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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 42 (2001), No. 2, 267--279

Persistent URL: `http://dml.cz/dmlcz/119242`

# Kneser-type theorem for the Darboux problem in Banach spaces

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*Abstract.* In this paper we study the Darboux problem in some class of Banach spaces. The right-hand side of this problem is a Pettis-integrable function satisfying some conditions expressed in terms of measures of weak noncompactness. We prove that the set of all local pseudo-solutions of our problem is nonempty, compact and connected in the space of continuous functions equipped with the weak topology.

*Keywords:* Pettis integral, Fubini theorem, Darboux problem, measure of weak noncompactness

*Classification:* 35R20, 46G10

## 1. Introduction

In this paper we study the set of solutions for the Darboux problem in a Banach space $E$:

$$\begin{cases} \dfrac{\partial^2 z}{\partial x \partial y} = f(x, y, z(x, y)), \quad (x, y) \in \mathcal{P}, \\ z(x, 0) = 0, \\ z(0, y) = 0, \end{cases} \tag{1}$$

where $\mathcal{P} = \{(x, y) : 0 \le x \le a_1,\ 0 \le y \le a_2\}$. By $\frac{\partial^2 z}{\partial x \partial y}$ we will denote the second mixed pseudo-derivative and, consequently, we are looking for pseudo-solutions of the above problem.

In this paper we consider the case when the function $f$ is Pettis-integrable but not necessarily Bochner integrable. This problem generalizes already known results with Carathéodory solutions or weak solutions (cf. [6], [8], [15], [22]). We prove also a full Kneser-type theorem, i.e. we show that a set $S$ of all pseudo-solutions is nonempty, compact and connected in the space $C(J, E)$ endowed with its weak topology, where the set $J$ is a rectangle included in $\mathcal{P}$. By $C(J, E)$ we denote the space of all continuous functions from $J = \langle 0, \alpha_1 \rangle \times \langle 0, \alpha_2 \rangle$ into $E$ and, consequently, $(C(J, E), \omega)$ is the space $C(J, E)$ with the weak topology $\sigma(C(J, E), C(J, E)^*)$.

The problem (1) has been studied by many specialists, for instance Alexiewicz and Orlicz [1], Negrini [19], Dawidowski and Kubiaczyk [9], DeBlasi and Myjak [4], Górniewicz and Pruszko [13], Górniewicz, Bryszewski and Pruszko [14], Bugajewski and Szufla [5].

The key point of the problem (1) is the Fubini theorem. The classical version of this theorem for the Lebesgue integral remains valid for the Bochner integral but fails for the Pettis integral.

In the proof of the existence theorem we use a new Fubini type theorem for some Pettis integrable functions (recently obtained by Michalak in [17]).

Throughout the paper, $(E, \|\cdot\|)$ will denote a real Banach space and $E^*$ its topological dual. We set $(E,\omega) = (E, \sigma(E, E^*))$ the space $E$ with its weak topology, $B_r = \{x \in E : \|x\| \leq r\}$. By $\langle a, b\rangle$ we denote a closed interval in $\mathbb{R}$.

A function $f : E \longrightarrow E$ will be called weakly-weakly sequentially continuous iff for each weakly convergent sequence $(x_n)$ in $E$ the sequence $(f(x_n))$ is weakly convergent. Some comparison results between different concepts of the continuity can by found in [2] (cf. also [20]).

By $(\mathcal{P})\displaystyle\int$ and $\displaystyle\int$ we will denote the Pettis integral and Lebesgue integral, respectively.

For any bounded subset $A$ of $E$ we denote by $\beta(A)$ the DeBlasi measure of weak noncompactness of $A$, i.e.

$$\beta(A) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon + W,\ W - \text{weakly compact subset of } E\}.$$

Let us recall some facts that will be used in the sequel.

**Lemma 1** ([18])**.** *Let $H \subset C(\mathcal{P}, E)$ be a family of strongly equicontinuous functions. Then*

$$\beta(H(\mathcal{P})) = \sup_{(x,y)\in\mathcal{P}} \beta(H(x,y)) \tag{2}$$

*and the function*

$$(x,y) \mapsto v(x,y) = \beta(H(x,y))$$

*is continuous on $\mathcal{P}$.*

**Lemma 2** ([7])**.** *Let $(X,d)$ be a metric space and let $f : X \longrightarrow (E,\omega)$ be sequentially continuous. If $A \subset X$ is a connected subset in $X$, then $f(A)$ is a connected subset in $(E,\omega)$.*

We will need the following fixed-point theorem

**Proposition 1** ([16], cf. also [20]). *Let $X$ be a metrizable locally convex topological vector space, $D$ a closed convex subset of $X$, and let $F$ be a weakly sequentially continuous map of $D$ into itself. If for some $x \in D$ the implication*

$$\overline{V} = \overline{\mathrm{conv}}(\{x\} \cup F(V)) \Longrightarrow V \text{ is relatively weakly compact,} \tag{3}$$

*holds for every subset $V$ of $D$, then $F$ has a fixed point.*

## 2. Pseudo-solutions and the Fubini theorem

Fix arbitrary $x^* \in E^*$ and consider functions $z : \mathcal{P} \longrightarrow E$ and $x^*z : \mathcal{P} \longrightarrow \mathbb{R}$. We will investigate the following problem

$$\begin{cases} \frac{\partial^2(x^*z)}{\partial x \partial y}(x,y) = x^* f(x,y,z(x,y)), \ (x,y) \in \mathcal{P}, \\ z(x,0) = 0, \\ z(0,y) = 0. \end{cases} \tag{1'}$$

By a pseudo-solution of (1) we understand an absolutely continuous function $z : \mathcal{P} \to E$ such that

$$\begin{aligned} z(x,0) &= 0, \qquad 0 \le x \le a_1, \\ z(0,y) &= 0, \qquad 0 \le y \le a_2, \end{aligned}$$

where $z$ has second mixed pseudo-derivative $\frac{\partial^2}{\partial x \partial y}(x^*z)$ for each $x^* \in E^*$ (cf. [7], [8], [21]) and $z$ satisfies (1′) a.e.

Now, consider the problem

$$z(x,y) = (\mathcal{P}) \int_0^y (\mathcal{P}) \int_0^x f(s,t,z(s,t))\, ds\, dt, \quad (x,y) \in \mathcal{P}. \tag{4}$$

This problem is equivalent to (1) in the following sense: $z$ is a solution of (4) iff $z$ is a pseudo-solution of (1). Indeed, under the assumption that $f(\cdot,\cdot,z(\cdot,\cdot))$ is Pettis-integrable for each $z \in C(\mathcal{P}, E)$ and the Fubini theorem holds in $E$ for the Pettis integral (see next part of this section) we have:

$$\frac{\partial}{\partial x}\left(\frac{\partial(x^*z)}{\partial y}\right)(x,y) = x^* f(x,y,z(x,y))$$

for each $x^* \in E$ a.e. on $\mathcal{P}$. Therefore

$$\frac{\partial(x^*z)}{\partial y}(x,y) = \int_0^x x^* f(s,y,z(s,y))\,ds,$$
$$\frac{\partial(x^*z)}{\partial y}(x,y) = x^*\left((\mathcal{P})\int_0^x f(s,y,z(s,y))\,ds\right),$$
$$x^*z(x,y) = \int_0^y x^*\left((\mathcal{P})\int_0^x f(s,t,z(s,t))\,ds\right)dt,$$
$$x^*z(x,y) = x^*\left((\mathcal{P})\int_0^y (\mathcal{P})\int_0^x f(s,t,z(s,t))\right)ds\,dt,$$
$$z(x,y) = (\mathcal{P})\int_0^y (\mathcal{P})\int_0^x f(s,t,z(s,t))\,ds\,dt.$$

Hence, in fact the problem (4) appears.

Since the Fubini theorem fails, in general, we fix a possible large class of spaces for which this useful theorem holds.

Let $X$ be a weakly compactly generated (WCG) Banach space containing no isomorphic copy of $l_1$ and let $E = X^*$. Such spaces will be called (FP)-spaces (Fubini-Pettis spaces). For WCG-spaces see [11], for instance.

*Remark.* The class of (FP)-spaces contains all reflexive Banach spaces but of course not only such spaces. The James space $J$, the James tree space $JT$ or the dual to the last space which is nonreflexive and nonseparable are also interesting examples of nonreflexive (FP)-spaces (see [11]). A full characterization for separable spaces (hence WCG Banach spaces) containing no isomorphic copy of $l_1$ is given in Theorem 4.1 [11].

Functions $f, g : \mathcal{P} \longrightarrow E$ are scalarly equivalent if $x^*f = x^*g$ a.e. on $\mathcal{P}$ for every $x^* \in E^*$.

**Theorem 1** ([17] Fubini theorem)**.** *For every Pettis-integrable function $f : \mathcal{P} \to E$ such that $f$ is bounded and $E$ is an (FP)-space there exists a function $f_1 : \mathcal{P} \to E$ scalarly equivalent to $f$ such that*

(i) *the function $s \mapsto f_1(s,t)$ is Pettis-integrable for a.a. $t \in \langle 0, a_2\rangle$,*

(ii) *the function $t \mapsto f_1(s,t)$ is Pettis-integrable for a.a. $s \in \langle 0, a_1\rangle$,*

(iii) $$(\mathcal{P})\iint_{A\times B} f\,ds\,dt = (\mathcal{P})\iint_{A\times B} f_1\,ds\,dt = (\mathcal{P})\int_B\left((\mathcal{P})\int_A f_1(s,t)\,ds\right)dt$$
$$= (\mathcal{P})\int_A\left((\mathcal{P})\int_B f_1(s,t)\,dt\right)ds$$

*for every measurable subsets $A \subset \langle 0, a_1\rangle$, $B \subset \langle 0, a_2\rangle$.*

It will cause no confusion if we use the same letter $f$ to designate our function $f$ and $f_1$ which is scalarly equivalent to the function $f$.

The assumption that $E$ is an (FP)-space is really essential and cannot be omitted (see [17]).

## 3. Main results

Now, we are able to prove a Kneser-type theorem for the problem (1).
Let $B = \{x \in E : \|x\| \le b\}$ and $\mathcal{P} = \langle 0, a_1\rangle \times \langle 0, a_2\rangle$.

Assume that $f : \mathcal{P} \times B \longrightarrow E$ is such that

$$\|f(x,y,z)\| \le M \quad \text{for} \quad (x,y) \in \mathcal{P},\ z \in B,\ M \ge 0.$$

We will assume in the sequel that $E$ is an (FP)-space.

Choose positive numbers $d_1$, $d_2$ is such a way that

$$d_1 \le a_1, \quad d_2 \le a_2 \quad \text{and} \quad M \cdot d_1 \cdot d_2 < b/2.$$

Put $K = \langle 0, d_1\rangle \times \langle 0, d_2\rangle$.

Now define a set

$$\begin{aligned}\widetilde{B} = \{z \in C(K,E) : z(K) \subset B, &\|z(x,y) - z(\overline{x},\overline{y})\| \\ \le M \cdot d_2 \cdot |x - \overline{x}| + M \cdot d_1 \cdot |y - \overline{y}| &\text{ for each } \ (x,y), (\overline{x},\overline{y}) \in K\}.\end{aligned}$$

Note that $\widetilde{B}$ is nonempty, closed, bounded, convex and equicontinuous in $C(K,E)$.

**Theorem 2.** *Assume that for each strongly absolutely continuous function $z : K \longrightarrow E$, $f(\cdot,\cdot,z(\cdot,\cdot))$ is Pettis-integrable and $f(x,y,\cdot)$ is weakly-weakly sequentially continuous. Moreover, assume that there exists a continuous nondecreasing function $h$ such that the function $u$ identically equal to zero is the unique continuous solution of the inequality*

$$0 \le u(x,y) \le \int_0^y \int_0^x h(u(s,t))\,ds\,dt, \quad (x,y) \in K.$$

*If the function $f$ satisfies*

$$\beta(f(K \times X)) \le h(\beta(X)) \quad \textit{for each} \quad X \subset B, \tag{5}$$

*then the set $S$ of all pseudo-solutions of the Darboux problem (1) defined on $K$ is nonempty, compact and connected in $(C(K,E),\omega)$.*

Proof: **I.** Recall that the problem (1) is equivalent to (4). Put

$$F(z)(x,y) = (\mathcal{P})\int_0^x (\mathcal{P})\int_0^y f(s,t,z(s,t))\,ds\,dt, \quad (x,y) \in K,\ z \in \widetilde{B}.$$

By our assumptions there exists an integral $(\mathcal{P})\iint\limits_K f(x,y,z(x,y))\,dx\,dy$ and, by Fubini theorem, the operator $F$ is well-defined. We will show that $F:\widetilde{B}\longrightarrow\widetilde{B}$.

Fix arbitrary $x^*\in E^*$, $\|x^*\|\le 1$, $(x,y),(\overline{x},\overline{y})\in K$. Then

$$
\begin{aligned}
&|x^*(F(z)(x,y)-F(z)(\overline{x},\overline{y}))|\\
&=\Bigg|x^*\Bigg[(\mathcal{P})\int_0^x\Bigg((\mathcal{P})\int_0^y f(s,t,z(s,t)\,ds)\,dt\Bigg)\\
&\qquad\qquad -(\mathcal{P})\int_0^{\overline{x}}\Bigg((\mathcal{P})\int_0^{\overline{y}} f(s,t,z(s,t))\,ds\Bigg)\,dt\Bigg]\Bigg|\\
&=\Bigg|x^*\Bigg[(\mathcal{P})\iint\limits_{\langle 0,x\rangle\times\langle 0,y\rangle} f(s,t,z(s,t))\,ds\,dt\\
&\qquad\qquad -(\mathcal{P})\iint\limits_{\langle 0,\overline{x}\rangle\times\langle 0,\overline{y}\rangle} f(s,t,z(s,t))\,ds\,dt\Bigg]\Bigg|\\
&=\Bigg|x^*\Bigg[(\mathcal{P})\iint\limits_{\langle \overline{x},x\rangle\times\langle 0,y\rangle} f(s,t,z(s,t))\,ds\,dt\\
&\qquad\qquad +(\mathcal{P})\iint\limits_{\langle 0,\overline{x}\rangle\times\langle \overline{y},y\rangle} f(s,t,z(s,t))\,ds\,dt\Bigg]\Bigg|\\
&\le\int_{\overline{x}}^x\int_0^y|x^*f(s,t,z(s,t))|\,ds\,dt+\int_0^{\overline{x}}\int_{\overline{y}}^y|x^*f(s,t,z(s,t))|\,ds\,dt\\
&\le M\cdot|x-\overline{x}|\cdot y+M\cdot|y-\overline{y}|\cdot\overline{x}\\
&\le M\cdot d_2\cdot|x-\overline{x}|+M\cdot d_1\cdot|y-\overline{y}|<b.
\end{aligned}
$$

Thus

$$\|F(z)(x,y)-F(z)(\overline{x},\overline{y})\|\le M\cdot d_2\cdot|x-\overline{x}|+M\cdot d_1\cdot|y-\overline{y}|$$

and $F:\widetilde{B}\longrightarrow\widetilde{B}$.

Using the Lebesgue dominated convergence theorem for the Pettis integral (see [12]), we deduce that $F$ is weakly-weakly sequentially continuous.

Now we will prove that the implication (3) holds for subsets of $\widetilde{B}$.

Let $V\subset\widetilde{B}$ be such that for some $z\in\widetilde{B}$

$$\overline{V}=\overline{\operatorname{conv}}(\{z\}\cup F(V)).$$

From the definition of $F$ and by Lemma 1 it follows that the function

$$v:(x,y)\mapsto\beta(V(x,y))$$

is continuous on $K$.

For fixed $(x,y) \in K$ we divide intervals $\langle 0,x\rangle$ and $\langle 0,y\rangle$ into $m$ and $n$ parts respectively

$$0 = x_0 < x_1 < \ldots < x_m = x,$$
$$0 = y_0 < y_1 < \ldots < y_n = y,$$

where $x_i = \frac{i \cdot x}{m}$, $\quad y_j = \frac{j \cdot y}{n} \quad (i = 1, \ldots , m \quad , \quad j = 1, \ldots , n)$.

Let $P_{ij} = \{(x,y) : x_{i-1} \le x \le x_i \ , y_{j-1} \le y \le y_j\}$ and

$$V(P_{ij}) = \{u(x,y) : u \in V,\ (x,y) \in P_{ij}\}.$$

By Lemma 1 and the continuity of $v$ there is a point $(\xi_i, \eta_j) \in P_{ij}$ such that

$$\beta(V(P_{ij})) = \sup\{\beta(V(\xi,\eta)) : (\xi,\eta) \in P_{ij}\} = v(\xi_i, \eta_j). \tag{6}$$

On the other hand, by the mean value theorem and using Fubini theorem we obtain

$$\begin{aligned} F(z)(x,y) &= (\mathcal{P})\int_0^x (\mathcal{P})\int_0^y f(s,t,z(s,t))\,ds\,dt = (\mathcal{P}) \iint\limits_{\langle 0,x\rangle\times\langle 0,y\rangle} f(s,t,z(s,t))\,ds\,dt \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} (\mathcal{P}) \iint\limits_{P_{ij}} f(s,t,z(s,t))\,ds\,dt \\ &\subset \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (x_{i+1}-x_i)\cdot(y_{i+1}-y_i)\cdot \overline{\text{conv}} f(K\times V(P_{ij})) \end{aligned}$$

for each $z \in V$. Therefore

$$F(V)(x,y) \subset \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (x_{i+1}-x_i)\cdot(y_{i+1}-y_i)\cdot \overline{\text{conv}} f(K\times V(P_{ij})).$$

By (6) and the corresponding properties of $\beta$ (see [3]) it follows that

$$\beta(F(V)(x,y)) \le \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (x_{i+1}-x_i)\cdot(y_{j+1}-y_j)\cdot h(v(\xi_i,\eta_j)).$$

But if $m,n$ tends to infinity then the last sum tends to the integral

$$\int_0^x \int_0^y h(v(s,t))\,ds\,dt.$$

Thus

$$\beta(F(V)(x,y)) \le \int_0^x \int_0^y h(v(s,t))\,ds\,dt \quad \text{for} \quad (x,y) \in K. \tag{7}$$

Because $\overline{V} = \overline{\text{conv}}(\{z\} \cup F(V))$ by properties of the measure of weak compactness, we have $\beta(V) = \beta(\overline{V}) = \beta(F(V))$, $\beta(V(x,y)) = \beta(F(V)(x,y))$ and finally by (7)

$$v(x,y) \le \int_0^x \int_0^y h(v(s,t))\,ds\,dt \qquad \text{for} \qquad (x,y) \in K.$$

By our assumptions on $h$, this inequality implies that $v(x,y) = 0$ for $(x,y) \in K$. So $V(x,y)$ is relatively weakly compact in $E$ and since $V \subset \widetilde{B}$, $V$ is equicontinuous and by the Ascoli theorem $V$ is compact in $(C(K,E),\omega)$.

By Proposition 1 the map $F$ has a fixed point in $\widetilde{B}$, which is a pseudo-solution of (1).

As $S = F(S)$, using the same arguments one gets that $S$ is relatively compact in $(C(K,E),\omega)$ and by the Eberlein-Šmulian theorem weakly compact in $C(K,E)$.

**II.** For any $\eta > 0$ denote by $S_\eta$ the set of all functions $z : K \longrightarrow E$ satisfying the following conditions

(i) $z(0,0) = 0$,
$\|z(x,y) - z(\overline{x},\overline{y})\| \le 2Md_1|x - \overline{x}| + 2Md_2|y - \overline{y}|$ for $(x,y)$, $(\overline{x},\overline{y}) \in K$,

(ii) $\displaystyle\sup_{(x,y)\in K} \|z(x,y) - (\mathcal{P})\int_0^x ((\mathcal{P})\int_0^y f(\xi,\eta,z(\xi,\eta))\,d\xi)\,d\eta\| < \eta.$

The set $S_\eta$ is nonempty as $S \subset S_\eta$. Now, we will prove the connectedness of $S_\eta$. In fact, we will show that this set can be presented in the form $W = \bigcup_{z\in S_\eta} T_z \cup V$, where the sets $V$ and $T_z$ are connected and the sets $V \cap T_z$ are nonempty.

We are in a position to construct the set $V$.

Let $\beta_1 = \min\left(d_1, \frac{\eta}{4Md_2}\right)$, $\beta_2 = \min\left(\frac{\eta}{4Md_1}\right)$ and
$P_{\varepsilon_1,\varepsilon_2} = \{(x,y) : 0 \le x \le \varepsilon_1,\ 0 \le y \le \varepsilon_2\}$ for any $\varepsilon_1 \in (0,\beta_1)$, $\varepsilon_2 \in (0,\beta_2)$.

We define a function $v(\cdot,\cdot,\varepsilon_1,\varepsilon_2)$ by the formula

$$v(x,y,\varepsilon_1,\varepsilon_2) = \begin{cases} 0 & \text{for } (x,y) \in P_{\varepsilon_1,\varepsilon_2}, \\ (\mathcal{P})\displaystyle\int_0^{x-\varepsilon_1} \left((\mathcal{P})\int_0^{y-\varepsilon_2} f(\xi,\eta,v(\xi,\eta,\varepsilon_1,\varepsilon_2))\,d\xi\right) d\eta & \\ & \text{for } (x,y) \in P_{2\varepsilon_1,2\varepsilon_2} \setminus P_{\varepsilon_1,\varepsilon_2}. \end{cases}$$

Because $v(x,y,\varepsilon_1,\varepsilon_2) = 0$ for $(x,y) \in P_{\varepsilon_1,\varepsilon_2}$, the integral

$$\begin{aligned}(\mathcal{P})\int_0^{x-\varepsilon_1} &((\mathcal{P})\int_0^{y-\varepsilon_2} f(\zeta,\eta,v(\zeta,\eta,\varepsilon_1,\varepsilon_2))\,d\zeta)\,d\eta \\ &= (\mathcal{P})\int_0^{x-\varepsilon_1} ((\mathcal{P})\int_0^{y-\varepsilon_2} f(\zeta,\eta,0)\,d\zeta)\,d\eta\end{aligned}$$

is well-defined for $(x,y) \in P_{2\varepsilon_1,2\varepsilon_2} \setminus P_{\varepsilon_1,\varepsilon_2}$.

Consider the following case:

$\varepsilon_1 < \overline{x} < x < 2\varepsilon_1, \varepsilon_2 < \overline{y} < y < 2\varepsilon_2$.

For $x^* \in E^*$ with $\|x^*\| \le 1$ we have

$$\begin{aligned}
&|x^*[v(x,y,\varepsilon_1,\varepsilon_2) - v(\overline{x},\overline{y},\varepsilon_1,\varepsilon_2)]| \\
&\quad = \left| x^*\left[ (\mathcal{P})\int_0^{x-\varepsilon_1} (\mathcal{P})\int_0^{y-\varepsilon_2} f(\zeta,\eta,0)\,d\zeta\,d\eta \right.\right. \\
&\qquad\qquad \left.\left. - (\mathcal{P})\int_0^{\overline{x}-\varepsilon_1} (\mathcal{P})\int_0^{\overline{y}-\varepsilon_2} f(\zeta,\eta,0)\,d\zeta\,d\eta \right]\right| \\
&\quad = \left| x^*\left( (\mathcal{P})\int_{\overline{x}-\varepsilon_1}^{x-\varepsilon_1} (\mathcal{P})\int_{\overline{y}-\varepsilon_2}^{y-\varepsilon_2} + (\mathcal{P})\int_{\overline{x}-\varepsilon_1}^{x-\varepsilon_1} (\mathcal{P})\int_0^{y-\varepsilon_2} \right.\right. \\
&\qquad\qquad \left.\left. + (\mathcal{P})\int_0^{x-\varepsilon_1} (\mathcal{P})\int_{\overline{y}-\varepsilon_2}^{y-\varepsilon_2} \right)\right| \\
&\quad \le \int_{\overline{x}-\varepsilon_1}^{x-\varepsilon_1} \int_{\overline{y}-\varepsilon_2}^{y-\varepsilon_2} |x^*(f(\zeta,\eta,0))|\,d\zeta\,d\eta \\
&\qquad\qquad + \int_{\overline{x}-\varepsilon_1}^{x-\varepsilon_1} \int_0^{y-\varepsilon_2} |x^*(f(\zeta,\eta,0))|\,d\zeta\,d\eta \\
&\qquad\qquad + \int_0^{x-\varepsilon_1} \int_{\overline{y}-\varepsilon_2}^{y-\varepsilon_2} |x^*(f(\zeta,\eta,0))|\,d\zeta\,d\eta \\
&\quad \le M[(|y-\overline{y}| \cdot |x-\overline{x}|) + M\beta_2|x-\overline{x}| + M\beta_1|y-\overline{y}|] \\
&\quad \le 2M\beta_2|x-\overline{x}| + M\beta_1|y-\overline{y}| \\
&\quad \le 2M[\beta_2|x-\overline{x}| + \beta_1|y-\overline{y}|].
\end{aligned}$$

Hence

$$\|v(x,y,\varepsilon_1,\varepsilon_2) - v(\overline{x},\overline{y},\varepsilon_1,\varepsilon_2)\| \le 2M\beta_2|x-\overline{x}| + 2M\beta_1|y-\overline{y}|. \tag{8}$$

In the remaining cases for $(x,y)$, $(\overline{x},\overline{y}) \in P_{2\varepsilon_1,2\varepsilon_2} - P_{\varepsilon_1,\varepsilon_2}$ we can obtain the same estimation.

So our function $v$ is strongly continuous on $P_{2\varepsilon_1,2\varepsilon_2} - P_{\varepsilon_1,\varepsilon_2}$ and by our assumptions the integral

$$(\mathcal{P})\int_0^{x-\varepsilon_1} ((\mathcal{P})\int_0^{y-\varepsilon_2} f(\zeta,\eta,v(\zeta,\eta,\varepsilon_1,\varepsilon_2))\,d\zeta)\,d\eta$$

exists on $P_{3\varepsilon_1,3\varepsilon_2} \setminus P_{2\varepsilon_1,2\varepsilon_2}$.

By induction we can prove that this integral exists on $K \setminus P_{\varepsilon_1,\varepsilon_2}$.

Now we define the function $v$ on the whole $K$:

$$v(x,y,\varepsilon_1,\varepsilon_2) = \begin{cases} 0 & \text{for } (x,y) \in P_{\varepsilon_1,\varepsilon_2}, \\ (\mathcal{P}) \int_0^{x-\varepsilon_1} ((\mathcal{P}) \int_0^{y-\varepsilon_2} f(\zeta,\eta,v(\zeta,\eta,\varepsilon_1,\varepsilon_2))\, d\zeta)\, d\eta & \\ & \text{for } (x,y) \in K \setminus P_{\varepsilon_1,\varepsilon_2}. \end{cases}$$

Similarly as in the proof of (8) we can show that the function $v$ is strongly continuous on $K$ and satisfies the condition (i).

Furthermore for $x^* \in E^*$, $\|x^*\| \le 1$, one gets

$$\begin{aligned} &|x^*(v(x,y,\varepsilon_1,\varepsilon_2) - (\mathcal{P}) \int_0^x (\mathcal{P}) \int_0^y f(\zeta,\eta,v(\zeta,\eta,\varepsilon_1,\varepsilon_2))\, d\zeta\, d\eta)| \\ &\le \left\{ \begin{array}{l} \left| \int_0^x \int_0^y x^*(f(\zeta,\eta,v(\zeta,\eta,\varepsilon_1,\varepsilon_2)))\, d\zeta\, d\eta \right| \\ \text{for } (x,y) \in P_{\varepsilon_1,\varepsilon_2} \\ \left| \int_0^x \int_{y-\varepsilon_2}^y x^* f \right| + \left| \int_{x-\varepsilon_1}^x \int_0^y x^* f \right| + \left| \int_{x-\varepsilon_1}^x \int_{y-\varepsilon_2}^y x^* f \right| \\ \text{for } (x,y) \in K \setminus P_{\varepsilon_1,\varepsilon_2} \end{array} \right\} \\ &\le \left\{ \begin{array}{l} M\varepsilon_1\varepsilon_2 \\ Md_1\varepsilon_2 + Md_2\varepsilon_1 + M\varepsilon_1\varepsilon_2 \end{array} \right\} \\ &\le 2Md_1\varepsilon_2 + 2Md_2\varepsilon_1 < \eta, \end{aligned}$$

where the supremum is taken over all $x^* \in E^*$ such that $\|x^*\| \le 1$ so $v(\cdot,\cdot,\varepsilon_1,\varepsilon_2)$ satisfies (ii).

Put $v(\cdot,\cdot,\varepsilon_1,\varepsilon_2) = v_{\varepsilon_1,\varepsilon_2}$, then

$$v_{\varepsilon_1,\varepsilon_2}(x,y) = \begin{cases} 0 & \text{for } (x,y) \in P_{\varepsilon_1,\varepsilon_2}, \\ F(v_{\varepsilon_1,\varepsilon_2})(x-\varepsilon_1, y-\varepsilon_2) & \text{for } (x,y) \in K \setminus P_{\varepsilon_1,\varepsilon_2}. \end{cases}$$

Define the set $V = \{v_{\varepsilon_1\varepsilon_2}(\cdot,\cdot) : 0 < \varepsilon_1 < \beta_1,\ 0 < \varepsilon_2 < \beta_2\}$. The set $V$ is a connected set in $(C(K,E),\omega)$. To prove this, we will show that a mapping $(\varepsilon_1,\varepsilon_2) \longrightarrow v_{\varepsilon_1,\varepsilon_2}(\cdot,\cdot)$ is sequentially continuous from $(0,\beta_1) \times (0,\beta_2)$ into $(C(K,E),\omega)$.

Let $0 < \varepsilon_1 < \delta_1 < d_1$ and $0 < \varepsilon_2 < \delta_2 < d_2$ (in other cases the argumentation given below is similar).

For $(x,y) \in P_{\varepsilon_1,\varepsilon_2}$ and $x^* \in E^*$

$$|x^*(v_{\varepsilon_1\varepsilon_2}(x,y) - v_{\delta_1\delta_2}(x,y))| = 0. \tag{9}$$

For $(x, y) \in P_{\delta_1,\delta_2} \setminus P_{\varepsilon_1\varepsilon_2}$ one gets as in (8) that

$$(10) \quad |x^*(v_{\varepsilon_1\varepsilon_2}(x,y) - v_{\delta_1\delta_2}(x,y))| \le \|x^*\|\{2Md_1(\delta_2 - \varepsilon_2) + 2Md_2(\delta_1 - \varepsilon_1)\}.$$

Let $(\varepsilon_1^n, \varepsilon_2^n)$ be a sequence such that $(\varepsilon_1^n, \varepsilon_2^n) \longrightarrow (\varepsilon_1^0, \varepsilon_2^0)$, $\varepsilon_i^n \ge \varepsilon_i^0$, $i = 1, 2$. By (10) it follows that $v_{\varepsilon_1^n\varepsilon_2^n}(x,y)$ converges weakly to $v_{\varepsilon_1^0,\varepsilon_2^0}(x,y)$ uniformly for $(x,y) \in K$. So $F(v_{\varepsilon_1^n\varepsilon_2^n})(x,y) \to F(v_{\varepsilon_1^0\varepsilon_2^0})(x,y)$ weakly on $K$, the map $(\varepsilon_1, \varepsilon_1) \longrightarrow v_{\varepsilon_1,\varepsilon_2}(\cdot,\cdot)$ from $K$ into $(C(K,E),\omega)$ is sequentially continuous (cf. [18, Lemma 1.9]).

Therefore by Lemma 2, the set $V = \{v_{\varepsilon_1\varepsilon_2}(\cdot,\cdot) \ : \ 0 < \varepsilon_1 < \beta_1, \ 0 < \varepsilon_2 < \beta_2\}$ is connected in $(C(K,E),\omega)$.

Now we define the set $T_z$. Let $z \in S_\eta$. Choose $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\sup_{(x,y)\in K} \|z(x,y) - \int_0^x \int_0^y f(\zeta,\eta,z(\zeta,\eta))\,d\zeta\,d\eta\| + 2Md_2\varepsilon_1 + 2Md_1\varepsilon_2 < \eta.$$

For any $(p_1, p_2) \in K$ we define a function $y(\cdot,\cdot,p_1,p_2)$ by the formula

$$y(x,y,p_1,p_2) = \begin{cases} z(x,y) & \text{for } 0 \le x \le p_1, \quad 0 \le y \le p_2, \\ z(p_1,p_2) & \text{for } (x,y) \in P_{p_1+\varepsilon_1}, p_2+\varepsilon_2, \\ & p_1 \le x \le \min(d_1, p_1+\varepsilon_1), \\ & p_2 \le y \le \min(d_2, p_2+\varepsilon_2), \\ z(x,p_2) + (\mathcal{P})\int_0^x (\mathcal{P})\int_{p_2}^{y-\varepsilon_2} f(\zeta,\eta,y(\zeta,\eta,p_1,p_2))\,d\zeta\,d\eta & \\ & \text{for } 0 < x \le p_1, \ \min(d_1, p_2+\varepsilon_2) < y < d_2, \\ z(p_1,y) + (\mathcal{P})\int_{p_1}^{x-\varepsilon_1} (\mathcal{P})\int_0^y f(\zeta,\eta,y(\zeta,\eta,p_1,p_2))\,d\zeta\,d\eta & \\ & \text{for } \min(d_1, p_1+\varepsilon_1) < x < d_1, \quad 0 < y < p_2, \\ z(p_1,p_2) + (\mathcal{P})\int_{p_1}^{x-\varepsilon_1} (\mathcal{P})\int_{p_2}^{x-\varepsilon_2} f(\zeta,\eta,y(\zeta,\eta,p_1,p_2))\,d\zeta\,d\eta & \\ & \text{for } \min(d_1, p_1+\varepsilon_1) < x < d_1, \\ & \min(d_2, p_2+\varepsilon_2) < y < d_2. \end{cases}$$

Applying the same argument as above with $y(\cdot,\cdot,p_1,p_2)$, one shows that $y(\cdot,\cdot,p_1,p_2) \in S_\eta$ for each $(p_1,p_2) \in K$ and that the mapping $(p_1,p_2) \longrightarrow y(\cdot,\cdot,p_1,p_2)$ from $K$ into $(C(K,E),\omega)$ is sequentially continuous.

Consequently, by Lemma 2, the set $T_z = \{y(\cdot,\cdot,p_1,p_2) : 0 < p_1 < \beta_1, \ 0 < p_2 < \beta_2\}$ is connected in $(C(K,E),\omega)$.

As $y(\cdot,\cdot,0,0) = v(\cdot,\cdot,\varepsilon_1,\varepsilon_2) \in V \cap T_z$, the set $V \cup T_z$ is connected, and therefore the set

$$W = \bigcup_{z\in S_\eta} T_z \cup V \quad \text{is connected in} \quad (C(K,E),\omega).$$

Moreover $S_\eta \subset W$, because $z = y(\cdot,\cdot,\beta_1,\beta_2) \in T_z$ for each $z \in S_\eta$. On the other hand $W \subset S_\eta$, since $T_z \subset S_\eta$ and $V \subset S_\eta$. Finally $S_\eta = W$ is a connected subset of $(C(K,E),\omega)$.

**III.** Suppose that the set $S$ is not connected.

As $S$ is weakly compact, there exist nonempty weakly compact sets $W_1$ and $W_2$ such that $S = W_1 \cup W_2$ and $W_1 \cap W_2 = \emptyset$. Consequently there exist two disjoint weakly open sets $U_1$, $U_2$ such that $W_1 \subset U_1$, $W_2 \subset U_2$. Suppose that for every $n \in N$ there exists a $u_n \in V_n \setminus U$, where $V_n = \overline{S_{\frac{1}{n}}}^{\omega}$ and $U = U_1 \cup U_2$. Put $H = \overline{\{u_n \,:\, n \in N\}}^{\omega}$.

Since $u_n - F(u_n) \longrightarrow 0$ in $C(K, E)$ as $n \to \infty$ and $H(x,y) \subset \{u_n(x,y) - F(u_n)(x,y) : u_n \in H\} + F(H)(x,y)$, an analysis similar to that in part **I.** shows that there exists $u_0 \in H$ such that $u_0 = F(u_0)$, i.e. $u_0 \in S$. On the other hand, $H \subset (C(K,E),\omega) \setminus U$, as $U$ is weakly open, so $u_0 \in S \setminus U$, a contradiction.

Therefore, there is $m \in N$ such that $V_m \subset U$.

Since $U_1 \cap V_m \neq \emptyset \neq U_2 \cap V_m$, $V_m$ is not connected, a contradiction with the connectedness of each $V_n$. Consequently, $S$ is connected in $(C(K,E),\omega)$. □

## Remarks

If $f(\cdot, x)$ is scalarly measurable, $f(t, \cdot)$ is weakly-weakly continuous, $f$ is bounded and $E$ is a $WCG$ space then for each absolutely strongly continuous function $x(\cdot)$ $f(\cdot, x(\cdot))$ is Pettis-integrable, so our assumption on $f$ seems to be natural.

One can easily prove that the integral of a weakly continuous function is weakly differentiable with respect to the right endpoint of the integration interval and its derivative equals the integral at the same point. In this case a pseudo-solution is, actually, a weak solution ([22]). Moreover, in separable Banach spaces our pseudo-solutions are also strong Carathéodory solutions.

It seems to be natural to point out that $S$ is strongly equicontinuous in $C(K,E)$, so $S$ is a continuum in $C_\omega(K,E)$ (cf. [6], [7]). A very interesting lecture about the structure of the solution sets, including the Darboux problem, one can find in [10].

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(*Received* May 5, 2000)