gamma$, $y_{\gamma,k} \in \Sigma$, basic open subsets $U_\gamma(\cdot)$, $U_\gamma(\cdot,k) \subset X_\gamma$, satisfying the following conditions (2)–(5): for $n \in \omega$,

(2) $\{E_\theta : \theta \in \Theta_n\}$ is $\sigma$-locally finite in $\Sigma$,

(3) for $n \geq 1$,

(a) for each $\theta_- \in \Theta_{n-1}$, $\{C_\gamma : \gamma \in \Gamma_{\theta_-}\}$ is a $\sigma$-locally finite collection of Čech-complete closed subsets of $E_{\theta_-}$, induced by $\mathcal{A}(\theta_-)_0$ and $\Gamma^+_{\theta_-} = \{\gamma \in \Gamma_{\theta_-} : C_\gamma \cap C_{\theta_-} \neq \emptyset$ and $C_\gamma \times \prod_{\lambda \in \Lambda - R_\gamma}\{x^*_\lambda\}$ satisfy (**)$\}$ and $\Gamma^-_{\theta_-} = \Gamma_{\theta_-} - \Gamma^+_{\theta_-}$,

(b) for each $\gamma \in \Gamma_{\theta_-}$, $\Xi_\gamma = \Xi^+_\gamma \oplus \Xi^-_\gamma$ is defined and $\Theta^+_n = \{\theta = (\gamma,\xi) : \gamma \in \Gamma^+_{\theta_-}, \xi \in \Xi^+_\gamma, \theta_- \in \Theta^-_{n-1}\}$, $\Theta^-_n = \{\theta = (\gamma,\xi) : \gamma \in \Gamma_{\theta_-}, \xi \in \Xi^-_\gamma, \theta_- \in \Theta^-_{n-1}\}$, $\Theta_n = \Theta^+_n \oplus \Theta^-_n$

and for each $\theta = (\gamma,\xi) \in \Theta_n$,

(c) $\theta_- < \theta$,

(d) $R_{\theta_-} \subset R_\gamma$,

(4) for each $\theta \in \Theta^+_n$, $E_\theta \subset \bigcup\{p^{-1}_{\Lambda-\gamma}(W_s) : s \in M(p_\gamma(E_\theta))\}$,

(5) for each $\theta = (\gamma,\xi) \in \Theta^-_n$, $\gamma \in \Gamma_{\theta_-}$, $\xi \in \Xi^-_\gamma$, $\theta_- \in \Theta^-_{n-1}$,

(a) $E_{\theta_-}$ is a topped, $R_{\gamma_-}$-cylindrically closed set in $\Sigma$, where $E_{\theta_-} = \prod_{\lambda \in R_{\gamma_-}} E_{\theta,\lambda} \times \Sigma_{\Lambda-\gamma_-}$ such that $(E_{\theta_-}, C_{\theta_-}, (\mathcal{A}(\theta_-)_j), x_{\theta_-}) \in \mathcal{B}$,

(b) $E_{\theta_-} = \bigcup\{E_\mu : \mu \in \Theta_n$ with $\theta_- < \mu\}$,

(c) for each $x \in X_\gamma, \{U_\gamma(x,k) : k \in \omega\}$ is a neighborhood base at $x$, consisting of basic open subsets in $X_\gamma$, such that $U_\gamma(x) = U_\gamma(x,0)$ and $U_\gamma(x,k+1) \subset U_\gamma(x,k)$ for each $k \in \omega$,

(d) $P_\theta = \{\lambda \in R_{\theta_-} : \alpha(E_{\theta,\lambda}) < \alpha(E_{\theta_-,\lambda})\}$,

(e) if $\gamma \in \Gamma^+_{\theta_-}$, then

(e-1) $x_\theta = (x_{\theta,\lambda}) \in p_\gamma(E_{\theta_-})$,

(e-2) $p_\gamma(E_\theta) \subset U_\gamma(x_\theta)$,

(e-3) $p^\gamma_{\gamma_-}(U_\gamma(x_\theta)) \subset U_{\gamma_-}(x_{\theta_-})$, where $n \geq 2$ and $\theta = (\gamma,\xi)$ and $\theta_- = (\gamma_-,\xi')$ for $\gamma_- \in \Gamma_\mu$, $\xi \in \Xi^-_\gamma$, $\mu \in \Theta^-_{n-2}$ and $\xi' \in \Xi^-_\mu$,

(e-4) $y_{\theta,k} \in p^{-1}_\gamma(U_\gamma(x_\theta,k)) - \bigcup\{p^{-1}_{\Lambda-\gamma}(W_s) : s \in M(U_\gamma(x_\theta,k))\}$ for each $k \in \omega$,

(e-5) if $\lambda \in R_{\theta_-}$ with $x_{\theta,\lambda} \notin p_\lambda(C_{\theta_-})$, then $\lambda \in P_\theta$,

(e-6) $R_\theta = \bigcup\{\langle \mathrm{Supp}(y_{\mu,j})\rangle_k : \mu \leq \theta$ and $j,k \leq n\} \cup R_\gamma$

and if $\gamma \in \Gamma^-_{\theta_-}$, then

(e-6) $R_\theta = R_\gamma = R_{\theta_-}$,

(e-7) $x_\theta = x_{\theta_-}$,

(f) $C_\theta = \mathrm{Top}(E_\theta)$ and $(\mathcal{A}(\theta)_j)$ is a sequence of open (in $C_\theta$) covers of $C_\theta$ such that for $R \subset R_\gamma$,

(f-1) $p^\gamma_R(\mathcal{A}(\theta)_j)$ is a complete sequence of open (in $p^\gamma_R(C_\theta)$) covers of $p^\gamma_R(C_\theta)$,

(f-2) if $R \subset R_{\theta_-}$ with $R \cap P_\theta = \emptyset$, then $p^\gamma_R(\mathcal{A}(\theta)j)$ refines $p^{\theta_-}_R(\mathcal{A}(\theta_-)_{j+1})$ for each $j \in \omega$,

(g) $(E_\theta, C_\theta, (\mathcal{A}(\theta)_j), x_\theta) \in \mathcal{B}$.

Take an arbitrary $\lambda_0 \in \Lambda$ and a $\sigma$-locally finite closed cover $\mathcal{E}' = \{E'_\theta : \theta \in \Theta^-_0\}$ of $X_{\lambda_0}$ such that for each $\theta \in \Theta^-_0$, $E'_\theta$ is topped. Put $\Theta^+_0 = \{\emptyset\}$ and $\Theta_0 = \Theta^+_0 \oplus \Theta^-_0 = \Theta^-_0$. For each $\theta \in \Theta^-_0$, let $E_\theta = p^{-1}_{\lambda_0}(E'_\theta)$ and let $C_\theta = \mathrm{Top}(E_\theta)$. Take a complete sequence $(\mathcal{A}(\theta)_j)$ of open (in $C_\theta$) covers of $C_\theta$ and a $x_\theta \in C_\theta$. Let $(U_{\lambda_0}(x_\theta, k))$ be a countable, open neighborhood base at $x_\theta$ in $X_{\lambda_0}$ such that $U_{\lambda_0}(x_\theta, k+1) \subset U_{\lambda_0}(x_\theta, k)$ for each $k \in \omega$. Then, for each $\theta \in \Theta^-_0$, $(E_\theta, C_\theta, (\mathcal{A}(\theta)_j), x_\theta) \in \mathcal{B}$.

Let $n \in \omega$ and assume that for each $k \leq n$, we have already obtained a collection $\{E_\theta : \theta \in \Theta_k\}$ and other collections, satisfying (2)–(5). Take a $\theta_- \in \Theta^-_n$. Let $(E_{\theta_-}, C_{\theta_-}, \mathcal{A}(\theta_-)_j, x_{\theta_-}) \in \mathcal{B}$, where $E = \prod_{\lambda \in R_{\gamma_-}} E_\lambda \times \Sigma_{\Lambda - R_{\gamma_-}}$. For each $A \in \mathcal{A}(\theta_-)_0$, take an open subset $B_A$ in $\prod_{\lambda \in R_{\gamma_-}} E_\lambda$ such that $B_A \cap C_E = A$. By Lemma 2(2), there is a $\sigma$-locally finite collection $\mathcal{H}(E) = \{H'_\gamma : \gamma \in \Gamma_{\theta_-}\}$ of topped, closed rectangle subsets of $\prod_{\lambda \in R_{\gamma_-}} E_\lambda$ such that $\prod_{\lambda \in R_{\gamma_-}} E_\lambda = \bigcup \mathcal{H}(E)$ and for each $\gamma \in \Gamma_{\theta_-}$, $H'_\gamma$ is contained in some member of $\{B_A : A \in \mathcal{A}(\theta_-)_0\} \cup \{\prod_{\lambda \in R_{\gamma_-}} E_\lambda - C_{\theta_-}\}$.

For each $\gamma \in \Gamma_{\theta_-}$, let $C_\gamma = \mathrm{Top}(H'_\gamma \times \Sigma_{\Lambda - \gamma_-})$ and $\Gamma^+_{\theta_-} = \{\gamma \in \Gamma_{\theta_-} : C_\gamma \cap C_{\theta_-} \neq \emptyset$ and $C_\gamma \times \prod_{\Lambda - R_{\gamma_-}} \{x^*_\lambda\}$ satisfy (**)$\}$ and $\Gamma^-_{\theta_-} = \Gamma_{\theta_-} - \Gamma^+_{\theta_-}$.

Let $\gamma \in \Gamma^+_{\theta_-}$. Then there is a finite collection $\mathcal{B}$ of cylindrically open subsets in $\Sigma$ such that $C_\gamma \times \prod_{\lambda \in \Lambda - \gamma_-} \{x^*_\lambda\} \subset \bigcup \mathcal{B}$ and for each $B \in \mathcal{B}$, there is a $G \in \mathcal{G}$ such that $\overline{B} \subset G$. Define $R_\gamma$ as before and $H_\gamma = H'_\gamma \times X_{\gamma - \gamma_-}$. Let

$$\begin{aligned} \Omega = \{V : V \ &\text{is an open subset in} \ X_\gamma \ \text{meeting} \ H_\gamma \ \text{such that} \\ &p^{-1}_\gamma(V) \subset \bigcup \{p^{-1}_{\Lambda - \gamma}(W_s) : s \in M(V)\}\} \ \text{and} \\ \Phi_\gamma = H_\gamma - \bigcup \Omega. \end{aligned}$$

It is clear that $p_\gamma(C_\gamma \times \prod_{\lambda \in \Lambda - R_{\gamma_-}} \{x^*_\lambda\}) \cap \Phi_\gamma = \emptyset$. By Lemma 2(2), every open cover of $H_\gamma$ has a $\sigma$-locally finite refinement, consisting of topped, closed rectangles. The rest of the construction is similar to that in the proofs of Theorem 1, [TY, Theorem 4.2] and [Y2, Theorem 4].

Let $\Theta = \bigcup_{n\in\omega} \Theta_n$, $\Theta^+ = \bigcup_{n\geq 1} \Theta_n^+$ and $\mathcal{E} = \{E_\theta : \theta \in \Theta^+\}$. It follows from (2) and (5) that $\mathcal{E}$ is a $\sigma$-locally finite collection of cylindrically closed sets in $\Sigma$ and for each $\theta \in \Theta^+, E_\theta \subset \bigcup\{p_{\Lambda-\gamma}^{-1}(W_s) : s \in M(p_\gamma(E_\theta))\}$. By Lemma 4, it suffices to prove that $\mathcal{E}$ is a cover of $\Sigma$. Assume that there is a point $x = (x_\lambda) \in \Sigma - \bigcup \mathcal{E}$. By (5)(b), we can inductively choose a sequence $\{\theta_n : n \in \omega\}$ such that $\theta_n = (\gamma_n, \xi_n) \in \Theta_n$, $\gamma_n \in \Gamma_{\theta_{n-1}}$, $\xi_n \in \Xi_{\gamma_n}^-$, $\theta_{n-1} < \theta_n$, $n \geq 1$ and $x \in E_{\theta_n}$ for each $n \in \omega$. By the same proof of Claim 2 in Theorem 1, $\{n \in \omega : \gamma_n \in \Gamma_{\theta_{n-1}}^+\}$ is infinite. Then there is also a mapping $\phi : \omega \to \omega$ such that for each $n \in \omega$, $\phi(n) < \phi(n+1)$, $\gamma_{\phi(n)} \in \Gamma^+_{\theta_{\phi(n)-1}}$ and for each $k$ $(\phi(n) < k < \phi(n+1))$, $\gamma_k \in \Gamma_{k-1}^-$.

As the proof of Theorem 1, we also omit the index letter $\theta$, $\gamma$ and $\phi$ for simplicity. Let $Q = \bigcup_{n\in\omega} R_n$. As the same way as Claim 4 in Theorem 1, for each $\lambda \in Q$, there is an $m_\lambda \geq 1$ such that for each $n > m_\lambda$, $\lambda \in R_{n-1} - P_n$, $x_{n,\lambda} \in p_\lambda(C_n) \subset p_\lambda(C_{n-1})$. (We similarly use $C_n$.) Let us define $F_k^m$ (we use $E_k^m$ in the proof of Theorem 1) and $K_k$ for each $k, m \geq 1$ with $m > k$ as before. Then there is a point $(z_k) \in \varprojlim\{K_k, p_k^{k+1}\}$. Define a point $z = (z_\lambda) \in \Sigma$ such that $p_k(z) = z_k$ for each $k \in \omega$ and $z_\lambda = x_\lambda^*$ otherwise. Then we have a contradiction in the same argument as [Y2, Lemma 7]. □

By Theorem 1, 2, we have

**Theorem 3.** *If $\Sigma$ is a $\Sigma$-product of first countable, paracompact Čech-scattered spaces, then it is shrinking.*

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