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A note on Briot-Bouquet-Bernoulli differential subordination

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Abstract. Let p, q be analytic functions in the unit disk $\mathcal{U}$. For $\alpha \in [0,1)$ the authors consider the differential subordination and the differential equation of the Briot-Bouquet type:

$$p^{1-\alpha}(z) + \frac{zp'(z)}{\delta p^{\alpha}(z) + \lambda p(z)} \prec h(z), \quad z \in \mathcal{U},$$

$$q^{1-\alpha}(z) + \frac{nzq'(z)}{\delta q^{\alpha}(z) + \lambda q(z)} = h(z), \quad z \in \mathcal{U},$$

with $p(0) = q(0) = h(0) = 1$. The aim of the paper is to find the dominant and the best dominant of the above subordination. In addition, the authors give some particular cases of the main result obtained for appropriate choices of functions h.

Keywords: differential subordinations, Briot-Bouquet-Bernoulli differential subordination

Classification: 34A25, 30C35, 30C45

1. Introduction

By $\mathcal{P}$ we denote the class of Carathéodory functions, that is the class of functions p analytic in the unit disk $\mathcal{U}$ and such that $p(0) = 1$, $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$.

Suppose that functions g and G are analytic in the unit disc $\mathcal{U}$. The function g is said to be *subordinate* to G, written $g \prec G$ (or $g(z) \prec G(z)$, $z \in \mathcal{U}$), if G is univalent in $\mathcal{U}$, $g(0) = G(0)$ and $g(\mathcal{U}) \subset G(\mathcal{U})$. When the function $g(z) = \psi(p(z), zp'(z))$ with ψ, p has the appropriate normalization, then the subordination is called *the first-order differential subordination.*

Now, we recall some results from the theory of differential subordinations, developed by Miller and Mocanu (cf. e.g. [1], [2], [3], [4]) useful for further investigation.

Lemma 1.1 ([1], [4]). *Let q be analytic in $\mathcal{U}$, $q(z) = a + q_n z^n + \cdots$. Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ be a function defined in some domain $D \subset \mathbb{C}^2$, such that for a function h, analytic and univalent in $\mathcal{U}$, $q(0) = h(0) = a$, it satisfies*

$$\psi(q(\zeta_0), m\zeta_0 q'(\zeta_0)) \notin h(\mathcal{U}) \ \ \textit{when} \ \ m \geq n,\ z \in \mathcal{U},\ |\zeta_0| = 1. \tag{1.1}$$

If p is analytic in $\mathcal{U}$, $p(z) = a + p_n z^n + \cdots$, and

$$\psi(p(z), zp'(z)) \prec h(z),\ z \in \mathcal{U}, \tag{1.2}$$

where $\psi(p(0), 0) = h(0)$, then $p \prec q$ in $\mathcal{U}$.

The above result remains true for the special case when $q \equiv h$.

A univalent function q is called a *dominant of the solution of the differential subordination*, or more simply a *dominant*, if $p \prec q$ for all p satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be *the best dominant* of (1.2).

Lemma 1.2 ([1], [4]). *Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ be such that the differential equation*

$$\psi(q(z), nzq'(z)) = h(z),\ z \in \mathcal{U}, \tag{1.3}$$

where h is analytic and univalent in $\mathcal{U}$, with $\psi(q(0), 0) = h(0)$, has a univalent solution q, with $q(0) = h(0)$. If p is an analytic function in $\mathcal{U}$ and satisfies

$$\psi(p(z), nzp'(z)) \prec h(z),\ z \in \mathcal{U}, \tag{1.4}$$

$\psi(p(0), 0) = h(0)$, and

$$\psi(q(\zeta_0), m\zeta_0 q'(\zeta_0)) \notin h(\mathcal{U}) \quad \text{when} \quad m \geq n,\ z \in \mathcal{U},\ |\zeta_0| = 1, \tag{1.5}$$

then $p \prec q$ in $\mathcal{U}$ and q is the best dominant of (1.2).

(The results are in fact much more general than those mentioned in Lemma 1.1 and Lemma 1.2 but the above versions are sufficient for our consideration.)

Lemma 1.3 ([4]). *Let β and γ be complex constants, and let h be convex* (*univalent*) *in $\mathcal{U}$, with $\operatorname{Re}(\beta h(z) + \gamma) > 0$ and $q(0) = h(0)$, and let $n \in \mathbb{N}$. Also, suppose that the differential equation*

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z) \tag{1.6}$$

has an univalent solution q. If p is analytic in $\mathcal{U}$ with $p(0) = h(0)$ and satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \tag{1.7}$$

then $p \prec q \prec h$ in $\mathcal{U}$, and q is the best dominant of (1.7).

Let c be a complex number such that $\operatorname{Re} c > 0$, let n be a positive integer, and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + 2\operatorname{Re}\frac{c}{n}} + \operatorname{Im} c\right].$$

If $R(z)$ is the univalent function in $\mathcal{U}$ defined by

$$R(z) = 2C_n \frac{z}{1-z^2}$$

then the *open door function* is defined by

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n \frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2-(z+b)^2}, \tag{1.8}$$

where $b = R^{-1}(c)$. From its definition we see that $R_{c,n}$ is univalent in $\mathcal{U}$, $R_{c,n}(0) = c$ and $R_{c,n}(\mathcal{U})$ is a complex plane with slits along the half-lines $\operatorname{Re} w = 0$ and $|\operatorname{Im} w| \geq C_n$. The definition and the name of that function is due to Miller and Mocanu, its extensive applications can be found in their monograph [4].

Lemma 1.4 ([4])**.** *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}[\beta+\gamma] > 0$, and let $n \in \mathbb{N}$. Let $R_{\beta+\gamma,n}$ be as given in* (1.8)*, and let h be analytic in $\mathcal{U}$ with $h(0) = 1$. If*

$$\beta h(z) + \gamma \prec R_{\beta+\gamma,n}(z),$$

then the solution q of (1.6) *with $q(0) = 1$ is analytic in $\mathcal{U}$, satisfies $\operatorname{Re}[\beta q(z)+\gamma] > 0$, and is given by*

$$q(z) = z^{\frac{\gamma}{n}} [H(z)]^{\frac{\beta}{n}} \left(\frac{\beta}{n}\int_0^z [H(t)]^{\frac{\beta}{n}} t^{\frac{\gamma}{n}-1}\, dt\right)^{-1} - \frac{\gamma}{\beta}, \tag{1.9}$$

where

$$H(z) = z \exp \int_0^z \frac{h(t)-1}{t}\, dt. \tag{1.10}$$

Lemma 1.5 ([4])**.** *Let h be analytic in $\mathcal{U}$ with $h(0) = 1$, let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}[\beta+\gamma] > 0$, $n \in \mathbb{N}$. Let $R_{\beta+\gamma,n}$ be as given in* (1.8) *and*

$$\beta h(z) + \gamma \prec R_{\beta+\gamma,n}(z).$$

If q is the analytic solution of the Briot-Bouquet differential equation (1.6) *as given in* (1.9)*, and if h is convex or $Q(z) = \frac{zq'(z)}{\beta q(z)+\gamma}$ is starlike, then q and h are univalent, $\operatorname{Re}[\beta q(z)+\gamma] > 0$ and $\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0$. Furthermore if p is analytic, $p(0) = 1$ and p satisfies* (1.7) *then $p \prec q$ and q is the best dominant of* (1.7)*.*

Lemmas 1.3, 1.4 and 1.5 characterize properties of subordinations known as the subordinations of the Briot-Bouquet type, described in detail in papers [1], [2], [3], [4].

In this note we will discuss in some sense the general case of the Briot-Bouquet differential subordination involved in the Bernoulli differential equation.

2. Main results

For $\alpha \in [0,1)$ and for a function $p(z) = 1 + p_1 z + \cdots$ let us consider the first order differential subordination and the differential equation of the form

$$p^{1-\alpha}(z) + \frac{zp'(z)}{\delta p^{\alpha}(z) + \lambda p(z)} \prec h(z), \quad z \in \mathcal{U}, \tag{2.1}$$

$$q^{1-\alpha}(z) + \frac{nzq'(z)}{\delta q^{\alpha}(z) + \lambda q(z)} = h(z), \quad z \in \mathcal{U}, \tag{2.2}$$

with $p(0) = h(0) = 1$ and $p^{1-\alpha}(0) = q^{1-\alpha}(0) = 1$, h univalent in $\mathcal{U}$, and $\lambda, \delta \in \mathbb{C}$, $\lambda \neq 0$, $n \in \mathbb{N}$. In the sequel we choose the principal branch of the powers.

Observe first, that by setting $\alpha = 0$ in (2.1) and (2.2), we obtain the Briot-Bouquet subordination, and the Briot-Bouquet equation, respectively (cf. [4]).

Theorem 2.1. *Let $\delta, \lambda \in \mathbb{C}$, $\lambda \neq 0$, $\alpha \in [0,1)$ and let h be univalent and convex in $\mathcal{U}$. If $p(0) = p^{1-\alpha}(0) = h(0) = 1$, $\mathrm{Re}(\lambda h(z) + \delta) > 0$ and if p is an analytic function that satisfies*

$$p^{1-\alpha}(z) + \frac{zp'(z)}{\delta p^{\alpha}(z) + \lambda p(z)} \prec h(z), \quad z \in \mathcal{U}, \tag{2.3}$$

then $p^{1-\alpha} \prec h$. If moreover $h^{\frac{1}{1-\alpha}}$ is univalent in $\mathcal{U}$ then $p \prec h^{\frac{1}{1-\alpha}}$.

Proof: Set $P(z) = p^{1-\alpha}(z)$. Then $P(0) = 1 = h(0)$ and $zP'(z) = (1-\alpha)zp'(z)p^{-\alpha}(z)$. Hence

$$\begin{aligned} p^{1-\alpha}(z) + \frac{zp'(z)}{\delta p^{\alpha}(z) + \lambda p(z)} &= p^{1-\alpha}(z) + \frac{zp'(z)}{p^{\alpha}(z)(\delta + \lambda p^{1-\alpha}(z))} \\ &= P(z) + \frac{zP'(z)}{\lambda(1-\alpha)P(z) + \delta(1-\alpha)}. \end{aligned}$$

Setting $\beta = \lambda(1-\alpha)$ and $\gamma = \delta(1-\alpha)$, and taking into account the assumption, we have $\mathrm{Re}(\beta h(z) + \gamma) > 0$. Thus the subordination (2.3) becomes the well known Briot-Bouquet differential subordination (cf. e.g. [2], [3], [4]), from which we conclude $P \prec h$, it means $p^{1-\alpha} \prec h$.

Now, we will prove that $p \prec h^{\frac{1}{1-\alpha}}$. Put $\psi(r,s) = r^{1-\alpha} + \frac{s}{\delta r^{\alpha} + \lambda r}$. Then, by Lemma 1.1, it suffices to show that $\psi(h^{\frac{1}{1-\alpha}}(\zeta_0), \frac{m}{1-\alpha}\zeta_0 h'(\zeta_0) h^{\frac{\alpha}{1-\alpha}}(\zeta_0)) \notin h(\mathcal{U})$. Set

$$\begin{aligned} \psi_0 &= \psi\left(h^{\frac{1}{1-\alpha}}(\zeta_0), \frac{m}{1-\alpha}\zeta_0 h'(\zeta_0) h^{\frac{\alpha}{1-\alpha}}(\zeta_0)\right) \\ &= h(\zeta_0) + \frac{m\zeta_0 h'(\zeta_0) h^{\frac{\alpha}{1-\alpha}}(\zeta_0)}{(1-\alpha)\left[\delta h^{\frac{\alpha}{1-\alpha}}(\zeta_0) + \lambda h^{\frac{1}{1-\alpha}}(\zeta_0)\right]} \\ &= h(\zeta_0) + \frac{m\zeta_0 h'(\zeta_0)}{\gamma + \beta h(\zeta_0)} \end{aligned}$$

with $|\zeta_0| = 1$, $m \geq 1$. Since $\mathrm{Re}(\lambda h(z) + \delta) > 0$, so that $\mathrm{Re}(\beta h(z) + \gamma) > 0$, we obtain

$$\mathrm{Re}\,\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} = \mathrm{Re}\,\frac{m}{\gamma + \beta h(\zeta_0)} > 0,$$

or equivalently

$$\left|\mathrm{Arg}\,\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)}\right| < \frac{\pi}{2}\,.$$

By the fact that $h(\mathcal{U})$ is convex, $h(\zeta_0) \in h(\partial\mathcal{U})$ and $\zeta_0 h'(\zeta_0)$ is the outer normal to $h(\partial\mathcal{U})$ at $h(\zeta_0)$, we conclude that $\psi_0 \notin h(\mathcal{U})$ and therefore, in view of Lemma 1.1, we have $p \prec h^{\frac{1}{1-\alpha}}$, as asserted. □

Remark 2.1. Observe that the substitution $p^{1-\alpha} = q$ in the proof of Theorem 2.1 is of the Bernoulli type, so that the title of the presented note is justified.

Theorem 2.2. *Let $\delta, \lambda \in \mathbb{C}$, $\lambda \neq 0$, $\alpha \in [0, 1)$, $n \in \mathbb{N}$, and let h be univalent and convex in $\mathcal{U}$, and $\mathrm{Re}\,(\lambda h(z) + \delta) > 0$. Suppose that the differential equation*

$$q^{1-\alpha}(z) + \frac{nzq'(z)}{\delta q^{\alpha}(z) + \lambda q(z)} = h(z), \quad z \in \mathcal{U}, \tag{2.4}$$

with $q(0) = q^{1-\alpha}(0) = h(0) = 1$ has an univalent solution q which satisfies $q^{1-\alpha} \prec h$, where $q, q^{1-\alpha}$ are univalent functions. If p is analytic, $\psi(p(0), 0) = h(0) = p(0) = p^{1-\alpha}(0) = 1$, and

$$p^{1-\alpha}(z) + \frac{zp'(z)}{\delta p^{\alpha}(z) + \lambda p(z)} \prec h(z), \quad z \in \mathcal{U}, \tag{2.5}$$

then $p^{1-\alpha} \prec q^{1-\alpha}$ and $p \prec q$. The function q is the best dominant of (2.5).

Proof: The first part of the assertion is a direct consequence of properties of the Briot-Bouquet subordination. Thus we only need to show that $p \prec q$ and q is the best dominant of (2.5). In this case, condition (1.5) reduces to show that

$$\psi_0 = \psi(q(\zeta_0), m\zeta_0 q'(\zeta_0)) = q^{1-\alpha}(\zeta_0) + m\frac{\zeta_0 q'(\zeta_0)}{\delta q^{\alpha}(\zeta_0) + \lambda q(\zeta_0)} \notin h(\mathcal{U}),$$

when $|\zeta_0| = 1$, and $m \geq n$. Using (2.4) we obtain

$$\psi_0 = q^{1-\alpha}(\zeta_0) + \frac{m}{n}\left[h(\zeta_0) - q^{1-\alpha}(\zeta_0)\right].$$

From $q^{1-\alpha} \prec h$ we obtain $q^{1-\alpha}(\zeta_0) \in h(\mathcal{U})$. Using this together with the fact that $h(\mathcal{U})$ is a convex domain and $m/n \geq 1$, we deduce that $\psi_0 \notin h(\mathcal{U})$. Therefore, by Lemma 1.2 we conclude that $p \prec q$, and q is the best dominant of (2.5). □

Theorem 2.3. *Let* $\alpha \in [0,1)$ *and* $n \in \mathbb{N}$. *Let* $\delta, \lambda \in \mathbb{C}$ *be such that* $\lambda \neq 0$, $\mathrm{Re}(\lambda+\delta) > 0$, *and let* h *be analytic in* $\mathcal{U}$ *with* $h(0) = 1$. *Let* $R_{\lambda+\delta,n}$ *be as given in* (1.8) *and*

$$\delta + \lambda h(z) \prec R_{\lambda+\delta,n}(z).$$

If q *is the analytic solution of equation* (2.4) *and if* h *is convex or* $Q(z) = \frac{zq'(z)}{\delta q^{\alpha}(z)+\lambda q(z)}$ *is starlike then* $q^{1-\alpha}$ *and* h *are univalent, and*

$$\mathrm{Re}[\delta q^{\alpha}(z) + \lambda q(z)] > 0, \qquad \mathrm{Re}\, \frac{zh'(z)}{Q(z)} > 0.$$

The proof will be omitted because the statements of Theorem 2.3 are immediate consequences of Lemma 1.5.

When setting $G(z) = q^{1-\alpha}(z)$ in (2.4) we get the Briot-Bouquet equation

$$G(z) + \frac{nzG'(z)}{(1-\alpha)\lambda G(z) + \delta(1-\alpha)} = h(z). \tag{2.6}$$

Let

$$H(z) = z \exp \int_0^z \frac{h(t)-1}{t}\, dt. \tag{2.7}$$

Then, by making use of Lemma 1.4, the solution of equation (2.6) is given by
(2.8)

$$G(z) = z^{\frac{\delta(1-\alpha)}{n}} [H(z)]^{\frac{\lambda(1-\alpha)}{n}} \left(\frac{\lambda(1-\alpha)}{n} \int_0^z [H(t)]^{\frac{\lambda(1-\alpha)}{n}} t^{\frac{\delta(1-\alpha)}{n}-1}\, dt \right)^{-1} - \frac{\delta}{\lambda}.$$

Observe that the univalence of $G = q^{1-\alpha}$ does not have to imply the univalence of q, however in some cases we may obtain the function q being univalent, see below.

3. Special cases

Let us consider some special cases.

Theorem 3.1. *Let* $\alpha \in [0,1/2]$. *If* q *is an analytic function in* $\mathcal{U}$ *such that* $q(0) = q^{1-\alpha}(0) = 1$ *and*

$$q^{1-\alpha}(z) + \frac{(1-\alpha)zq'(z)}{q(z)} = \frac{1+z}{1-z}\,, \tag{3.1}$$

then q is an univalent solution of (3.1) *of the form*

$$q(z) = \left(\frac{1}{1-z}\right)^{\frac{1}{1-\alpha}}.$$

If moreover p is an analytic function in $\mathcal{U}$ with $p(0) = p^{1-\alpha}(0) = 1$ and

$$p^{1-\alpha}(z) + \frac{(1-\alpha)zp'(z)}{p(z)} \prec \frac{1+z}{1-z}, \tag{3.2}$$

then $p \prec q$ and q is the best dominant of (3.2).

Proof: Using the differential equation (2.6) with $h(z) = \frac{1+z}{1-z}$, $n = 1$, $\delta = 0$, $\lambda = \frac{1}{1-\alpha}$, and in view of (2.6) and (2.7), we have $H(z) = \frac{z}{(1-z)^2}$ and $G(z) = q^{1-\alpha}(z) = \frac{1}{1-z}$. The function $\frac{1}{1-z}$ maps the unit disk onto $\left\{w \in \mathbb{C} : \operatorname{Re} w > \frac{1}{2}\right\}$ and for $\alpha \le \frac{1}{2}$ we have $q(z) = \left(\frac{1}{1-z}\right)^{\frac{1}{1-\alpha}}$ which is univalent in $\mathcal{U}$. Thus, by Theorem 2.2 the assertion follows. □

Theorem 3.2. *Let $\alpha \in [0,1)$ and let $\lambda, \delta, A \in \mathbb{C}$ with $\lambda \neq 0, \operatorname{Re}(\lambda + \delta) > 0$. Assume also*

$$\begin{cases} \operatorname{Re}[\lambda(1+AB) + \delta(1+B^2)] \ge |\lambda A + \bar{\lambda} B + 2B \operatorname{Re} \delta| & \text{for } B \in (-1, 0], \\ \operatorname{Re}[\lambda(1-A) + 2\delta] \ge 0 \ \text{ and } \ \lambda(1+A) > 0 & \text{for } B = -1. \end{cases} \tag{3.3}$$

If p is an analytic function in $\mathcal{U}$ with $p(0) = p^{1-\alpha}(0) = 1$ which satisfies

$$p^{1-\alpha}(z) + \frac{zp'(z)}{\delta p^{\alpha}(z) + \lambda p(z)} \prec \frac{1+Az}{1+Bz},$$

and q is a solution of the differential equation

$$q^{1-\alpha}(z) + \frac{zq'(z)}{\delta q^{\alpha}(z) + \lambda q(z)} = \frac{1+Az}{1+Bz},$$

then

$$p^{1-\alpha}(z) \prec q^{1-\alpha}(z) \prec \frac{1+Az}{1+Bz},$$

where $q^{1-\alpha}(z)$ is univalent, and q is of the form

$$q(z) = \left(\frac{1}{\lambda(1-\alpha)g_n(z)} - \frac{\delta}{\lambda}\right)^{\frac{1}{1-\alpha}} \tag{3.4}$$

where

$$g_n(z) = \begin{cases} {}_2F_1\left(\frac{\lambda(1-\alpha)}{n}\left(1 - \frac{A}{B}\right), 1, \frac{(1-\alpha)(\lambda+\delta)}{n} + 1; \frac{Bz}{1+Bz}\right) \frac{1}{(1-\alpha)(\delta+\lambda)} & \text{if } B \neq 0, \\ {}_1F_1\left(1, \frac{(1-\alpha)(\lambda+\delta)}{n} + 1; -\frac{(1-\alpha)\lambda Az}{n}\right) \frac{1}{(1-\alpha)(\delta+\lambda)} & \text{if } B = 0. \end{cases}$$

PROOF: Setting $h(z)=\frac{1+Az}{1+Bz}$ in (2.6), where $A\in\mathbb{C}$, $B\in[-1,0]$, $A\neq B$, and assuming that $\lambda,\delta\in\mathbb{C}$, $\lambda\neq 0$, $\operatorname{Re}(\delta+\lambda)>0$, we have

$$\lambda h(z)+\delta=\frac{\lambda+\delta+(\lambda A+\delta B)z}{1+Bz}\,.$$

The function $h(z)=\frac{1+Az}{1+Bz}$ is convex, hence, in view of (2.7), we obtain

$$H(z)=\begin{cases} z\exp\displaystyle\int_0^z\frac{\frac{1+At}{1+Bt}-1}{t}\,dt=z[1+Bz]^{\frac{A-B}{B}} & \text{for}\ \ B\neq 0,\\ z\exp(Az) & \text{for}\ \ B=0.\end{cases}\tag{3.5}$$

Hence

$$q^{1-\alpha}(z)=\begin{cases}\dfrac{z^{(1-\alpha)(\delta+\lambda)}[1+Bz]^{(1-\alpha)\lambda\frac{A-B}{B}}}{(1-\alpha)\lambda\int_0^z[1+Bt]^{(1-\alpha)\lambda\frac{A-B}{B}}t^{(1-\alpha)(\delta+\lambda)-1}\,dt}-\dfrac{\delta}{\lambda}, & B\neq 0,\\ \dfrac{z^{(1-\alpha)(\delta+\lambda)}\exp[(1-\alpha)\lambda Az]}{(1-\alpha)\lambda\int_0^z\exp[(1-\alpha)\lambda At]t^{(1-\alpha)(\delta+\lambda)-1}\,dt}-\dfrac{\delta}{\lambda}, & B=0,\end{cases}\tag{3.6}$$

and rewriting functions in terms of the confluent and Gaussian hypergeometric functions, the assertion follows. □

Remark 3.1. Setting $n=1$, $\delta=0$, $\lambda=\frac{1}{1-\alpha}$ and $A=-1$, $B=0$ in (3.5) and (3.6) we get

$$H(z)=ze^{-z}\quad\text{and}\quad q^{1-\alpha}(z)=\frac{z}{e^z-1}=:F(z),$$

where $q^{1-\alpha}$ is a solution of

$$q^{1-\alpha}(z)+\frac{(1-\alpha)zq'(z)}{q(z)}=1-z.\tag{3.7}$$

The function F is called the Bernoulli function and its Taylor series is

$$F(z)=1-\frac{z}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}z^{2k},$$

where B_{2k} are Bernoulli numbers. The properties of F are known, for instance Mocanu ([5]) proved that F is analytic for $|z|\leq 2\pi$, convex and univalent in $\mathcal{U}$.

Taking into account the above remark we may state the following:

Theorem 3.3. *Let* $\alpha \in [0,1)$. *If* q *is an analytic and univalent function in* $\mathcal{U}$ *such that* $q(0) = q^{1-\alpha}(0) = 1$ *and*

$$q^{1-\alpha}(z) + \frac{(1-\alpha)zq'(z)}{q(z)} = 1 - z,$$

then

$$q(z) = \left(\frac{z}{e^z - 1}\right)^{\frac{1}{1-\alpha}} = F^{\frac{1}{1-\alpha}}(z).$$

If moreover p *is an analytic function in* $\mathcal{U}$ *with* $p(0) = p^{1-\alpha}(0) = 1$ *and*

$$p^{1-\alpha}(z) + \frac{(1-\alpha)zp'(z)}{p(z)} \prec 1 - z$$

then $p \prec q$ *and* q *is the best dominant.*

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