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# Complete hypersurfaces with constant scalar curvature in a sphere

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*Abstract.* In this paper, by using Cheng-Yau's self-adjoint operator $\Box$, we study the complete hypersurfaces in a sphere with constant scalar curvature.

*Keywords:* hypersurface, sphere, scalar curvature
*Classification:* 53C42, 53A10

## 1. Introduction

Let $S^{n+1}$ be an $(n+1)$-dimensional unit sphere with constant sectional curvature 1, let $M^n$ be an $n$-dimensional hypersurface in $S^{n+1}$, and $e_1, \dots, e_n$ a local orthonormal frame field on $M^n$, $\omega_1, \dots, \omega_n$ its dual coframe field. Then the second fundamental form of $M^n$ is

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j. \tag{1}$$

Further, near any given point $p \in M^n$, we can choose a local frame field $e_1, \dots, e_n$ so that at $p$, $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i k_i\omega_i \otimes \omega_j$. Then the Gauss equation says

$$R_{ijij} = 1 + k_i k_j, \quad i \neq j. \tag{2}$$

$$n(n-1)(R-1) = n^2H^2 - |h|^2, \tag{3}$$

where $R$ is the normalized scalar curvature, $H = \frac{1}{n}\sum_i k_i$ the mean curvature and $|h|^2 = \sum_i k_i^2$ the norm square of the second fundamental form of $M^n$.

As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature $H$ in $S^{n+1}$ by use of J. Simons' method, for example, see [1], [3], [4], [6], [9], etc.

On the other hand, Cheng-Yau [2] introduced a new self-adjoint differential operator $\Box$ to study the hypersurfaces with constant scalar curvature. Later, Li [5] obtained interesting rigidity results for hypersurfaces with constant scalar curvature in space-forms using the Cheng-Yau's self-adjoint operator $\Box$.

In the present paper, we use Cheng-Yau's self-adjoint operator $\Box$ to study the complete hypersurfaces in a sphere with constant scalar curvature, and prove the following theorem:

**Theorem.** *Let $M^n$ be an $n$-dimensional ($n \geq 3$) complete hypersurface with constant normalized scalar curvature $R$ in $S^{n+1}$. If*

(1) $\bar{R} = R - 1 \geq 0$,

(2) *the mean curvature $H$ of $M^n$ satisfies*

$$\bar{R} \leq \sup H^2 \leq \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big],$$

*then either*

$$\sup H^2 = \bar{R}$$

*and $M^n$ is a totally umbilical hypersurface; or*

$$\sup H^2 = \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big],$$

*and $M^n = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{n(\bar{R}+1)}}$.*

## 2. Preliminaries

Let $M^n$ be an $n$-dimensional complete hypersurface in $S^{n+1}$. We choose a local orthonormal frame $e_1, \dots, e_{n+1}$ in $S^{n+1}$ such that at each point of $M^n$, $e_1, \dots, e_n$ span the tangent space of $M^n$ and form an orthonormal frame there. Let $\omega_1, \dots, \omega_{n+1}$ be its dual coframe. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1; \quad 1 \leq i, j, k, \dots \leq n.$$

Then the structure equations of $S^{n+1}$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{4}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{5}$$

$$K_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{6}$$

Restricting these forms to $M^n$, we have

$$\omega_{n+1} = 0. \tag{7}$$

From Cartan's lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{8}$$

From these formulas, we obtain the structure equations of $M^n$:

(9) $$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

(10) $$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

(11) $$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where $R_{ijkl}$ are the components of the curvature tensor of $M^n$ and

(12) $$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$$

is the second fundamental form of $M^n$. We also have

(13) $$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj},$$

(14) $$n(n-1)(R-1) = n^2H^2 - |h|^2,$$

where $R$ is the normalized scalar curvature, and $H$ the mean curvature.

Define the first and the second covariant derivatives of $h_{ij}$, say $h_{ijk}$ and $h_{ijkl}$ by

(15) $$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj},$$

(16) $$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}.$$

Then we have the Codazzi equation

(17) $$h_{ijk} = h_{ikj},$$

and the Ricci's identity

(18) $$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}.$$

For a $C^2$-function $f$ defined on $M^n$, we define its gradient and Hessian $(f_{ij})$ by the following formulas

(19) $$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}.$$

The Laplacian of $f$ is defined by $\Delta f = \sum_i f_{ii}$.

Let $\phi = \sum_{ij} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on $M^n$, where

$$\phi_{ij} = nH\delta_{ij} - h_{ij}. \tag{20}$$

Following Cheng-Yau [2], we introduce the operator $\Box$ associated to $\phi$ acting on any $C^2$-function $f$ by

$$\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{21}$$

Since $\phi_{ij}$ is divergence-free, it follows [2] that the operator $\Box$ is self-adjoint relative to the $L^2$ inner product of $M^n$, i.e.

$$\int_{M^n} f\Box g = \int_{M^n} g\Box f. \tag{22}$$

We can choose a local frame field $e_1, \ldots e_n$ at any point $p \in M^n$, such that $h_{ij} = k_i\delta_{ij}$ at $p$, and by use of (21) and (14), we have

$$\begin{aligned} \Box(nH) &= nH\Delta(nH) - \sum_i k_i(nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH)_{ii} \\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|h|^2 - n^2|\nabla H|^2 - \sum_i k_i(nH)_{ii}. \end{aligned} \tag{23}$$

On the other hand, through a standard calculation by use of (17) and (18), we get

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{24}$$

Putting (24) into (23), we have

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{25}$$

From (11), we have $R_{ijij} = 1 + k_ik_j,\ \ i \neq j$, and by putting this into (25), we obtain

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + n|h|^2 - n^2H^2 - |h|^4 + nH\sum_i k_i^3. \tag{26}$$

Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$. We have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2, \tag{27}$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{28}$$

From (26)–(28), we get

$$\begin{aligned}\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 \\ + |Z|^2(n + nH^2 - |Z|^2) + nH\sum_i \mu_i^3.\end{aligned} \tag{29}$$

We need the following algebraic lemma due to M. Okumura [7] (see also [1]).

**Lemma 2.1.** *Let $\mu_i$, $i = 1, \dots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta =$ constant $\geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{30}$$

*and the equality holds in* (30) *if and only if at least* $(n-1)$ *of the* $\mu_i$ *are equal.*

By use of Lemma 2.1, we have

$$\begin{aligned}\Box(nH) \geq \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 \\ + (|h|^2 - nH^2)(n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}).\end{aligned} \tag{31}$$

## 3. Proof of Theorem

The following lemma is essentially due to Cheng-Yau [2] (see also [5]).

**Lemma 3.1.** *Let $M$ be an $n$-dimensional hypersurface in $S^{n+1}$. Suppose that the normalized scalar curvature $R =$ constant and $R \geq 1$. Then $|\nabla h|^2 \geq n^2|\nabla H|^2$.*

From the assumption of Theorem that $R$ is constant and $\bar{R} = R - 1 \geq 0$ and Lemma 3.1 we have

$$\Box(nH) \geq (|h|^2 - nH^2)(n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}). \tag{32}$$

By Gauss equation (14) we know that

$$|Z|^2 = |h|^2 - nH^2 = \frac{n-1}{n}(|h|^2 - n\bar{R}). \tag{33}$$

From (32) and (33) we have

$$\Box(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_H(|h|), \tag{34}$$

where

$$\phi_H(|h|) = n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}.$$

By (33) we can write $\phi_H(|h|)$ as

$$\phi_{\bar{R}}(|h|) = n+2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})}. \tag{35}$$

Therefore (34) becomes

$$\Box(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_{\bar{R}}(|h|). \tag{36}$$

It is a direct check that our assumption

$$\sup H^2 \leq \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big]$$

is equivalent to

$$\sup |h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}\Big[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\Big], \tag{37}$$

i.e.

$$\begin{aligned}(n + 2(n-1)\bar{R} - \frac{n-2}{n}\sup|h|^2)^2 \\ \geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R} + \sup|h|^2)(\sup|h|^2 - n\bar{R}).\end{aligned} \tag{38}$$

But it is clear from (37) that (38) is equivalent to

$$\begin{aligned}n + 2(n-1)\bar{R} - \frac{n-2}{n}\sup|h|^2 \\ \geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + \sup|h|^2)(\sup|h|^2 - n\bar{R})}.\end{aligned} \tag{39}$$

So under the hyperthesis that

$$\sup H^2 \leq \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big],$$

we have

$$\phi_{\bar{R}}(\sqrt{\sup |h|^2}) \geq 0. \tag{40}$$

On the other hand,

$$\begin{aligned} \Box(nH) &= \sum_{i,j}(nH\delta_{ij} - nh_{ij})(nH)_{ij} = \sum_i (nH - nh_{ii})(nH)_{ii} \\ &= n\sum_i H(nH)_{ii} - n\sum_i k_i(nH)_{ii} \leq (|H|_{\max} - C)\Delta(nH), \end{aligned} \tag{41}$$

where $|H|_{\max}$ is the maximum of the mean curvature $H$ and $C = \min k_i$ is the minimum of the principal curvatures of $M^n$.

Now we need the following maximum principle at infinity for complete manifolds due to Omori [8] and Yau [10]:

**Lemma 3.2.** *Let $M^n$ be an $n$-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to R$ a smooth function bounded from below. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M^n$ such that*

(i) $|\nabla f|(p_\varepsilon) < \varepsilon$,
(ii) $\Delta f(p_\varepsilon) > -\varepsilon$,
(iii) $\inf f \leq f(p_\varepsilon) \leq \inf f + \varepsilon$.

Since the scalar curvature of $M$ is a constant, from the hypothesis that $\bar{R} \leq \sup H^2 \leq \frac{1}{n^2}[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}]$, and Gauss equation (14), we know the squared norm $|h|^2$ of the second fundamental form is bounded from above, from (11) we know that the sectional curvature is bounded from below. So we may apply Lemma 3.2 to the smooth function $f$ on $M^n$ defined by

$$f = \frac{1}{\sqrt{1+(nH)^2}}.$$

It is immediate to check that

$$|\nabla f|^2 = \frac{1}{4}\frac{|\nabla(nH)^2|^2}{(1+(nH)^2)^3} \tag{42}$$

and that

$$\Delta f = -\frac{1}{2}\frac{\Delta(nH)^2}{(1+(nH)^2)^{3/2}} + \frac{3}{4}\frac{|\nabla(nH)^2|^2}{(1+(nH)^2)^{5/2}}. \tag{43}$$

By Lemma 3.2 we can find a sequence of points $p_k$, $k \in N$ in $M^n$, such that

$$\lim_{k\to\infty} f(p_k) = \inf f, \quad \Delta f(p_k) > -\frac{1}{k}, \quad |\nabla f|^2(p_k) < \frac{1}{k^2}. \tag{44}$$

Using (44) in equations (42) and (43) and the fact that

$$\lim_{k\to\infty} (nH)(p_k) = \sup_{p\in M^n} (nH)(p), \tag{45}$$

we get

$$-\frac{1}{k} \le -\frac{1}{2}\frac{\Delta(nH)^2}{(1+(nH)^2)^{3/2}}(p_k) + \frac{3}{k^2}(1+(nH)^2(p_k))^{1/2}. \tag{46}$$

Hence we obtain

$$\frac{\Delta(nH)^2}{(1+(nH)^2)^2}(p_k) < \frac{2}{k}\left(\frac{1}{\sqrt{1+(nH)^2(p_k)}} + \frac{3}{k}\right). \tag{47}$$

On the other hand, by (36) and (41), we have

$$\frac{n-1}{n}(|h|^2 - n\bar{R})\phi_{\bar{R}}(|h|) \le \square(nH) \le n(|H|_{\max} - C)\Delta(nH). \tag{48}$$

At points $p_k$ of the sequence given in (44), this becomes

$$\begin{aligned} \frac{n-1}{n}(|h|^2(p_k) - n\bar{R})\phi_{\bar{R}}(|h|(p_k)) &\le \square(nH(p_k)) \\ &\le n(|H|_{\max} - C)\Delta(nH)(p_k). \end{aligned} \tag{49}$$

Letting $k \to \infty$ and using (47) we have that the right hand side of (49) goes to zero, so we have either $\frac{n-1}{n}(\sup|h|^2 - n\bar{R}) = 0$, i.e. $\sup H^2 = \bar{R}$, or $\phi_{\bar{R}}(\sqrt{\sup|h|^2}) = 0$.

If $\sup|h|^2 = n\bar{R}$, by (33) $|Z|^2 = \frac{n-1}{n}(|h|^2 - n\bar{R})$ we have $\sup|Z|^2 = \frac{n-1}{n}(\sup|h|^2 - n\bar{R}) = 0$, hence $|Z|^2 = 0$ and $M^n$ is totally umbilical.

If $\phi_{\bar{R}}(\sqrt{\sup|h|^2}) = 0$, it is easy to prove that

$\sup H^2 = \frac{1}{n^2}[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}]$, hence equalities hold in (30) and Lemma 3.1, and it follows that $k_i =$ constant for all $i$ and $(n-1)$ of the $k_i$'s are equal. After renumberation if necessary, we can assume that

$$k_1 = k_2 = \cdots = k_{n-1}, \quad k_1 \ne k_n.$$

Therefore, $M^n$ is a isoparametric hypersurface in $S^{n+1}$ with two distinct principal curvatures, hence $M^n = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $k_1 = \cdots = k_{n-1} = \sqrt{1-r^2}/r$, $k_n = -r/\sqrt{1-r^2}$. From (14), it is easy to see that $n(n-1)\bar{R} = (n-1)(n-2-nr^2)/r^2$, thus $r = \sqrt{\frac{n-2}{n(\bar{R}+1)}}$. This completes the proof of Theorem.

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