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Peter Vassilev Danchev

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# Commutative group algebras of highly torsion-complete abelian $p$-groups

Peter Danchev

*Abstract.* A new class of abelian $p$-groups with all high subgroups isomorphic is defined. Commutative modular and semisimple group algebras over such groups are examined. The results obtained continue our recent statements published in Comment. Math. Univ. Carolinae (2002).

*Keywords:* high subgroups, torsion-complete groups, group algebras, direct factors

*Classification:* 20C07, 16S34, 16U60, 20K

## Introduction

Suppose $G$ is an abelian $p$-group with first Ulm subgroup $G^1 = G^{p^\omega} = \bigcap_{n<\omega} G^{p^n}$, with socle $G[p]$ and with some high subgroup $H_G$ that is a subgroup of $G$ maximal with respect to the ratio intersection $\bigcap G^1 = 1$; if $G$ is separable (equivalently $G^1 = 1$ by [15]) or divisible, it is obviously valid that $H_G = G$ or $H_G = 1$, respectively.

As usual, for $i \geq 0$, the cardinal number $f_i(G)$ is the $i$-th Ulm-Kaplansky invariant of $G$ defined as in [15].

Throughout the text, let $R$ denote a commutative unitary ring, and $RG$ the group ring regarded as an $R$-algebra of $G$ over $R$. For further use, let $S(RG)$ be the normed Sylow $p$-component of $RG$, i.e. the group of all normalized $p$-units in $RG$. The notation and terminology not explicitly given here shall in general be the same as that used in [15] and [7], [12].

In [9] and [11] we have investigated the *Isomorphism* and the *Direct Factor Problems* in modular group algebras of torsion-complete and semi-complete abelian $p$-groups, respectively.

Enlarging the class of direct sums of cyclic $p$-groups, we have settled in [7] these questions but about the high subgroups in commutative modular group rings.

The main purpose which motivates the present work is to describe the group algebra when the high subgroups of its group basis belong to certain group classes. In [7], $H_G$ was assumed to be a direct sum of cyclics while here $H_G$ will be taken torsion-complete.

## Highly torsion-complete abelian groups

**Definition.** The abelian $p$-group $G$ is said to be highly torsion-complete if some $H_G$ is torsion-complete.

Applying the well-known and documented Kulikov-Papp's criterion for torsion-completeness (see [15, p. 25, Theorem 68.4]) together with the purity of the high subgroups of torsion groups ([17] or [15]), we write $G = H_G \times G/H_G$. Since $G/H_G$ is divisible by [17], it is selfevident that the new group class coincides with the class of finite direct products of divisible $p$-groups and torsion-complete $p$-groups. Further, exploiting [17], it follows from the above established direct decomposition that all high subgroups are isomorphic to $H_G$ whence all of them are torsion-complete and isomorphic. We note also that because $f_i(G) = f_i(H_G)$ for every high subgroup $H_G$ of $G$ (see, for example, [15]) and the Ulm-Kaplansky functions determine up to an isomorphism the torsion-complete group [15], we may conclude once again that all high subgroups of $G$ are isomorphic if they are torsion-complete.

Hence, we can write $G = G_d \times G_r$ where $G_d$ is the maximal divisible subgroup of $G$ and the reduced part $G_r$ is torsion-complete; $H_G \cong G_r$.

Suppose now $G$ is an abelian $p$-group with high subgroup $H_G$ which is a direct sum of torsion-complete groups (whose high subgroups are semi-complete, in particular). By virtue of [18, p. 174] if $H'_G$ is another high subgroup in $G$, then $H_G[p]$ and $H'_G[p]$ are isometric, i.e. they are isomorphic as valuated vector spaces over the simple field $F_p$ of $p$-elements. That is why, according to a characterization lemma for torsion-completeness from [19] and to a classification isomorphism theorem due to Hill archived in [15], we may deduce that $H'_G$ is a direct sum of torsion-complete groups too and even more, $H'_G \cong H_G$. So, all high subgroups are isomorphic in this case.

We begin now with a series of group-theoretic assertions, starting with a new confirmation ([7, Lemma 5]).

**Lemma 1.** *If $C$ is a pure subgroup of an abelian $p$-group $G$, then for any high subgroup $H_C$ of $C$ we have $H_C = H_G \cap C$ for some high subgroup $H_G$ of $G$.*

Proof: Since $H_C \cap G^1 = H_C \cap C^1 = 1$ it holds that $H_C \subseteq H_G$ for some $H_G$. Therefore $H_C \subseteq H_G \cap C$. On the other hand, $(H_G \cap C) \cap C^1 = H_G \cap C^1 \subseteq H_G \cap G^1 = 1$, hence the maximality of $H_C$ leads us to the desired equality. □

**Lemma 2.** *If $C$ is an isotype subgroup in an abelian $p$-group $G$, then*

$$(CG^{p^\omega})[p] = C[p]G^{p^\omega}[p].$$

Proof: Choose $x \in (CG^{p^\omega})[p]$ and we write $x = cg$ with $c^p = g^{-p}$ where $c \in C$ and $g \in G^{p^\omega}$. Furthermore, $c^p \in C \cap G^{p^{\omega+1}} = C^{p^{\omega+1}}$, i.e. $c \in C^{p^\omega} C[p]$. Finally $x \in G^{p^\omega}[p]C[p]$, as claimed. □

**Lemma 3.** *Assume $C \leq G$ are abelian $p$-groups. Then $C$ is pure in $G$ if and only if $H_C$ is pure in $H_G$.*

Proof: For the necessity, bearing in mind that $H_C$ is pure in $C$ (see, for instance, [17] or [15]) and employing the transitivity of the purity, we detect that $H_C$ must be pure in $G$ hence in $H_G$ for some high subgroup in $G$.

For the sufficiency, using again the above cited purity property of high subgroups, we extract that $H_C$ is pure in $G$. Moreover, $C[p] = H_C[p] \times C^{p^\omega}[p]$. Thus, in view of the modular law and [17], $C[p] \cap G^{p^n} = C^{p^\omega}[p](H_C[p] \cap G^{p^n}) = C^{p^\omega}[p] \times H_C^{p^n}[p] = C^{p^n}[p]$. Finally, inspired by [15], $C$ is pure in $G$, as expected. □

The following are generalizations to ([7, Lemma 7 and Theorem 9]).

**Lemma 4.** *Let $G$ be an abelian $p$-group and $C$ its pure subgroup. Then*

$$(H_G C)[p] = H_G[p]C[p].$$

Proof: Given $x \in (H_G C)[p]$, write $x = hc$ with $h^{-p} = c^p$ for $h \in H_G$ and $c \in C$. Certainly, consuming [17] and Lemma 1, $c^p \in C^p \cap H_{G^p} = H_{C^p} = (H_C)^p$ where $H_G$ can be chosen so that it contains some $H_C$. That is why $c \in H_C C[p]$ and $x \in H_G[p]C[p]$, as desired. □

**Proposition 1.** *Suppose $G$ is an abelian $p$-group and $C$ its balanced subgroup. Then*

$$H_G C/C = H_{G/C} \cong H_G/H_C.$$

Proof: Use the idea from [7] and Lemmas 1 to 4. □

We come now to the central section entitled

## Group algebras of highly torsion-complete groups

Now, we are in a position to proceed by proving the first main assertion.

**Theorem 1.** *$S(RG)$ is highly torsion-complete if and only if $G$ is algebraically compact, provided $R$ is a first kind field with respect to $p$ of characteristic distinct from $p$ such that its spectrum contains all natural numbers or $R$ is a perfect field of characteristic $p > 0$.*

Proof: **Case 1.** $R$ is of the first type of char $R \neq p$.

Invoking [21], $H_{S(RG)} \cong S(R(G/G^1))$. Consequently, for the first implication, conforming with [12] (see also [10]), $G/G^1$ is bounded. So, via [4], $G$ is algebraically compact.

For the reverse implication, the algebraical compactness of $G$ yields elementarily that $G/G^1$ is bounded, hence $S(R(G/G^1))$ is bounded as well by [21]. Therefore $H_{S(RG)}$ is bounded whence torsion-complete, as required.

**Case 2.** $R$ is of the second type of char $R = p$.

Complying with [14], $H_{S(RG)/G}$ is a direct sum of cyclics. On the other hand, by using Lemma 3 and the fact that $G$ is pure in $S(RG)$, we observe that $H_G$ is pure in $H_{S(RG)}$. But Proposition 1 or [7, Theorem 9] implies that $H_{S(RG)}/H_G \cong H_{S(RG)/G}$, so a theorem due to L. Kulikov [15] assures the isomorphism $H_{S(RG)} \cong H_G \times H_{S(RG)/G}$.

Thus, considering the necessity, [15] ensures that $H_{S(RG)}/H_G$ is bounded. Then, for some positive integer $k$, we get the relation $H^{p^k}_{S(RG)} \subseteq H_G$, i.e., from Lemma 3 we obtain $H^{p^k}_{S(RG)} = H^{p^k}_G$. It is not hard to see that $H^{p^t}_G = 1$ for some $t \in \mathbb{N}$, hence $H_G$ is bounded. Indeed, since $S(RH_G) \cap S^{p^\omega}(RG) = S(RH_G) \cap S(R^{p^\omega} G^{p^\omega}) = S(R^{p^\omega}(H_G \cap G^{p^\omega})) = 1$, we deduce that $S(RH_G)$ is contained in $H_{S(RG)}$. Thus $S^{p^k}(RH_G) = S(R^{p^k} H^{p^k}_G) = H^{p^k}_G$, hence it follows easily that $H^{p^k}_G$ has power less than or equal to 2. Finally $H^{p^{k+1}}_G = 1$ which gives our claim. But as we have mentioned above, $G/H_G$ is divisible, which is equivalent to $G = H_G G^p$. Henceforth $G^{p^t} = G^{p^{t+1}}$, i.e. $G^{p^t}$ is divisible. Next, taking into account [15], we establish that $G$ is a direct product of a divisible group with a bounded group or, in other words, $G$ is algebraically compact, as stated.

The sufficiency is shown as follows. In the spirit of [3], the algebraic compactness of $G$ guarantees that $S(RG)$ is algebraically compact, so $H_{S(RG)}$ is bounded, hence torsion-complete. □

**Remark.** The fields of the first point do exist. Indeed, such are the field of all rationals $\mathbb{Q}$ and its algebraic extension $\mathbb{Q}(\xi_{q^i})$ where $\xi_{q^i}$ are the primitive $q^i$-roots of unity and $q$ is a prime, $q \neq p$.

We concentrate now on the verification of the second major attainment, namely:

**Theorem 2.** *If $G$ is highly torsion-complete and $R$ has prime characteristic $p$ such that $RG \cong RA$ as $R$-algebras for any group $A$, then $G \cong A$.*

Proof: Write down $G = G_d \times G_r$ where $G_d$ is divisible and $G_r$ is torsion-complete. Consulting with [8], $G_d \cong A_d$ and $RG_r \cong RA_r$. Besides, [9] is applicable to infer that $G_r \cong A_r$. Finally, $G = G_d \times G_r \cong A_d \times A_r = A$, as formulated. The proof is complete. □

After this, we improve two identical theorems from [5] and [6], respectively.

**Theorem 3.** *Let $G$ be an abelian group whose high subgroup $H_G$ is a direct sum of torsion-complete groups (in particular, it is semi-complete). If $\mathbb{F}_pA \cong \mathbb{F}_pG$ as $\mathbb{F}_p$-algebras for an arbitrary group $A$, then $H_A \cong H_G$.*

Proof: Foremost we remark that the $\mathbb{F}_p$-isomorphism $\mathbb{F}_pA \cong \mathbb{F}_pG$ yields that $H_A[p]$ and $H_G[p]$ are isometrically isomorphic (see [8]). Thus, by [15], it is enough to show that $H_A$ is a direct sum of torsion-complete groups. In fact, we may presume that $\mathbb{F}_pG = \mathbb{F}_pM$ for some group $M \cong A$. Furthermore, $S(\mathbb{F}_pG) = S(\mathbb{F}_pM)$ and $H_{S(\mathbb{F}_pG)} = H_{S(\mathbb{F}_pM)}$. Owing to [14], both $H_{S(\mathbb{F}_pG)/G}$ and $H_{S(\mathbb{F}_pM)/M}$ are direct sums of cyclic groups. The application of Proposition 1 or [7, Theorem 9] yields that $H_{S(\mathbb{F}_pG)/G} \cong H_{S(\mathbb{F}_pG)}/H_G$ and $H_{S(\mathbb{F}_pM)/M} \cong H_{S(\mathbb{F}_pM)}/H_M$. So, we can apply Lemma 3 and the above cited claim of Kulikov [15] on direct factors to infer $H_{S(\mathbb{F}_pG)} \cong H_G \times H_{S(\mathbb{F}_pG)/G}$ and $H_{S(\mathbb{F}_pM)} \cong H_M \times H_{S(\mathbb{F}_pM)/M}$. By what we have just shown above, $H_G \times H_{S(\mathbb{F}_pG)/G} \cong H_M \times H_{S(\mathbb{F}_pM)/M}$, hence in view of [15] the direct factor $H_M$ is a direct sum of torsion-complete groups.

Now, since $A \cong M$ we derive that $A[p]$ and $M[p]$ are isometric. But $A^{p^n}[p] = H_A^{p^n}[p] \times A^{p^\omega}[p]$ and $M^{p^n}[p] = H_M^{p^n}[p] \times M^{p^\omega}[p]$ for each $n \geq 0$. And so, $H_A[p]$ and $H_M[p]$ are isomorphic as filtered vector spaces. Consequently, the commentaries from the previous group-theoretic section trivially lead us to $H_A \cong H_M$, as needed. The proof is complete. $\square$

The following is actual.

**Question.** Can the restriction on the coefficient field being a simple field be removed?

## Concluding discussion

Utilizing the same technique as above or an analogous approach, other classes of abelian $p$-groups can be considered such as abelian groups with $p^\lambda$-high subgroups ($\lambda$ an arbitrary ordinal) which are totally projective or which belong to different classes of crucial abelian $p$-torsion groups (see, for example, [20] or [16]). However, this is a problem of some other possible study.

In this aspect, whether we may assume that the above argued affirmations and those from [13] remain true for an abelian $p$-group with a high subgroup which is $p^{\omega+n}$-projective ($n \in \mathbb{N}_0$); does it follow then that all high subgroups are isomorphic? In virtue of [18, p. 174], this holds provided all of them are $p^{\omega+n}$-projective (cf. [1] and [2], as well).

**Corrigenda:** In [7, p. 424, line 17-16(-)] the phrase “Moreover, no every high subgroup of $S(RG)$ is of the above kind” should be written and read as “Moreover, no every high subgroup of $V(RG)$ is of the above kind”.

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## References

[1] Culter D.O., *Primary abelian groups having all high subgroups isomorphic*, Proc. Amer. Math. Soc. (3) **83** (1981), 467–470.

[2] Culter D.O., *High subgroups of primary abelian groups of length $\omega + n$*, J. Algebra **83** (1983), 502–509.

[3] Danchev P.V., *Normed unit groups and direct factor problem for commutative modular group algebras*, Math. Balkanica (2-3) **10** (1996), 161–173.
[4] Danchev P.V., *Isomorphism of commutative semisimple group algebras*, Math. Balkanica (1-2) **11** (1997), 51–55.
[5] Danchev P.V., *Isomorphism of modular group algebras of direct sums of torsion-complete abelian p-groups*, Rend. Sem. Mat. Univ. Padova **101** (1999), 51–58.
[6] Danchev P.V., *Commutative group algebras of cardinality* $\aleph_1$, Southeast Asian Bull. Math. (4) **25** (2001), 589-598.
[7] Danchev P.V., *Commutative modular group algebras of p-mixed and p-splitting abelian* $\Sigma$*-groups*, Comment. Math. Univ. Carolinae **43** (2002), 419–428.
[8] Danchev P.V., *Invariants for group algebras of abelian groups*, Rend. Circolo Mat. Palermo (2) **51** (2002), 391–402.
[9] Danchev P.V., *Isomorphism of commutative group algebras of closed p-groups and p-local algebraically compact groups*, Proc. Amer. Math. Soc. **130** (2002), 1937–1941.
[10] Danchev P.V., *Sylow p-subgroups of abelian group rings*, Serdica Math. J. **29** (2003), 33–44.
[11] Danchev P.V., *Isomorphic commutative group algebras of primary semi-complete abelian groups*, to appear.
[12] Danchev P.V., *Torsion completeness of Sylow p-groups in semi-simple group rings*, Acta Math. Sinica (2003).
[13] Danchev P.V., *Commutative group algebras of* $p^{\omega+n}$*-projective abelian groups*, submitted.
[14] Danchev P.V., $S(RG)/G_p$ *is a* $\Sigma$*-group*, in press.
[15] Fuchs L., *Infinite Abelian Groups*, I and II, Mir, Moscow, 1974 and 1977 (in Russian).
[16] Hill P.D., *The equivalence of high subgroups*, Proc. Amer. Math. Soc. **88** (1983), 207–211.
[17] Irwin J.M., *High subgroups of abelian torsion groups*, Pacific J. Math. **11** (1961), 1375–1384.
[18] Irwin J.M., Cellars R.M., Snabb T., *A functorial generalization of Ulm's theorem*, Comment. Math. Univ. St. Pauli **27** (1978), 155–177.
[19] Irwin J.M., Richman F., Walker E.A., *Countable direct sums of torsion complete groups*, Proc. Amer. Math. Soc. **17** (1966), 763–766.
[20] Irwin J.M., Snabb T., Cellars R.M., *The torsion product of totally projective p-groups*, Comment. Math. Univ. St. Pauli **29** (1980), 1–5.
[21] Mollov T.Zh., *Sylow p-subgroups of the group of normed units of semisimple group algebras of uncountable abelian p-groups*, Pliska Stud. Math. Bulgar. **8** (1986), 34–46 (in Russian).

13, General Kutuzov Street, block 7, floor 2, flat 4, 4003 Plovdiv, Bulgaria
*E-mail*: pvdanchev@yahoo.com

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