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# Subgroups and products of $\mathbb{R}$-factorizable $P$-groups

Constancio Hernández, Michael Tkachenko

*Abstract.* We show that *every* subgroup of an $\mathbb{R}$-factorizable abelian $P$-group is topologically isomorphic to a *closed* subgroup of another $\mathbb{R}$-factorizable abelian $P$-group. This implies that closed subgroups of $\mathbb{R}$-factorizable $P$-groups are not necessarily $\mathbb{R}$-factorizable. We also prove that if a Hausdorff space $Y$ of countable pseudocharacter is a continuous image of a product $X = \prod_{i\in I} X_i$ of $P$-spaces and the space $X$ is pseudo-$\omega_1$-compact, then $nw(Y) \le \aleph_0$. In particular, direct products of $\mathbb{R}$-factorizable $P$-groups are $\mathbb{R}$-factorizable and $\omega$-stable.

*Keywords:* $P$-space, $P$-group, pseudo-$\omega_1$-compact, $\omega$-stable, $\mathbb{R}$-factorizable, $\aleph_0$-bounded, pseudocharacter, cellularity, $\aleph_0$-box topology, $\sigma$-product

*Classification:* Primary 54H11, 22A05, 54G10; Secondary 54A25, 54C10, 54C25

## 1. Introduction

The main subject of this article are $P$-*groups*, that is, topological groups in which all $G_\delta$-sets are open. It is known that $P$-groups are peculiar in many respects. For example, every $P$-group $G$ has a local base at the identity of open subgroups and if $G$ is $\aleph_0$-bounded, it has a local base at the identity of open normal subgroups [15, Lemma 2.1]. Weak compactness type conditions substantially improve the properties of $P$-groups. The following result proved in [15] demonstrates this phenomenon and will be frequently used in the article.

**Theorem 1.1** ([15, Theorem 4.16 and Corollary 4.14]). *For a $P$-group $G$, the following conditions are equivalent:*

(1) *$G$ is $\mathbb{R}$-factorizable;*
(2) *$G$ is pseudo-$\omega_1$-compact;*
(3) *$G$ is $\omega$-stable;*
(4) *$G$ is $\aleph_0$-bounded and every continuous homomorphic image $H$ of $G$ with $\psi(H) \le \aleph_1$ is Lindelöf.*

*In addition, every $\mathbb{R}$-factorizable $P$-group $G$ satisfies $c(G) \le \aleph_1$.*

All terms that appear in Theorem 1.1 are explained in the next subsection.

Subgroups of $\mathbb{R}$-factorizable $P$-groups need not be $\mathbb{R}$-factorizable (see [13, Example 2.1] or [15, Example 3.28]). It is an open problem whether every $\aleph_0$-bounded $P$-group is topologically isomorphic to a subgroup of an $\mathbb{R}$-factorizable

$P$-group (see Problem 4.1). We show, however, that *every* subgroup of an $\mathbb{R}$-factorizable abelian $P$-group can be embedded as a *closed* subgroup into another $\mathbb{R}$-factorizable abelian $P$-group (see Theorem 2.5). Hence closed subgroups of $\mathbb{R}$-factorizable $P$-groups can fail to be $\mathbb{R}$-factorizable. This is the main result of Section 2.

By [15, Theorem 5.5], direct products of $\mathbb{R}$-factorizable $P$-groups are $\mathbb{R}$-factorizable. In Theorem 3.7, we present a purely topological result about a special representation of continuous maps of products of $P$-spaces which generalizes Theorem 5.5 of [15]. It implies, in particular, that for any product of $P$-spaces, the properties of being $\omega$-stable and pseudo-$\omega_1$-compact are equivalent.

**1.1 Notation and terminology.** All spaces and topological groups are assumed to be Hausdorff unless a different axiom of separation is specified explicitly.

Let $\{X_i : i \in I\}$ be a family of topological spaces. A subset $B$ of the product $X = \prod_{i\in I} X_i$ is called a *box* in $X$ if it has the form $B = \prod_{i\in I} B_i$, where $B_i \subseteq X_i$ for each $i \in I$. Given a box $B \subseteq X$, we define the set $\operatorname{coord} B \subseteq I$ by

$$\operatorname{coord} B = \{i \in I : B_i \neq X_i\}.$$

The $\aleph_0$-*box topology* of the product $X$ is the topology generated by all boxes of the form $U = \prod_{i\in I} U_i$, where $|\operatorname{coord} U| \leq \aleph_0$ and each $U_i$ is open in $X_i$. Clearly, the Tychonoff topology of the space $X$ is generated by open boxes $U$ with $|\operatorname{coord} U| < \aleph_0$.

For every nonempty set $J \subseteq I$, we put $X_J = \prod_{i\in I} X_i$ and denote by $\pi_J$ the projection of $X$ onto $X_J$. Given a map $f\colon X \to Y$, we say that $f$ *depends only on a set* $J \subseteq I$ if $f(x) = f(y)$ for all $x, y \in X$ satisfying $\pi_J(x) = \pi_J(y)$.

Pick a point $a \in X$ and, for every $x \in X$, put

$$\operatorname{supp}(x) = \{i \in I : x_i \neq a_i\}.$$

Then the subset

$$\sigma(a) = \{x \in X : \operatorname{supp}(x) \text{ is finite}\}$$

of $X$ is called the $\sigma$-*product* of the family $\{X_i : i \in I\}$ with center at $a$.

Let $G = \prod_{i\in I} G_i$ be a direct product of groups. For every $x \in G$, we set $\operatorname{supp} x = \{i \in I : x_i \neq e_i\}$, where $e_i$ is the identity of $G_i$. Then the $\sigma$-product $\sigma(e) \subseteq G$ is a subgroup of $G$, where $e$ is the identity of $G$.

Suppose that $Y$ is a space. We say that $Y$ is a $P$-*space* if every countable intersection of open sets is open in $Y$. Let $\tau$ be an infinite cardinal. A subset $Z \subseteq Y$ is said to be $G_\tau$-*dense in* $Y$ if $Z$ intersects every nonempty $G_\tau$-set in $Y$.

A space $Y$ is called $\omega$-*stable* if every continuous image $Z$ of $Y$ which admits a coarser second countable Tychonoff topology satisfies $nw(Z) \leq \aleph_0$. In general, let $\tau \geq \aleph_0$. A space $Y$ is called $\tau$-*stable* if every continuous image $Z$ of $Y$ which admits a coarser Tychonoff topology of weight$\leq \tau$ satisfies $nw(Z) \leq \aleph_0$. If $Y$

is $\tau$-stable for $\tau \geq \aleph_0$, then $Y$ is said to be stable. It is known that arbitrary products and $\sigma$-products of second countable spaces are $\omega$-stable [1, Corollary 13].

A space $Y$ is said to be *pseudo-$\omega_1$-compact* if every locally finite (equivalently, discrete) family of open sets in $Y$ is countable. Lindelöf spaces as well as spaces of countable cellularity are pseudo-$\omega_1$-compact.

A topological group $G$ is called *$\aleph_0$-bounded* if it can be covered by countably many translates of any neighborhood of the identity. We also say that $G$ is *$\mathbb{R}$-factorizable* if every continuous real-valued function $f$ on $G$ can be represented in the form $f = h \circ \varphi$, where $\varphi\colon G \to H$ is a continuous homomorphism onto a second countable topological group $H$ and $h$ is a continuous real-valued function on $H$. Every $\mathbb{R}$-factorizable group is $\aleph_0$-bounded, but not vice versa [13], [14].

The kernel of a homomorphism $p\colon G \to H$ is $\ker p$. The minimal subgroup of a group $G$ containing a set $A \subseteq G$ is denoted by $\langle A \rangle$.

As usual, $w(Y)$, $nw(Y)$, $\psi(Y)$, $L(Y)$, and $c(Y)$ are the weight, network weight, pseudocharacter, Lindelöf number and cellularity of a space $Y$, respectively.

The set of all positive integers is denoted by $\mathbb{N}$, while $\mathbb{Z}$ is the additive group of integers.

## 2. Subgroups of $\mathbb{R}$-factorizable $P$-groups

Here we show that an arbitrary subgroup of an $\mathbb{R}$-factorizable abelian $P$-group is topologically isomorphic to a closed subgroup of another $\mathbb{R}$-factorizable abelian $P$-group. This result enables us to conclude that closed subgroups of $\mathbb{R}$-factorizable $P$-groups are not necessarily $\mathbb{R}$-factorizable. Since, by Theorem 1.1, $\mathbb{R}$-factorizability and pseudo-$\omega_1$-compactness coincide for $P$-groups, this makes $\mathbb{R}$-factorizable $P$-groups look like pseudocompact groups: every subgroup of a pseudocompact group is topologically isomorphic to a closed subgroup of another pseudocompact group [4]. This analogy between $\mathbb{R}$-factorizable $P$-groups and pseudocompact groups will be extended in Section 3.

We start with several auxiliary facts.

**Lemma 2.1.** *Suppose that $G$ is an $\mathbb{R}$-factorizable $P$-group, and let $H$ be a $G_{\omega_1}$-dense subgroup of $G$. Then $H$ is $\mathbb{R}$-factorizable.*

Proof: By Theorem 1.1, $G$ satisfies $c(G) \leq \aleph_1$. Therefore, the dense subgroup $H$ of $G$ also satisfies $c(H) \leq \aleph_1$. Let $f\colon H \to \mathbb{R}$ be a continuous function. By Schepin's theorem in [12], one can find a quotient homomorphism $\pi\colon H \to K$ onto a topological group $K$ with $\psi(K) \leq \aleph_1$ and a continuous function $g\colon K \to \mathbb{R}$ such that $f = g \circ \pi$. Observe that $H \subseteq G \subseteq \varrho G = \varrho H$, where $\varrho G$ and $\varrho H$ denote the Raĭkov completions of $G$ and $H$, respectively. Now, consider the continuous homomorphic extension $\hat{\pi}\colon \varrho H \to \varrho K$ of $\pi$, and take the restriction $\tilde{\pi} = \hat{\pi}\restriction_G\colon G \to \varrho K$ of $\hat{\pi}$ to $G$. Since $H$ is $G_{\omega_1}$-dense in $G$, the image $K = \tilde{\pi}(H)$ is $G_{\omega_1}$-dense in $\tilde{\pi}(G)$. We claim that $\tilde{\pi}(G) = K$.

Indeed, $\psi(K) \le \aleph_1$ implies that there exists a family $\{U_\alpha : \alpha < \omega_1\}$ of open sets in $\tilde{\pi}(G)$ such that $\{e\} = K \cap \bigcap_{\alpha\in\omega_1} U_\alpha$, where $e$ is the identity of $\varrho K$. If $P = \bigcap_{\alpha\in\omega_1} U_\alpha \setminus \{e\} \neq \emptyset$, then $P$ is a nonempty $G_{\omega_1}$-set in $\tilde{\pi}(G)$ that does not intersect $K$, which is a contradiction. Thus, $\psi(\tilde{\pi}(G)) \le \aleph_1$. Since every fiber of $\tilde{\pi}$ is a $G_{\omega_1}$-set in $G$, the group $H$ intersects all fibers of $\tilde{\pi}$. Hence we have $\tilde{\pi}(G) = \tilde{\pi}(H) = K$. So, $\tilde{f} = g \circ \tilde{\pi}$ is a continuous extension of $f$ to $G$. This implies that $H$ is $C$-embedded in $G$ and, hence, $H$ is $\mathbb{R}$-factorizable by [7, Theorem 2.4]. □

Pseudo-$\omega_1$-compactness is not a productive property, not even in the class of $P$-spaces (one can modify Novak's construction in [11] to produce a counterexample). The following lemma shows the difference between $P$-spaces and $P$-groups.

**Lemma 2.2.** *A finite product of $\mathbb{R}$-factorizable $P$-groups is pseudo-$\omega_1$-compact (equivalently, $\mathbb{R}$-factorizable).*

Proof: Let $G = G_1 \times \cdots \times G_n$, where each $G_i$ is an $\mathbb{R}$-factorizable $P$-group. Then $G$ is also a $P$-group. Hence we can assume that $n = 2$. Note that the factors $G_1$ and $G_2$ are $\aleph_0$-bounded, and so is the product group $G$. So, by Theorem 1.1, it suffices to verify that every continuous homomorphic image $H$ of $G$ with $\psi(H) \le \aleph_1$ is Lindelöf. Let $p: G \to H$ be a corresponding homomorphism. Then one can apply [14, Lemma 3.7] to find, for every $i = 1, 2$, a continuous homomorphism $f_i: G_i \to K_i$ onto a topological group $K_i$ with $\psi(K_i) \le \aleph_1$ such that $\ker f_1 \times \ker f_2 \subseteq \ker p$. Refining topologies of the groups $K_i$, we can assume that the homomorphisms $f_1$ and $f_2$ are open. Then $K_1$ and $K_2$ are $P$-groups by [15, Lemma 2.1] and the product homomorphism $f = f_1 \times f_2$ of $G$ onto $K = K_1 \times K_2$ is open. From our choice of the homomorphisms $f_1$ and $f_2$ it follows that there exists a homomorphism $\varphi: K \to H$ such that $p = \varphi \circ f$. Since $f$ is open, the homomorphism $\varphi$ is continuous. By Theorem 1.1, the $P$-groups $K_1$ and $K_2$ are Lindelöf, and so is the product group $K$ by Noble's theorem in [10]. Hence the group $H = \varphi(K)$ is Lindelöf as well. This finishes the proof. □

The next result has several applications in this section and in Section 3.

**Lemma 2.3.** *The following conditions are equivalent for a product space $X = \prod_{i\in I} X_i$:*

(a) *$X$ is pseudo-$\omega_1$-compact;*
(b) *the product $X_J = \prod_{i\in J} X_i$ is pseudo-$\omega_1$-compact for each finite set $J \subseteq I$;*
(c) *every $\sigma$-product $\sigma(a) \subseteq X$ is pseudo-$\omega_1$-compact;*
(d) *every $\sigma$-product $\sigma(a) \subseteq X$ endowed with the relative $\aleph_0$-box topology is pseudo-$\omega_1$-compact.*

Proof: It clear that (a) ⇒ (b). Since, for each $a \in X$, $\sigma(a)$ is dense in $X$ when $X$ carries the usual product topology and the $\aleph_0$-box topology is finer than the product topology of $X$, we have that (c)$\Rightarrow$(a) and (d)$\Rightarrow$(c)$\Rightarrow$(b). Therefore, it suffices to show that (b)$\Rightarrow$(d).

Let $\{U_\alpha : \alpha < \omega_1\}$ be a collection of nonempty open sets in $\sigma(a)$. We shall show that this family cannot be discrete. Without loss of generality, we may assume that $U_\alpha = \sigma \cap V_\alpha$ for each $\alpha < \omega_1$, where $V_\alpha$ has the form $\prod_{i\in I} V_{\alpha,i}$, the sets $V_{\alpha,i}$ are open in $X_i$ and $\operatorname{coord} V_\alpha \leq \aleph_0$. Take a point $x_\alpha \in U_\alpha$. Since $x_\alpha \in \sigma(a)$, the point $a(i) \in X_i$ is an element of $V_{\alpha,i}$ for all $i \in I \setminus J_\alpha$, where $J_\alpha = \operatorname{supp}(x_\alpha)$ is a finite subset of $I$. Now we apply the $\Delta$-lemma in order to find a subset $A$ of $\omega_1$ of cardinality $\aleph_1$ and a finite set $J \subseteq I$ such that $J_\alpha \cap J_\beta = J$ whenever $\alpha, \beta \in A$ and $J_\alpha \neq J_\beta$. Since the space $X_J = \prod_{i\in J} X_i$ is pseudo-$\omega_1$-compact, there exists a point $y \in X_J$ such that every neighborhood of $y$ intersects infinitely many elements of the family $\{\prod_{i\in J} V_{\alpha,i} : \alpha \in A\}$. Define a point $x \in \sigma(a)$ by

$$x(i) = \begin{cases} y(i) & \text{if } i \in J; \\ a(i) & \text{if } i \in I \setminus J. \end{cases}$$

It is easy to see that $\pi_J(x) = y$ and every neighborhood of $x$ intersects an infinite number of elements of $\{U_\alpha : \alpha \in A\}$. Hence the space $\sigma(a)$ is pseudo-$\omega_1$-compact. □

The equivalence of (a) and (b) in the above lemma should be a known result, but the authors have not found a corresponding reference in the literature.

**Corollary 2.4.** *Let $\Pi = \prod_{i\in I} G_i$ be a direct product of $\mathbb{R}$-factorizable $P$-groups. Then $\sigma(e) \subseteq \Pi$, endowed with the relative $\aleph_0$-box topology, is an $\mathbb{R}$-factorizable $P$-group.*

Proof: It is clear that $\sigma(e)$ is a $P$-group. Therefore, $\sigma(e)$ is $\mathbb{R}$-factorizable by Theorem 1.1, Lemma 2.2 and Lemma 2.3. □

We now have all necessary tools to deduce the main result of this section about closed embeddings into $\mathbb{R}$-factorizable $P$-groups.

**Theorem 2.5.** *Suppose that $G$ is an $\mathbb{R}$-factorizable abelian $P$-group. If $H$ is an arbitrary subgroup of $G$, then $H$ can be embedded as a closed subgroup into another $\mathbb{R}$-factorizable abelian $P$-group.*

Proof: Let $\mathbb{Z}$ be the discrete group of integers. Clearly, $G \times \mathbb{Z}$ is an $\mathbb{R}$-factorizable abelian $P$-group that contains an isomorphic copy of $G$. Replacing $G$ by $G \times \mathbb{Z}$, if necessary, we may assume that $G$ contains an element $g$ of infinite order, $g \neq 0_G$.

Let $\lambda = |G| \cdot \aleph_2$ and put $\kappa = \lambda$ if $\lambda$ is a regular cardinal or $\kappa = \lambda^+$, otherwise. Consider the group

$$\sigma = \{x \in G^\kappa : |\operatorname{supp} x| < \aleph_0\}$$

endowed with the relative $\aleph_0$-box topology inherited from $G^\kappa$. Then $\sigma$ is an $\mathbb{R}$-factorizable abelian $P$-group by Corollary 2.4 and, clearly, $|\sigma| = \kappa$. Let $\sigma \setminus \{0_\sigma\} =$

$\{x_\alpha : \alpha < \kappa\}$. To every element $x_\alpha$, we assign an element $\tilde{x}_\alpha \in \sigma$ recursively as follows. Choose $\delta_0 > \max \operatorname{supp} x_0$ and define $\tilde{x}_0 \in \sigma$ by

$$\tilde{x}_0(\nu) = \begin{cases} x_0(\nu) & \text{if } \nu \neq \delta_0; \\ g & \text{if } \nu = \delta_0. \end{cases}$$

Suppose that we have already defined $\tilde{x}_\beta$ for each $\beta < \alpha$, where $\alpha < \kappa$. Choose $\delta_\alpha > \sup(\operatorname{supp} x_\alpha \cup \bigcup_{\beta<\alpha} \operatorname{supp} \tilde{x}_\beta)$ and define a point $\tilde{x}_\alpha \in \sigma$ by

$$\tilde{x}_\alpha(\nu) = \begin{cases} x_\alpha(\nu) & \text{if } \nu \neq \delta_\alpha; \\ g & \text{if } \nu = \delta_\alpha. \end{cases}$$

It is clear that $\delta_\alpha = \max \operatorname{supp} \tilde{x}_\alpha$. This finishes our construction.

Observe that the sequence $\{\delta_\alpha : \alpha < \kappa\}$ is strictly increasing (hence it is cofinal in $\kappa$) and $\tilde{x}_\beta(\delta_\alpha) = 0_G$ whenever $\beta < \alpha < \kappa$. Consider the subgroup $G_0 = \langle H_0 \cup B \rangle$ of $\sigma$, where

$$H_0 = \{x \in \sigma : x(0) \in H \text{ and } x(\nu) = 0_G \text{ for each } \nu \neq 0\}$$

and $B = \{\tilde{x}_\alpha : \alpha < \kappa\}$. We claim that the group $G_0$ is $\mathbb{R}$-factorizable and contains $H_0 \simeq H$ as a closed subgroup. It is easy to see that $H_0$ is closed in $G_0$ because it can be expressed as the intersection of the coordinate 0 axes with $G_0$. Indeed, suppose that $x \in G_0$ and $x(\nu) = 0_G$ for all $\nu > 0$. By the definition of $G_0$, $x$ has the form $x = h + k_1\tilde{x}_{\alpha_1} + \cdots + k_n\tilde{x}_{\alpha_n}$, where $h \in H_0$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$ and $k_i \in \mathbb{Z}$ for $i = 1, \dots, n$. Then $\tilde{x}_{\alpha_i}(\delta_{\alpha_n}) = 0_G$ for each $i < n$ and $\tilde{x}_{\alpha_n}(\delta_{\alpha_n}) = g$. Hence $k_n = 0$. If we proceed in the same way for $i = n-1, \dots, 1$, we obtain $k_n = \cdots = k_1 = 0$, whence $x = h$, with $h \in H_0$.

By Lemma 2.1, to prove that $G_0$ is $\mathbb{R}$-factorizable, it suffices to verify that $G_0$ is $G_{\omega_1}$-dense in $\sigma$. To this end, it is enough to show that if $x \in \sigma$, $C \subseteq \kappa$ and $|C| \leq \aleph_1$, then there exists $\alpha < \kappa$ such that $\tilde{x}_\alpha(\nu) = x(\nu)$ for each $\nu \in C$. Suppose that $x \in \sigma$ and choose $\beta < \kappa$ such that $\delta_\beta > \sup C$. Then choose $\alpha < \kappa$ such that $\beta \leq \alpha$ and $x_\alpha(\nu) = x(\nu)$ for each $\nu < \delta_\beta$. Then $\tilde{x}_\alpha(\nu) = x(\nu)$ for each $\nu \in C$. This implies that the group $G_0$ is $G_{\omega_1}$-dense in $\sigma$ and, therefore, $\mathbb{R}$-factorizable. □

**Corollary 2.6.** *Closed subgroups of $\mathbb{R}$-factorizable $P$-groups need not be $\mathbb{R}$-factorizable.*

Proof: According to [13, Example 3.1], there exist an $\mathbb{R}$-factorizable abelian $P$-group $G$ and a dense subgroup $H$ of $G$ such that $H$ is not $\mathbb{R}$-factorizable. By Theorem 2.5, $H$ is topologically isomorphic to a closed subgroup of another $\mathbb{R}$-factorizable $P$-group, so that closed subgroups of $\mathbb{R}$-factorizable $P$-groups are not necessarily $\mathbb{R}$-factorizable. □

It is known that all subgroups of compact groups as well as all subgroups of $\sigma$-compact groups are $\mathbb{R}$-factorizable [13], [14]. In the following definition, we introduce the class of groups with this property.

**Definition 2.7.** A topological group $G$ is called *hereditarily* $\mathbb{R}$*-factorizable* if all subgroups of $G$ are $\mathbb{R}$-factorizable.

**Theorem 2.8.** *Every hereditarily* $\mathbb{R}$*-factorizable* $P$*-group is countable and, therefore, discrete.*

PROOF: Suppose to the contrary that $G$ is an uncountable hereditarily $\mathbb{R}$-factorizable $P$-group and take a subset $A$ of $G$ of cardinality $\aleph_1$. It is clear that the $P$-group $H = \langle A \rangle$ has cardinality $\aleph_1$. Since $H$ is $\mathbb{R}$-factorizable and $L(H) \leq \aleph_1$, from [15, Corollary 3.34] it follows that $H$ is a Lindelöf group. In its turn, this implies that $w(H) \leq \aleph_1$ (see [15, Corollary 4.11]). If $w(H) = \aleph_1$, then by [7, Theorem 3.1], $H$ has a subgroup which fails to be $\mathbb{R}$-factorizable, thus contradicting the hereditary $\mathbb{R}$-factorizability of $G$. Hence, $w(H) = \aleph_0$. Since $H$ is a $P$-space, it is discrete and, consequently, $|H| = w(H) = \aleph_0$. This contradiction completes the proof. □

One can reformulate Theorem 2.8 by saying that every uncountable $P$-group $G$ contains a subgroup of size $\aleph_1$ which fails to be $\mathbb{R}$-factorizable. Indeed, if $G$ is $\mathbb{R}$-factorizable, this immediately follows from the above argument. Otherwise, by Theorem 1.1, $G$ contains a discrete family $\{U_\alpha : \alpha < \omega_1\}$ of nonempty open sets. Choose a subgroup $H$ of $G$ of size $\aleph_1$ such that $V_\alpha = H \cap U_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. Then the family $\{V_\alpha : \alpha < \omega_1\}$ of nonempty open sets is discrete in $H$, so that the group $H$ is not $\mathbb{R}$-factorizable by Theorem 1.1.

## 3. Continuous images

By [15, Theorem 5.5], an arbitrary direct product $G$ of $\mathbb{R}$-factorizable $P$-groups is $\mathbb{R}$-factorizable. Here we strengthen this result and show that every continuous map $f: G \to X$ to a Hausdorff space $X$ of countable pseudocharacter can be factored via a quotient homomorphism $\pi: G \to K$ onto a second countable topological group $K$. In fact, this follows from an even stronger result (see Theorem 3.7): if a Hausdorff space $Y$ of countable pseudocharacter is a continuous image of a product $X$ of $P$-spaces and $X$ is pseudo-$\omega_1$-compact, then $nw(Y) \leq \aleph_0$. In particular, the space $X$ is $\omega$-stable. We precede this result by a series of lemmas. The first of them is an analogue of Noble's theorem on $z$-closed projections [9], [10].

**Lemma 3.1.** *The Cartesian product* $X \times Y$ *of regular* $P$*-spaces* $X$ *and* $Y$ *is pseudo-*$\omega_1$*-compact if and only if* $X$ *and* $Y$ *are pseudo-*$\omega_1$*-compact and the projection* $p: X \times Y \to X$ *transforms clopen subsets of* $X \times Y$ *to clopen subsets of* $X$.

PROOF: Suppose that $X \times Y$ is pseudo-$\omega_1$-compact and let $W \subseteq X \times Y$ be a clopen set. If there exists a point $x_0 \in \overline{p(W)} \setminus p(W)$, take any point $y_0 \in Y$ and a neighborhood $W_0' = U_0' \times V_0$ of $(x_0, y_0)$, where $U_0'$ and $V_0$ are clopen sets, such that $W_0' \cap W = \emptyset$. Pick a point $(x_1, y_1) \in W$ with $x_1 \in U_0'$. Now we take neighborhoods $W_1 = U_1 \times V_1$ and $W_1' = U_1' \times V_1$ of $(x_1, y_1)$ and $(x_0, y_1)$, respectively, where $U_1$,

$U_1'$ and $V_1$ are clopen sets such that $W_1' \cap W = \emptyset$, $W_1 \subseteq W$ and $U_1 \cup U_1' \subseteq U_0'$. Suppose that for some $\alpha < \omega_1$, we have already chosen points $(x_\beta, y_\beta) \in W$ as well as clopen sets $W_\beta$ and $W_\beta'$ for each $\beta < \alpha$, such that $W_\beta = U_\beta \times V_\beta$ is a neighborhood of $(x_\beta, y_\beta)$ satisfying $W_\beta \subseteq W$ and $W_\beta' = U_\beta' \times V_\beta$ is a neighborhood of $(x_0, y_\beta)$ with $W_\beta' \cap W = \emptyset$, and where $U_\beta \cup U_\beta' \subseteq U_\gamma'$ if $\gamma < \beta < \alpha$. Choose $(x_\alpha, y_\alpha) \in W$ in such a way that $x_\alpha \in \bigcap_{\beta<\alpha} U_\beta'$. Then we can take neighborhoods $W_\alpha = U_\alpha \times V_\alpha$ and $W_\alpha' = U_\alpha' \times V_\alpha$ of $(x_\alpha, y_\alpha)$ and $(x_0, y_\alpha)$, respectively, such that $W_\alpha' \cap W = \emptyset$ and $W_\alpha \subseteq W$, and where $U_\alpha \cup U_\alpha' \subseteq \bigcap_{\beta<\alpha} U_\beta'$. This finishes our recursive construction.

Since $X \times Y$ is pseudo-$\omega_1$-compact, the family $\mathcal{F} = \{W_\alpha : \alpha < \omega_1\}$ has an accumulation point $(x, y) \in W$. We claim that $(x, y)$ is an accumulation point of the family $\mathcal{F}' = \{W_\alpha' : \alpha < \omega_1\}$. Indeed, let $\alpha_0 < \omega_1$ be arbitrary. Since $U_\alpha \cup U_\alpha' \subseteq U_\beta$ if $\beta < \alpha < \omega_1$ and each $U_\alpha'$ is clopen, we have $x \in \bigcap_{\alpha<\omega_1} U_\alpha'$. Let $U \times V$ be a neighborhood of $(x, y)$ in $X \times Y$. Since $y$ is an accumulation point of the family $\{V_\alpha : \alpha < \omega_1\}$, there exists $\alpha > \alpha_0$ such that $V \cap V_\alpha \neq \emptyset$. Clearly, $x \in U \cap U_\alpha'$, so that $(U \times V) \cap (U_\alpha' \times V_\alpha) \neq \emptyset$. Our claim is proved.

Thus, $(x, y) \in \overline{\bigcup \mathcal{F}} \cap \overline{\bigcup \mathcal{F}'} \neq \emptyset$. However, $\bigcup \mathcal{F} \subseteq W$ and $\bigcup \mathcal{F}' \subseteq (X \times Y) \setminus W = W'$, whence $\overline{\bigcup \mathcal{F}} \cap \overline{\bigcup \mathcal{F}'} \subseteq W \cap W' = \emptyset$. This contradiction shows that the set $p(W)$ is clopen in $X$.

Conversely, suppose that both spaces $X$ and $Y$ are pseudo-$\omega_1$-compact and $p: X \times Y \to X$ transforms clopen subsets of $X \times Y$ to clopen subsets of $X$. Suppose to the contrary that $X \times Y$ contains a discrete family $\{O_\alpha : \alpha < \omega_1\}$ of nonempty clopen sets. For every $\alpha < \omega_1$, put $W_\alpha = \bigcup_{\beta \geq \alpha} O_\beta$. Then we have a decreasing sequence $W_0 \supseteq W_1 \supseteq \cdots \supseteq W_\alpha \supseteq \ldots$, $\alpha < \omega_1$, of nonempty clopen subsets of $X \times Y$ with empty intersection. Each set $U_\alpha = p(W_\alpha)$ is clopen in $X$ and, since $X$ is pseudo-$\omega_1$-compact, the set $\bigcap_{\alpha<\omega_1} U_\alpha$ is nonempty. Let $x_0$ be an element of $\bigcap_{\alpha<\omega_1} U_\alpha$. The sets $V_\alpha = (\{x_0\} \times Y) \cap W_\alpha$ are clopen in the pseudo-$\omega_1$-compact space $\{x_0\} \times Y$. Hence $\bigcap_{\alpha<\omega_1} V_\alpha \subseteq \bigcap_{\alpha<\omega_1} W_\alpha$ is nonempty. This contradiction proves the lemma. □

**Lemma 3.2.** *Suppose that the product $X \times Y$ of $P$-spaces $X$ and $Y$ is pseudo-$\omega_1$-compact. If $W$ is a clopen set in $X \times Y$, then for every $x_0 \in p(W)$, there exists a clopen neighborhood $U$ of $x_0$ in $X$ such that $U \times V_{x_0} \subseteq W$, where $V_{x_0} = \{y \in Y : (x_0, y) \in W\}$.*

Proof: Set $O = (X \times V_{x_0}) \setminus W$. Since $V_{x_0}$ is clopen in $Y$, the set $O$ is clopen in $X \times Y$. From Lemma 3.1 it follows that $p(O)$ and $U = X \setminus p(O)$ are clopen sets in $X$, where $p: X \times Y \to X$ is the projection. Note that $x_0 \in U$ and if $(x, y) \in U \times V_{x_0}$, then $x \notin p(O)$. So, $(x, y) \in W$ and, hence, $U \times V_{x_0} \subseteq W$. □

The next result can be obtained by combining [8, Theorem 1.6] and the characterization of the so-called *approximation property* for products of two spaces given in [2]. We prefer, however, to supply the reader with a direct proof.

**Lemma 3.3.** *Suppose that the product $X = \prod_{i=1}^{k} X_i$ of $P$-spaces is pseudo-$\omega_1$-compact. If $W$ is a clopen set in $X$, then $W = \bigcup_{n\in\omega} \prod_{i=1}^{k} U_{n,i}$, where the sets $U_{n,i}$ are clopen in $X_i$ for all $n \in \omega$ and $i \leq k$.*

Proof: By Lemma 3.1, it suffices to consider the case $n = 2$. Let $W$ be a clopen subset of $X_1 \times X_2$. Then $W' = X \setminus W$ is clopen as well. For every $x \in X_1$, put

$$V_x = \{y \in X_2 : (x, y) \in W\} \ \text{ and } \ V'_x = \{y \in X_2 : (x, y) \in W'\}.$$

Then both sets $V_x$ and $V'_x$ are clopen in $X_2$ and $V'_x = X_2 \setminus V_x$. Consider the equivalence relation $\sim$ on $X_1$ defined by $x \sim y$ if and only if $V_x = V_y$. We claim that for every $x \in X_1$, the equivalence class $[x]$ of $x$ is open in $X_1$. Indeed, if $y \in [x]$, then $V_y = V_x = V$. Apply Lemma 3.2 to choose a clopen neighborhood $U$ of $y$ in $X_1$ such that $U \times V \subseteq W$ and $U \times V' \subseteq W'$, where $V' = X_2 \setminus V$. Then $V_z = V$ for each $z \in U$, so that $y \in U \subseteq [x]$. This proves that the set $[x]$ is open.

Since the space $X_1$ is pseudo-$\omega_1$-compact and the equivalence classes $[x]$ with $x \in X_1$ form a disjoint open cover of $X_1$, there exists a countable set $\{x_n : n \in \omega\} \subseteq X_1$ such that $X_1 = \bigcup_{n\in\omega} [x_n]$. It is clear that every set $U_{n,1} = [x_n]$ is clopen in $X_1$. Therefore, $W = \bigcup_{n\in\omega} U_{n,1} \times U_{n,2}$ is the required representation of $W$, where $U_{n,2} = V_{x_n}$ for each $n \in \omega$. □

It is well known (see [6]) that if a product space $X = \prod_{i\in I} X_i$ has countable cellularity, then every regular closed set in $X$ depends on at most countably many coordinates. In a sense, our next result is an analogue of this fact in the case when the product space $X$ is pseudo-$\omega_1$-compact and the factors $X_i$ are $P$-spaces.

**Lemma 3.4.** *Suppose that a product $X = \prod_{i\in I} X_i$ of $P$-spaces is pseudo-$\omega_1$-compact. Let $\sigma(a) \subseteq X$ be a $\sigma$-product endowed with the relative $\aleph_0$-box topology (finer than the usual subspace topology). Then every clopen subset of $\sigma(a)$ depends on at most countably many coordinates.*

Proof: It is clear that the space $\sigma(a)$ with the $\aleph_0$-box topology is a $P$-space. Let $U$ be a clopen subset of $\sigma(a)$. Then $V = \sigma(a) \setminus U$ is also clopen in $\sigma(a)$. Suppose that $\pi_J(U) \cap \pi_J(V) \neq \emptyset$ for every countable set $J \subseteq I$. Let us call a set $A \subseteq \sigma(a)$ *canonical* if $A$ has the form $\sigma(a) \cap P$, where $P$ is an $\aleph_0$-box in $X$. First, we prove the following auxiliary fact.

**Claim.** *Let $A \subseteq U$ and $B \subseteq V$ be canonical open sets in $\sigma(a)$ such that $U' = U \setminus \overline{A} \neq \emptyset$ and $V' = V \setminus \overline{B} \neq \emptyset$. Then $\pi_J(U') \cap \pi_J(V') \neq \emptyset$ for each countable set $J \subseteq I$.*

Indeed, there exists a nonempty countable set $C \subseteq I$ such that $A = \sigma(a) \cap \pi_C^{-1}\pi_C(A)$ and $B = \sigma(a) \cap \pi_C^{-1}\pi_C(B)$. Let $J$ be a countable subset of $I$. We can assume that $C \subseteq J$. Since $A \cap V = \emptyset = B \cap U$, we infer that

$$\pi_J(A) \cap \pi_J(V) = \emptyset \ \text{ and } \ \pi_J(B) \cap \pi_J(U) = \emptyset. \tag{1}$$

Note that the set $U' \cup A$ is dense in $U$ and $V' \cup B$ is dense in $V$. Since the restriction of $\pi_J$ to $\sigma(a)$ is an open map, from $\pi_J(U) \cap \pi_J(V) \neq \emptyset$ it follows that

$$\pi_J(U' \cup A) \cap \pi_J(V' \cup B) \neq \emptyset. \tag{2}$$

Note that $U' \subseteq U$ and $V' \subseteq V$, so (1) implies that $\pi_J(U') \cap \pi_J(B) = \emptyset$, $\pi_J(V') \cap \pi_J(A) = \emptyset$ and $\pi_J(A) \cap \pi_J(B) = \emptyset$. Therefore, from (2) it follows that $\pi_J(U') \cap \pi_J(V') \neq \emptyset$. This proves our claim.

We will construct by recursion three sequences $\{I_\alpha : \alpha < \omega_1\}$, $\{U_\alpha : \alpha < \omega_1\}$ and $\{V_\alpha : \alpha < \omega_1\}$ satisfying the following conditions for all $\beta, \gamma < \omega_1$:

(i) $I_\beta \subseteq I$, $|I_\beta| \leq \aleph_0$;
(ii) $I_\gamma \subseteq I_\beta$ if $\gamma < \beta$;
(iii) $U_\beta$ and $V_\beta$ are nonempty canonical clopen sets in $\sigma(a)$;
(iv) $U_\beta \subseteq U$, $V_\beta \subseteq V$ and $\pi_{I_\beta}(U_\beta) = \pi_{I_\beta}(V_\beta)$;
(v) $U_\gamma = \sigma(a) \cap \pi_{I_\beta}^{-1}\pi_{I_\beta}(U_\gamma)$ and $V_\gamma = \sigma(a) \cap \pi_{I_\beta}^{-1}\pi_{I_\beta}(V_\gamma)$ if $\gamma < \beta$;
(vi) $U_\gamma \cap U_\beta = \emptyset$ and $V_\gamma \cap V_\beta = \emptyset$ if $\gamma < \beta$.

To start, take a nonempty countable set $I_0 \subseteq I$ and choose canonical clopen sets $U_0$ and $V_0$ in $\sigma(a)$ such that $U_0 \subseteq U$, $V_0 \subseteq V$ and $\pi_{I_0}(U_0) \cap \pi_{I_0}(V_0) \neq \emptyset$. Taking smaller clopen sets, one can assume that $\pi_{I_0}(U_0) = \pi_{I_0}(V_0)$.

Suppose that at some stage $\alpha < \omega_1$, we have defined sequences $\{I_\beta : \beta < \alpha\}$, $\{U_\beta : \beta < \alpha\}$ and $\{V_\beta : \beta < \alpha\}$ satisfying conditions (i)–(vi). Since each $I_\beta$ is countable and the sets $U_\beta$, $V_\beta$ depend on countably many coordinates, there exists a countable set $I_\alpha \subseteq I$ such that $I_\beta \subseteq I_\alpha$, $U_\beta = \sigma(a) \cap \pi_{I_\alpha}^{-1}\pi_{I_\alpha}(U_\beta)$ and $V_\beta = \sigma(a) \cap \pi_{I_\alpha}^{-1}\pi_{I_\alpha}(V_\beta)$ for each $\beta < \alpha$. Let $U'_\alpha = U \setminus \overline{G}_\alpha$ and $V'_\alpha = V \setminus \overline{H}_\alpha$, where $G_\alpha = \bigcup_{\beta < \alpha} U_\beta$ and $H_\alpha = \bigcup_{\beta < \alpha} V_\beta$. Apply the above Claim to choose nonempty canonical clopen sets $U_\alpha \subseteq U'_\alpha$ and $V_\alpha \subseteq V'_\alpha$ such that $\pi_{I_\alpha}(U_\alpha) = \pi_{I_\alpha}(V_\alpha)$. An easy verification shows that the sequences $\{I_\beta : \beta \leq \alpha\}$, $\{U_\beta : \beta \leq \alpha\}$ and $\{V_\beta : \beta \leq \alpha\}$ satisfy conditions (i)–(vi) for all $\beta, \gamma \leq \alpha$, thus finishing our recursive construction.

Let $K = \bigcup_{\alpha < \omega_1} I_\alpha$. By (iv), the set $G = \bigcup_{\alpha < \omega_1} U_\alpha$ is contained in $U$ and $H = \bigcup_{\alpha < \omega_1} V_\alpha$ is contained in $V$, so that $\overline{G} \cap \overline{H} = \emptyset$. To obtain a contradiction, it suffices to show that the sets $G$ and $H$ have a common cluster point in $\sigma(a)$. From (v), (ii) and our definition of the sets $G$ and $H$ it follows that $G = \sigma(a) \cap \pi_K^{-1}\pi_K(G)$ and $H = \sigma(a) \cap \pi_K^{-1}\pi_K(H)$, so we can assume without loss of generality that $K = I$.

By Lemma 2.3, the $P$-space $\sigma(a)$ is pseudo-$\omega_1$-compact. Hence the family $\gamma = \{U_\alpha : \alpha < \omega_1\}$ has an accumulation point $x \in \sigma(a)$ and every neighborhood of $x$ in $\sigma(a)$ intersects uncountably many elements of $\gamma$. Let $O$ be a canonical open neighborhood of $x$ in $X$ and let $C = \operatorname{coord} O$. Since $|C| \leq \aleph_0$, (ii) implies that there exists $\beta < \omega_1$ such that $C \subseteq I_\beta$. There are uncountably many ordinals $\alpha < \omega_1$ such that $\beta \leq \alpha$ and $O \cap U_\alpha \neq \emptyset$. For every such an $\alpha < \omega_1$, let $z_\alpha$

be an arbitrary point of the set $\pi_{I_\alpha}(O \cap U_\alpha) \subseteq \pi_{I_\alpha}(O) \cap \pi_{I_\alpha}(U_\alpha)$. From (iv) it follows that $\pi_{I_\alpha}(U_\alpha) = \pi_{I_\alpha}(V_\alpha)$, so $z_\alpha \in \pi_{I_\alpha}(O) \cap \pi_{I_\alpha}(V_\alpha)$. Choose a point $z \in V_\alpha$ such that $\pi_{I_\alpha}(z) = z_\alpha$. Since $\operatorname{coord} O = C \subseteq I_\beta \subseteq I_\alpha$, we conclude that $z \in O \cap V_\alpha \neq \emptyset$. This immediately implies that $x$ is an accumulation point of the family $\{V_\alpha : \alpha < \omega_1\}$ and, hence, $x \in \overline{H}$. Thus, $x \in \overline{G} \cap \overline{H} \neq \emptyset$, which is a contradiction.

We have thus proved that $\pi_J(U) \cap \pi_J(V) = \emptyset$ for some nonempty countable subset $J$ of $I$, whence it follows that $U = \sigma(a) \cap \pi_J^{-1}\pi_J(U)$. In other words, $U$ depends only on the set $J$. □

A simple modification of the argument in the proof of Lemma 3.4 (combined with the $\Delta$-lemma) implies the following corollary.

**Corollary 3.5.** *Let $\{X_i : i \in I\}$ be a family of $P$-spaces such that the product $X = \prod_{i\in I} X_i$ is pseudo-$\omega_1$-compact. If $U$ and $V$ are open sets in $X$ and $\overline{U} \cap \overline{V} = \emptyset$, then there exists a nonempty countable set $J \subseteq I$ such that $\pi_J(U) \cap \pi_J(V) = \emptyset$.*

It is not clear whether one can find a countable set $J \subseteq I$ in Corollary 3.5 satisfying $\overline{\pi_J(U)} \cap \overline{\pi_J(V)} = \emptyset$.

**Lemma 3.6.** *Let $X = \prod_{i\in I}$ be a product space and $\sigma(a) \subseteq X$ be the $\sigma$-product with center at $a \in X$. Suppose that $\emptyset \neq J \subseteq I$ and that a continuous map $f: X \to Y$ to a Hausdorff space $Y$ satisfies $f(x) = f(y)$ whenever $x, y \in \sigma(a)$ and $\pi_J(x) = \pi_J(y)$. Then $f$ depends only on $J$.*

Proof: Let $x, y \in X$ satisfy $\pi_J(x) = \pi_J(y)$. Suppose to the contrary that $f(x) \neq f(y)$ and choose in $X$ disjoint open neighborhoods $U$ and $V$ of $x$ and $y$, respectively, such that $f(U) \cap f(V) = \emptyset$. We can assume without loss of generality that the sets $U$ and $V$ are canonical and $\operatorname{coord} U = C = \operatorname{coord} V$. Let us define two points $x^*, y^* \in X$ by

$$x^*(i) = \begin{cases} x(i) & \text{if } i \in C; \\ x^*(i) = a(i) & \text{if } i \in I \setminus C \end{cases}$$

and, similarly,

$$y^*(i) = \begin{cases} y(i) & \text{if } i \in C; \\ y^*(i) = a(i) & \text{if } i \in I \setminus C. \end{cases}$$

Then $x^*, y^* \in \sigma(a)$ and $\pi_J(x^*) = \pi_J(y^*)$, so that $f(x^*) = f(y^*)$. On the other hand, we have $x^* \in U$ and $y^* \in V$, whence $f(x^*) \in f(U)$ and $f(y^*) \in f(V)$. Since $f(U) \cap f(V) = \emptyset$, this implies that $f(x^*) \neq f(y^*)$, which is a contradiction. □

Let $f: X \to Y$ and $g: X \to Z$ be continuous maps, where $Y = f(X)$. We say that $f$ *is finer than* $g$ or, in symbols, $f \prec g$ if there exists a continuous map $\varphi: Y \to Z$ such that $g = \varphi \circ f$. The theorem below is the main result of this section.

**Theorem 3.7.** *Let $X = \prod_{i\in I} X_i$ be a product of $P$-spaces and $f\colon X \to Y$ be a continuous map onto a space $Y$ of countable pseudocharacter. If $X$ is pseudo-$\omega_1$-compact, then $f$ depends on at most countably many coordinates. In addition, one can find a countable set $C \subseteq I$ and, for each $i \in C$, a continuous map $h_i\colon X_i \to \mathbb{N}$ to the discrete space $\mathbb{N}$ such that $(\prod_{i\in C} h_i)\circ \pi_C \prec f$. Hence $nw(Y) \le \aleph_0$.*

PROOF: First, we show that $f$ depends on countably many coordinates. Choose any point $a \in X$ and denote by $\sigma(a)$ the $\sigma$-product of the spaces $X_i$ with center at $a$. Let $\sigma(a)$ carry the relative $\aleph_0$-box topology (which is finer than the subspace topology of $\sigma(a)$ inherited from $X$). By Lemma 2.3, the $P$-space $\sigma(a)$ is pseudo-$\omega_1$-compact. Since $\psi(Y) \le \aleph_0$, the set $F_y = f^{-1}(y) \cap \sigma(a)$ is clopen in $\sigma(a)$ for each $y \in Y$. Clearly, $\{F_y : y \in f(\sigma(a))\}$ is a partition of $\sigma(a)$ into disjoint clopen sets. Hence, the pseudo-$\omega_1$-compactness of $\sigma(a)$ implies that the image $Z = f(\sigma(a))$ is countable.

Given a nonempty set $J \subseteq I$, we denote by $\pi_J$ the projection of $X$ onto $X_J = \prod_{i\in J} X_i$. By Lemma 3.4, every set $F_y$ depends only on a countable number coordinates, that is, there exists a countable set $C(y) \subseteq I$ such that $F_y = \sigma(a) \cap \pi^{-1}_{C(y)}\pi_{C(y)}(F_y)$. Put $C = \bigcup_{y\in Z} C(y)$. Then $C$ is a countable subset of $I$ and $F_y = \sigma(a) \cap \pi_C^{-1}\pi_C(F_y)$ for each $y \in Z$. Therefore, if $x, y \in \sigma(a)$ and $\pi_C(x) = \pi_C(y)$, then $f(x) = f(y)$. Apply Lemma 3.6 to conclude that $f$ depends only on the set $C$. In other words, there exists a map $f_C\colon X_C \to Y$ such $f = f_C \circ \pi_C$. The map $f_C$ is continuous because the projection $\pi_C$ is open. We can assume, therefore, that $C = I$ (and $f_C = f$). In addition, we can assume that $I = \omega$, i.e., $X = \prod_{n\in\omega} X_n$ and that each factor $X_n$ is infinite.

For every $n \in \omega$, consider the subspace $K_n$ of $X$ defined by

$$K_n = \{x \in X : x(i) = a(i) \ \text{ for each } \ i > n\}.$$

Then $K_n \cong \prod_{i\le n} X_i$, so that $K_n$ is a pseudo-$\omega_1$-compact $P$-space. As above, it is easy to see that the image $f(K_n)$ is countable for each $n \in \omega$ and the set $F_{n,y} = K_n \cap f^{-1}(y)$ is clopen in $K_n$ for each $y \in f(K_n)$. By Lemma 3.3, every set $F_{n,y}$ can be represented as a countable union of basic open sets of the form $U_0 \times \cdots \times U_n$, where $U_i$ is a clopen subset of $X_i$ for each $i \le n$ (we identify $K_n$ and $X_0 \times \ldots \times X_n$). Since these representations of the sets $F_{n,y}$ involve only countably many clopen sets in each of the factors $X_0, \ldots, X_n$, one can find, for every $i \le n$, a continuous map $g_{n,i}\colon X_i \to \mathbb{N}$ to the discrete space $\mathbb{N}$ such that the direct product $p_n = \prod_{i\le n} g_{n,i}$ satisfies $p_n \prec f_n$, where $f_n = f\restriction_{K_n}$. For every $i \in \omega$, let $g_i$ be the diagonal product of the family $\{g_{n,i} : n \ge i\}$. Then the map $g_i\colon X_i \to \mathbb{N}^{\omega\setminus i}$ is continuous and, clearly, the product map $q_n = \prod_{i\le n} g_i$ satisfies $q_n \prec p_n \prec f_n$ for each $n \in \omega$. Again, the image $g_i(X_i)$ is countable and the fibers $g_i^{-1}(y)$, with $y \in g_i(X_i)$, form a partition of $X_i$ into clopen sets. Hence, for every $i \in \omega$, there exists a continuous onto map $h_i\colon X_i \to \mathbb{N}$ satisfying $h_i \prec g_i$. Let

$h = \prod_{i\in\omega} h_i\colon X \to \mathbb{N}^\omega$ be the direct product of the family $\{h_i : i \in \omega\}$. Note that each map $h_i$ is open and onto, and so is the map $h$.

Let us verify that $h \prec f$. Indeed, since $h_i \prec g_i$ for each $i \in \omega$, we have $\prod_{i\le n} h_i \prec \prod_{i\le n} g_i = q_n \prec f_n$ and, hence,

$$\phi_n = h\restriction_{K_n} = \prod_{i\le n} h_i \prec f_n \tag{3}$$

for all $n \in \omega$. First, we claim that $h^{-1}h(x) \subseteq f^{-1}f(x)$ for every $x \in X$. Suppose to the contrary that there exist points $x, y \in X$ such that $h(x) = h(y)$ but $f(x) \neq f(y)$. Choose in $Y$ disjoint neighborhoods $U_x$ and $U_y$ of $f(x)$ and $f(y)$, respectively. By the continuity of $f$, there are canonical open sets $V_x \ni x$ and $V_y \ni y$ in the product space $X$ such that $f(V_x) \subseteq U_x$ and $f(V_y) \subseteq U_y$. We can assume without loss of generality that $V_x = V_0^x \times \cdots \times V_n^x \times P_n$ and $V_y = V_0^y \times \cdots \times V_n^y \times P_n$, where $n \in \omega$, the sets $V_i^x, V_i^y$ are open in $X_i$ for $i = 0, \dots, n$ and $P_n = \prod_{i>n} X_i$. For every $n \in \omega$, denote by $r_n$ the retraction of $X$ onto $K_n$ defined by $r_n(x)(i) = x(i)$ if $i \le n$ and $r_n(x) = a(i)$ if $i > n$. Then $x' = r_n(x) \in V_x \cap K_n$ and $y' = r_n(y) \in V_y \cap K_n$. Therefore, from $f(x') \in f(V_x) \subseteq U_x$, $f(y') \subseteq f(V_y) \subseteq U_y$ and $U_x \cap U_y = \emptyset$ it follows that $f(x') \neq f(y')$. By (3), however, we have $h \prec \phi_n \circ r_n \prec f_n \circ r_n = f \circ r_n$ and, hence, the equality $h(x) = h(y)$ implies that $f(r_n(x)) = f(r_n(y))$ or, equivalently, $f(x') = f(y')$. This contradiction proves the claim. So, there exists a map $i\colon \mathbb{N}^\omega \to Y$ satisfying $f = i \circ h$. Since the map $h$ is open, $i$ is continuous. Therefore, $h \prec f$.

Finally, the space $\mathbb{N}^\omega$ is second countable, so that the image $Y = f(X) = i(\mathbb{N}^\omega)$ has a countable network. □

It is shown in [15, Lemma 3.29] that every $\omega$-stable space is pseudo-$\omega_1$-compact. For $P$-spaces, $\omega$-stability and pseudo-$\omega_1$-compactness are equivalent by [15, Proposition 3.30]. It turns out that this equivalence holds for arbitrary products of $P$-spaces.

**Corollary 3.8.** *Suppose that the product $X = \prod_{i\in I} X_i$ of $P$-spaces is pseudo-$\omega_1$-compact. Then the space $X$ is $\omega$-stable.*

Proof: Let $f\colon X \to Y$ be a continuous map onto a space $Y$ which admits a coarser second countable Tychonoff topology. Then $Y$ is Hausdorff and $\psi(Y) \le \aleph_0$, so that $nw(Y) \le \aleph_0$ by Theorem 3.7. □

By [1, Theorem 10], every $\sigma$-product of Lindelöf $P$-spaces is $\omega$-stable. The next corollary extends this result to products of Lindelöf $P$-spaces.

**Corollary 3.9.** *Every product of Lindelöf $P$-spaces is $\omega$-stable.*

Proof: By Noble's theorem in [10], finite products of Lindelöf $P$-spaces are Lindelöf (hence, pseudo-$\omega_1$-compact). Therefore, an arbitrary product $X = \prod_{i\in I} X_i$

of Lindelöf $P$-spaces is pseudo-$\omega_1$-compact by Lemma 2.3, and the required conclusion follows from Corollary 3.8. □

In general, the product of two pseudo-$\omega_1$-compact $P$-spaces can fail to be pseudo-$\omega_1$-compact. In the class of $P$-groups, however, pseudo-$\omega_1$-compactness becomes productive by Lemmas 2.2 and 2.3. This explains, in part, the strong factorization property of products of $\mathbb{R}$-factorizable $P$-groups given in the next theorem.

**Theorem 3.10.** *Let $G = \prod_{i\in I} G_i$ be a direct product of $\mathbb{R}$-factorizable $P$-groups. If $f\colon G \to Y$ is a continuous map onto a space $Y$ with $\psi(Y) \le \aleph_0$, then there exists a quotient homomorphism $\pi\colon G \to H$ onto a second countable topological group $H$ such that $\pi \prec f$. In particular, $nw(Y) \le \aleph_0$.*

Proof: By Lemmas 2.2 and 2.3, the group $G$ is pseudo-$\omega_1$-compact. Apply Theorem 3.7 to find a countable set $C \subseteq I$ and, for each $i \in C$, a continuous map $h_i\colon G_i \to \mathbb{N}$ such that $(\prod_{i\in C} h_i) \circ \pi_C \prec f$. Since the groups $G_i$ are $\mathbb{R}$-factorizable, for each $i \in C$ there exists a continuous homomorphism $p_i\colon G_i \to K_i$ onto a second countable group $K_i$ such that $p_i \prec h_i$. Note that the fibers $p_i^{-1}(y)$ are $G_\delta$-sets in $G_i$, so they are open in $G_i$. Clearly, the homomorphism $p_i$ remains continuous if we endow the group $K_i$ with the discrete topology. The group $G_i$ is pseudo-$\omega_1$-compact by Theorem 1.1, so the cover of $G_i$ by the fibers $p_i^{-1}(y)$, with $y \in K_i$, is countable. Hence the discrete group $K_i = p_i(G_i)$ is countable and the homomorphism $p_i$ is open.

Let $p$ be the direct product of the homomorphisms $p_i$, $i \in C$. Then the homomorphism $p\colon \prod_{i\in C} G_i \to \prod_{i\in C} K_i$ is continuous, open and the group $H = \prod_{i\in C} K_i$ is second countable. It is clear that the homomorphism $\varphi = p \circ \pi_C$ of $G$ to $H$ is continuous, open and satisfies $\varphi \prec (\prod_{i\in C} h_i) \circ \pi_C \prec f$. Therefore, there exists a continuous map $i\colon H \to Y$ such that $f = i \circ \varphi$ and, hence, $Y = i(H)$. This implies that $Y$ has a countable network. □

The following corollary to Theorem 3.10 is immediate. It was proved (by a different method) in [15].

**Corollary 3.11.** *Let $G$ be a direct product of $\mathbb{R}$-factorizable $P$-groups. Then the group $G$ is $\mathbb{R}$-factorizable and $\tau$-stable for $\tau \in \{\omega, \omega_1\}$.*

Proof: The $\mathbb{R}$-factorizability of $G$ follows directly from Theorem 3.10. In addition, $G$ is $\omega_1$-stable by [15, Theorem 3.9]. To conclude that $G$ is $\omega$-stable, apply Corollary 3.8 and Lemmas 2.2 and 2.3. □

By a theorem of Comfort and Ross [5], the class of pseudocompact groups is productive. Therefore, Corollary 3.11 extends a certain similarity in the permanence properties of $\mathbb{R}$-factorizable $P$-groups and pseudocompact groups mentioned in Section 2. In addition, the groups of both classes are $\omega$-stable. In fact, one can apply Lemma 5.9 of [14] to prove the following analogue of Theorem 3.10 for pseudocompact groups: if a regular space $Y$ of countable pseudocharacter is a continuous image of (a $G_\delta$-subset of) a pseudocompact group, then $nw(Y) \leq \aleph_0$.

## 4. Open problems

Here we formulate two open problems concerning Theorem 2.5.

**Problem 4.1.** *Is every $\aleph_0$-bounded $P$-group topologically isomorphic to a subgroup of an $\mathbb{R}$-factorizable $P$-group?*

**Problem 4.2.** *Does Theorem* 2.5 *remain valid in the non-abelian case?*

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