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# Representing lattices by homotopy groups of graphs

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*Abstract.* In this paper we represent every lattice by subgroups of free groups using the concept of the homotopy group of a graph.

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In this paper we present a method how to represent a given lattice $L$ as a sublattice of the subgroup lattice of a free group. The method is based on the idea of the homotopy group of a graph. Our construction is such that if the lattice $L$ is finite then the free group and all its subgroups representing the elements of $L$ are finitely generated. The first part of the proof is formulated in a more general way to enable further possible modifications of the proof replacing the free group by a finite group $\mathbf{G}$.

The first proof that every lattice can be embedded into the subgroup lattice of a group was given in [Wh]. The proof was later simplified by the author of the present note in [Tů2] using the solution of the word problem for HNN-extensions.

Let $\Omega$ be a set. By a *twist* on $\Omega$ we mean a bijection $t : A \to B$ between two subsets $A, B$ of $\Omega$. The set $A$ is called the *domain* of $t$ and denoted by $\mathrm{Dom}(t)$, while $B$ is called the *range* of $t$ and denoted by $\mathrm{Rng}(t)$. If $t : A \to B$ is a twist on $\Omega$, then the inverse mapping $t^{-1} : B \to A$ is also a twist on $\Omega$ and called the *inverse* of $t$. The value of a twist $t$ at a point $a \in \mathrm{Dom}(t)$ will be written as $at$.

By a *twisting structure* on $\Omega$ we mean a set $\mathcal{T} = \{t_i : i \in I\}$ of twists on $\Omega$ such that with every $t \in \mathcal{T}$ the inverse $t^{-1}$ of $t$ is also contained in $\mathcal{T}$.

The Cayley graph $\mathbf{G}(\mathcal{T})$ of a twisting structure $\mathcal{T}$ is defined as follows. The vertex set of $\mathbf{G}(\mathcal{T})$ is $\Omega$. The edge set of $\mathbf{G}(\mathcal{T})$ is the set $E = \{(a,t) : a \in \mathrm{Dom}(t), t \in \mathcal{T}\}$. If $e = (a,t) \in E$, then $a$ is the initial vertex $\alpha(e)$ of $e$ and $at$ is the terminal vertex $\omega(e)$ of $e$. Since $\mathcal{T}$ contains with every twist $t$ also the inverse $t^{-1}$ of $t$, with every edge $e = (a,t) \in E$ there is also the edge $(at, t^{-1}) \in E$. The edge $(at, t^{-1})$ is called the *inverse* of $e$ and denoted by $e^{-1}$. It is obvious that $\alpha(e^{-1}) = \omega(e)$ and $\omega(e^{-1}) = \alpha(e)$, thus the graph $\mathbf{G}(\mathcal{T})$ is a symmetric graph possibly with loops and parallel edges. We further define the value $\nu(e)$ of an edge $e = (a,t)$ as the twist $t$. Thus the Cayley graph $\mathbf{G}(\mathcal{T}) = (\Omega, E, \alpha, \omega, \nu)$ of a twisting structure $\mathcal{T}$ is a symmetric graph with edges valued by elements of $\mathcal{T}$. It

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is the union of the graphs of all partial bijections $t \in \mathcal{T}$. Note also that for every $t \in \mathcal{T}$ and $a \in \Omega$ there is at most one edge of $\mathbf{G}(\mathcal{T})$ with initial vertex $a$ and value $\nu(e) = t$.

By a *congruence* of a twisting structure $\mathcal{T}$ on $\Omega$ we mean an equivalence relation $\pi$ on $\Omega$ satisfying the following condition:

$$(*) \qquad \text{whenever} \quad (a,b) \in \pi,\ t \in \mathcal{T} \quad \text{and} \quad a,b \in \mathrm{Dom}(t), \text{ then also} \quad (at, bt) \in \pi.$$

Thus a congruence of $\mathcal{T}$ is a congruence of the partial algebra $(\Omega, \mathcal{T})$, where each $t \in \mathcal{T}$ is considered to be a partial unary operation on $\Omega$. Obviously the set $\mathbf{C}(\mathcal{T})$ of all congruences of the twisting structure $\mathcal{T}$ on $\Omega$ is closed under arbitrary intersections and contains the least and the greatest equivalence relations on $\Omega$. Thus $\mathbf{C}(\mathcal{T})$ when ordered by inclusion is a complete lattice. It is called the *congruence lattice* of $\mathcal{T}$. The meet $\pi \wedge \rho$ of two congruences $\pi, \rho$ of $\mathcal{T}$ is their set-theoretical intersection $\pi \cap \rho$, while their join $\pi \vee \rho$ in $\mathbf{C}(\mathcal{T})$ is the least equivalence relation on $\Omega$ containing the set-theoretical union $\pi \cup \rho$ and satisfying the condition $(*)$.

The following simple representation result was proved in [Tů1].

**Theorem 1.** *Every lattice $L$ can be represented as a sublattice of $\mathbf{C}(\mathcal{T})$ for some twisting structure $\mathcal{T}$.* □

For the sake of completeness we present the construction. Given a lattice $L$, we may assume that it has a least element 0. For any two non-zero elements $a < b$ of $L$ we define a twist $t_{a,b}$ on $L$ with $\mathrm{Dom}(t_{ab}) = \mathrm{Rng}(t_{ab}) = \{0, a, b\}$ and such that

$$bt_{ab} = b,\ at_{ab} = 0,\ 0t_{ab} = a.$$

Moreover, if $a, b \in L$ are two non-comparable and non-zero elements, then we define a twist $t_{ab}$ on $L$ such that $\mathrm{Dom}(t_{ab}) = \mathrm{Rng}(t_{ab}) = \{a, b, a \vee b\}$ and

$$at_{ab} = a,\ bt_{ab} = a \vee b,\ (a \vee b)t_{ab} = b.$$

Let the twisting structure $\mathcal{T}$ on $L$ consist of all twists of the form $t_{ab}$, where $0 \neq a, b \in L$. Then the mapping assigning to every $x \in L$ the partition of $L$ with one block the interval $[0, x]$ and the other blocks singletons is an embedding of $L$ into $\mathbf{C}(\mathcal{T})$.

The main purpose of this note is to investigate a canonical mapping $\Phi$ from $\mathbf{C}(\mathcal{T})$ into the subgroup lattice of the free group $\mathbf{F}(\mathcal{T})$. Here $\mathbf{F}(\mathcal{T})$ denotes the free group generated by the set $\mathcal{T}$ of free generators. Under sufficiently general conditions on $\mathcal{T}$ we can prove that the canonical mapping $\Phi$ is an embedding of any member from a large class of sublattices of $\mathbf{C}(\mathcal{T})$. To this end we recall some basic ideas from combinatorial group theory related to homotopy groups of graphs. By a graph we mean a quadruple $(V, E, \alpha, \omega)$, where $V, E$ are non-empty sets and $\alpha, \omega : E \to V$ are two incidence functions, $\alpha(e)$ is called the initial vertex of an edge $e \in E$ and $\omega(e)$ is the terminal vertex of $e$. By a path in the graph $(V, E, \alpha, \omega)$ we mean a sequence $p = e_1e_2 \cdots e_k$ of edges of $E$ such that $\omega(e_i) = \alpha(e_{i+1})$ for every $i = 1, 2, \ldots, k-1$. A path $p = e_1e_2 \cdots e_k$ is called a loop if $\omega(e_k) = \alpha(e_1)$, and a loop $p = e_1e_2 \cdots e_k$ is called a loop at a vertex $v \in V$ if $\alpha(e_1) = v$.

We will generalize these concepts to the Cayley graph of a twisting structure $\mathcal{T}$. Let $\pi$ be a congruence of $\mathcal{T}$. A sequence $p = e_1e_2 \cdots e_k$ of edges of the Cayley graph $\mathbf{C}(\mathcal{T})$ of $\mathcal{T}$ is called a $\pi$-path if $(\omega(e_i), \alpha(e_{i+1})) \in \pi$ for every $i = 1, 2, \ldots k-1$. A path $p = e_1e_2 \cdots e_k$ is called a $\pi$-loop if $(\alpha(e_1), \omega(e_k)) \in \pi$ and it is called a $\pi$-loop at a vertex $v \in \Omega$ if moreover $(v, \alpha(e_1)) \in \pi$. If $(\omega(e_i), \alpha(e_{i+1})) \in \tau$ for some relation $\tau$ on $\Omega$, then we say that $(\omega(e_i), \alpha(e_{i+1}))$ is a $\tau$-jump.

Suppose moreover that $\mathcal{T} \subset \mathbf{G}$, where $\mathbf{G}$ is a group. Then we can assign to every $\pi$-path $p = e_1e_2 \cdots e_k$ its $\mathbf{G}$-value

$$\nu(p) = \nu(e_1)\nu(e_2) \cdots \nu(e_k) \in \mathbf{G}.$$

The set of $\mathbf{G}$-values of all $\pi$-loops at a vertex $v \in \Omega$ in the Cayley graph $\mathbf{C}(\mathcal{T})$ of a twisting structure $\mathcal{T}$ is obviously a subgroup of $\mathbf{G}$. Indeed, $1 \in \mathbf{G}$ is the value of the empty $\pi$-path, if $g = \nu(p)$ for a $\pi$-loop $p = e_1e_2 \cdots e_k$ at $v$, then $g^{-1}$ is the value of the inverse path $p^{-1} = e_k^{-1} \cdots e_1^{-1}$. And if $g = \nu(p)$, $h = \nu(q)$, then $gh = \nu(pq)$. Thus we can define a mapping

$$\Phi_{\mathbf{G}} : \mathbf{C}(\mathcal{T}) \to \mathrm{Sub}(\mathbf{G})$$

from the congruence lattice of $\mathcal{T}$ into the subgroup lattice of $\mathbf{G}$ by

$$\Phi_{\mathbf{G}}(\pi) = \{\nu(p) : p \text{ is a } \pi\text{-loop at } v\}.$$

If $\pi \subseteq \rho$ are two congruences of $\mathcal{T}$, then obviously any $\pi$-loop at $v$ is also a $\rho$-loop at $v$, thus we get the following simple lemma.

**Lemma 2.** *The mapping $\Phi_{\mathbf{G}}$ is order-preserving.* □

We say that a twisting structure $\mathcal{T}$ is connected if its Cayley graph $\mathbf{G}(\mathcal{T})$ is connected. For connected twisting structures we have the following result.

**Theorem 3.** *If $\mathcal{T}$ is a connected twisting structure, then the mapping $\Phi_{\mathbf{G}}$ is join-preserving.*

PROOF: First of all we describe the join $\pi \vee \rho$ of two congruences $\pi, \rho \in \mathbf{C}(\mathcal{T})$. Set $\sigma_0 = \pi \cup \rho$. If $\sigma_{2i}$ is already defined for a natural number $i$, we define

$$\sigma_{2i+1} = \sigma_{2i} \cup \{(at, bt) : t \in \mathcal{T}, a, b \in \mathrm{Dom}(t), (a, b) \in \sigma_{2i}\},$$

and

$$\sigma_{2i+2} \text{ is the transitive closure of } \sigma_{2i+1}.$$

Obviously, any congruence $\sigma$ of $\mathcal{T}$ containing both $\pi$ and $\rho$ must contain also $\sigma_n$ for any natural number $n$. On the other hand,

$$\sigma = \bigcup_n \sigma_n$$

is an equivalence relation satisfying the condition $(*)$, hence a congruence of $\mathcal{T}$. Thus $\sigma = \pi \vee \rho$ in $\mathbf{C}(\mathcal{T})$.

Since $\Phi_{\mathbf{G}}$ is order-preserving, we get that

$$\Phi_{\mathbf{G}}(\pi) \vee \Phi_{\mathbf{G}}(\rho) \subseteq \Phi_{\mathbf{G}}(\pi \vee \rho)$$

for any two congruences $\pi, \rho$ of $\mathcal{T}$. To prove the opposite inclusion we have to show that the value $\nu(p)$ of any $(\pi \vee \rho)$-path $p$ at $v$ is contained in the subgroup of $\mathbf{G}$ generated by $\Phi_{\mathbf{G}}(\pi) \cup \Phi_{\mathbf{G}}(\rho)$.

So let $p = e_1 e_2 \cdots e_k$ be an arbitrary $(\pi \vee \rho)$-loop at $v$. Thus $(\omega(e_i), \alpha(e_{i+1})) \in \pi \vee \rho$ for any $i = 1, 2, \ldots, k-1$ as well as $(v, \alpha(e_1)), (\omega(e_k), v) \in \pi \vee \rho$. Since $\pi \vee \rho = \bigcup \sigma_n$, there exists a natural number $m$ such that

$$(v, \alpha(e_1)), (\omega(e_k), v) \in \sigma_m, (\omega(e_i), \alpha(e_{i+1})) \in \sigma_m$$

for every $i = 1, 2, \ldots, k-1$. Let us call such a $(\pi \vee \rho)$-loop $p = e_1 e_2 \cdots e_k$ at $v$ a $\sigma_m$-loop. We are going to prove that the value $\nu(p)$ of any $\sigma_m$-loop at $v$, $m \geq 1$, belongs to the subgroup of $\mathbf{G}$ generated by the values of $\sigma_{m-1}$-loops at $v$.

If $m$ is odd, let $i \leq k$ be such that $(\omega(e_i), \alpha(e_{i+1})) \in \sigma_m \setminus \sigma_{m-1}$. Thus there exist a twist $t \in \mathcal{T}$ and $a, b \in \sigma_{m-1}$ such that $at = \omega(e_i)$ and $bt = \alpha(e_{i+1})$. Consider the loop $p' = e_1 e_2 \cdots e_i (at, t^{-1})(b, t) e_{i+1} \cdots e_k$. Since $\omega(e_i) = \alpha(at, t^{-1})$, $(\omega(at, t^{-1}), \alpha(b, t)) = (a, b) \in \sigma_{m-1}$ and $\omega(b, t) = bt = \alpha(e_{i+1})$, $p'$ is also a $\sigma_m$-loop at $v$ and the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in $p'$ is one less than the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in $p$. Similarly, if $(v, \alpha(e_1)) \in \sigma_m \setminus \sigma_{m-1}$, then again there are a twist $t \in \mathcal{T}$ and $(a, b) \in \sigma_m \setminus \sigma_{m-1}$ such that $at = v$ and $bt = \alpha(e_1)$. Again the loop $p' = (at, t^{-1})(b, t)p$ is a $\sigma_m$-loop at $v$ (since $(v, t^{-1}) = (at, t^{-1})$) and the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in $p'$ is one less than the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in $p$. The case $(\omega(e_k), v) \in \sigma_m \setminus \sigma_{m-1}$ is treated in exactly the same way. In all cases, $\nu(p') = \nu(p)$ and the number of $(\sigma_m \setminus \sigma_{m-1})$-jumps in the path $p'$ is one less than the number of $(\sigma_m \setminus \sigma_{m-1})$-jumps in $p$. By a simple induction on the number of $(\sigma_m \setminus \sigma_{m-1})$-jumps in $p$ we prove that for every $\sigma_m$-loop $p$ at $v$ there exists a $\sigma_{m-1}$-loop $p''$ at $v$ such that $\nu(p'') = \nu(p)$. Thus if $m$ is odd, then the value $\nu(p)$ of any $\sigma_m$-loop $p$ at $v$ is equal to the value of a $\sigma_{m-1}$-loop at $v$.

If $m > 0$ is even and $p = e_1 e_2 \cdots e_k$ a $\sigma_m$-loop at $v$ that is not a $\sigma_{m-1}$-loop, then either there exists some $i = 1, 2, \ldots, k-1$ such that $(\omega(e_i), \alpha(e_{i+1})) \in \sigma_m \setminus \sigma_{m-1}$ or $(v, \alpha(e_1)) \in \sigma_m \setminus \sigma_{m-1}$ or $(\omega(e_k), v) \in \sigma_m \setminus \sigma_{m-1}$. Let the first of the three possibilities occur. Since $\sigma_m$ is the transitive closure of $\sigma_{m-1}$, there are elements $\omega(e_i) = a_1, a_2, \ldots, a_l = \alpha(e_{i+1})$ such that $(a_j, a_{j+1}) \in \sigma_{m-1}$. Since the Cayley graph $\mathbf{G}(\mathcal{T})$ of $\mathcal{T}$ is connected, there are paths $q_i$ in $\mathbf{G}(\mathcal{T})$ of $\mathcal{T}$ from $v$ to $a_i$. Then

$p' = e_1e_2\cdots e_iq_1^{-1}q_1q_2^{-1}\cdots q_{l-1}q_l^{-1}q_le_{i+1}\cdots e_k$ is again a $\sigma_m$-loop at $v$ in which the number of $\sigma_m\setminus\sigma_{m-1}$-jumps is one less than the number of $\sigma_m\setminus\sigma_{m-1}$-jumps in $p$. Moreover, $\nu(p')=\nu(p)$. The other two cases are treated in exactly the same way. Thus also in this case the value of any $\sigma_m$-loop at $v$ coincides with the value of a $\sigma_{m-1}$-loop at $v$.

Hence the value $\nu(p)$ of any $(\pi\vee\rho)$-loop at $v$ equals the value $\nu(p')$ of a $\sigma_0$-loop $p'$ at $v$. Recall that $\sigma_0=\pi\cup\rho$. Let $p'=f_1f_2\cdots f_l$. For every $i=1,2,\ldots,l$ let $q_i$ be a path in $\mathbf{G}(\mathcal{T})$ from $v$ to $\omega(f_i)$. Then

$$p''=f_1q_1^{-1}q_1f_2q_2^{-1}q_2f_3\cdots q_{l-1}^{-1}q_{l-1}f_l$$

is also a $\sigma_0$-loop at $v$. Obviously, $\nu(p'')=\nu(p)$. Finally, observe that each $q_{i-1}f_iq_i^{-1}$, $i=2,\ldots,l-1$ is either a $\pi$-loop or a $\rho$-loop, since $(\omega(f_{i-1}),\alpha(f_i))\in\sigma_0=\pi\cup\rho$. Thus $\nu(q_{i-1}f_iq_i^{-1})\in\Phi_{\mathbf{G}}(\pi)\cup\Phi_{\mathbf{G}}(\rho)$.

Similarly, we prove that also $\nu(f_1)q_1^{-1}\in\Phi_{\mathbf{G}}(\pi)\cup\Phi_{\mathbf{G}}(\rho)$ and $\nu(q_{l-1}f_l)\in\Phi_{\mathbf{G}}(\pi)\cup\Phi_{\mathbf{G}}(\rho)$. Thus $\nu(p'')\in\Phi_{\mathbf{G}}(\pi)\vee\Phi_{\mathbf{G}}(\rho)$. $\square$

In the rest of the paper we restrict ourselves to the case that $\mathbf{G}$ is the free group $\mathbf{F}$ freely generated by $\mathcal{T}$. Let $L$ be any non-empty collection of partitions on the set $\Omega$ closed under finite meets. We say that a twisting structure $\mathcal{T}$ on $\Omega$ is *balanced* with respect to $L$ if for every twist $t\in T$, an element $a\in\Omega$ and any two $x,y\in\mathrm{Dom}(t)$, whenever $(a,x)\in\pi\in L$ and $(a,y)\in\rho\in L$, then there exists some $z\in\mathrm{Dom}(t)$ such that $(a,z)\in\pi\wedge\rho\in L$. The following lemma from [Tů1] gives us a way to construct balanced sets.

**Lemma 4.** *Let $X$ be a set and $\Omega=\mathbf{S}_X$, the group of all permutations of $X$ of a finite type (i.e. generated by transpositions). For a set $Y\subset X$ let $S_Y$ be the subgroup of $\mathbf{S}_X$ consisting of all permutations $p$ such that $p(x)=x$ for every $x\in X\setminus Y$. Let $L$ be the set of partitions of $\Omega$ into left cosets of subgroups $S_Y$, $Y\subseteq X$. Then every left coset of every $S_Y$, $Y\subseteq X$, is balanced with respect to $L$.* $\square$

By modifying Example 2.5. and Proposition 2.8. of [Tů1] we get the following lemma.

**Lemma 5.** *For every lattice $\mathbf{L}$ there exist a set $\Omega$, a twisting structure $\mathcal{T}$ on $\Omega$ with finite domains and a lattice embedding $\phi:\mathbf{L}\to\mathbf{C}(\mathcal{T})$ such that the twisting structure $\mathcal{T}$ is balanced with respect to the lattice $L=Im(\phi)$.* $\square$

No we are ready to prove the following counterpart to Theorem 3.

**Theorem 6.** *Let $\mathcal{T}$ be a twisting structure on a set $\Omega$ and $L\subset\mathbf{C}(\mathcal{T})$ a sublattice of $\mathbf{C}(\mathcal{T})$. Suppose moreover that the domains of the elements of $\mathcal{T}$ are finite and that $\mathcal{T}$ is balanced with respect to $L$. Then the restriction of the canonical mapping $\Phi_{\mathbf{F}}$ to the lattice $L$ is meet-preserving.*

Proof: Let $\pi\in\mathbf{C}(\mathcal{T})$. First of all we prove that for any $\pi$-loop $p=e_1e_2\cdots e_k$ in $\mathbf{G}(\mathcal{T})$ at $v$ such that $\nu(e_1)\nu(e_2)\cdots\nu(e_k)$ is not a reduced word in $\mathbf{F}$ there exists a subpath $p' = e_{i_1} \cdots e_{i_l}$ of $p$ that is also a $\pi$-loop at $v$ and the word $\nu(e_{i_1}) \cdots \nu(e_{i_l})$ is reduced. Indeed, if $\nu(e_1)\nu(e_2) \cdots \nu(e_k)$ is not reduced, then there is some $i = 1, 2, \ldots, k-1$ such that $\nu(e_i) = t = \nu(e_{i+1})^{-1}$. Thus there are some $a, b \in \mathrm{Dom}(t)$ such that $e_i = (a, t)$ and $e_{i+1} = (bt, t^{-1})$. Moreover, since $p$ is a $\pi$-path, we have $(\omega(e_i), \alpha(e_{i+1})) = (at, bt) \in \pi$. Since $\pi \in \mathbf{C}(\mathcal{T})$, we have also $(att^{-1}, btt^{-1}) = (a, b) \in \pi$. But we have also $(\omega(e_{i-1}), \alpha(e_i)) = (\omega(e_{i-1}), a) \in \pi$ and $(\omega(e_{i+1}), \alpha(e_{i+2})) = (b, \alpha(e_{i+2})) \in \pi$, we get $(\omega(e_{i-1}), \alpha(e_{i+2})) \in \pi$. Thus we can delete from $p$ the edges $e_i, e_{i+1}$ and the remaining path $p'$ is still a $\pi$-loop at $v$. In this way we can subsequently delete from $p$ pairs of subsequent edges with mutually inverse values to get a $\pi$-loop $p'$ with required properties. Let us call such a $\pi$-path a *reduced* $\pi$-path.

Take now arbitrary congruences $\pi, \rho \in L$. Since $\Phi_{\mathbf{F}}$ is order-preserving, we have

$$\Phi_{\mathbf{F}}(\pi) \cap \Phi_{\mathbf{F}}(\rho) \supseteq \Phi_{\mathbf{F}}(\pi \cap \rho).$$

To prove the opposite inclusion take any reduced word $w = t_1 \cdots t_k \in \Phi_{\mathbf{F}}(\pi) \cap \Phi_{\mathbf{F}}(\rho)$. Then by the previous paragraph there is a reduced $\pi$-loop $p = e_1 e_2 \cdots e_k$ at $v$ such that $\nu(p) = w$. Hence $\nu(e_i) = t_i$ for every $i = 1, 2, \ldots, k$. Similarly, there is a reduced $\rho$-loop $q = f_1 f_2 \cdots f_l$ at $v$ such that $\nu(f_i) = t_i$ for every $i = 1, 2, \ldots, k$. Thus in particular, $(v, \alpha(e_1)) \in \pi$ and $(v, \alpha(f_1)) \in \rho$. Let us denote $\alpha(e_1) = a$ and $\alpha(f_1) = b$. Thus $a, b \in \mathrm{Dom}(t_1)$. Since $\mathrm{Dom}(t_1)$ is balanced with respect to $L$, there exists some $c \in \mathrm{Dom}(t_1)$ such that $(v, c) \in \pi \cap \rho$. Hence also $(a, c) \in \pi$ and $(b, c) \in \rho$. Since both $\pi$ and $\rho$ are congruences of $\mathcal{T}$, we get also $(at_1, ct_1) \in \pi$ and $(bt_1, ct_1) \in \rho$. Moreover, $(at_1, \alpha(e_2)) = (\omega(e_1), \alpha(e_2)) \in \pi$ and $(bt_1, \alpha(f_2)) = (\omega(f_1), \alpha(f_2)) \in \rho$, we get $(ct_1, \alpha(e_2)) \in \pi$ and $(ct, \alpha(f_2)) \in \rho$. Denote by $g_1$ the edge $(c, ct_1)$. Thus $g_1 e_2 \cdots e_k$ is another $\pi$-loop at $v$ and $g_1 f_2 \cdots f_k$ is another $\rho$-loop at $v$. Moreover, $(v, \alpha(g_1)) = (v, c) \in \pi \cap \rho$, and $\nu(g_1) = t_1$.

By repeating the same procedure with $ct_1$, $e_2$ and $f_2$ in place of $v$, $e_1$ and $f_1$, we get another edge $g_2$ that can replace $e_2$ in $p$ and $f_2$ in $q$ and satisfies $\nu(g_2) = \nu(e_2) = \nu(f_2)$ and $(\omega(g_1), \alpha(g_2)) \in \pi \cap \rho$. After $k$ steps we construct a $(\pi \cap \rho)$-loop $r = g_1 g_2 \cdots g_k$ at $v$ with $\nu(r) = w$. Hence $w \in \Phi_{\mathbf{F}}(\pi \cap \rho)$. □

Finally, connectedness of $\mathcal{T}$ also implies injectivity of $\Phi_{\mathbf{F}}$. Since the method of the proof will be also used in the proof of Theorem 8, we introduce some definitions here. If $\pi$ is a congruence of $\mathcal{T}$, we define the quotient $\mathcal{T}/\pi$ of $\mathcal{T}$ as follows. The twisting structure $\mathcal{T}/\pi$ will be defined on the set $\Omega/\pi$ of blocks of $\pi$. For any twist $t \in \mathcal{T}$ we define another twist $t_\pi$ on $\Omega/\pi$. The domain $\mathrm{Dom}(t_\pi)$ consists of all blocks of $\pi$ intersecting the domain $\mathrm{Dom}(t)$. If $a \in \mathrm{Dom}(t)$, then we define $[a]t_\pi = [at]$, where $[x]$ denotes the block of $\pi$ containing $x$. The definition of $t_\pi$ is correct since $\pi$ is a congruence of $\mathcal{T}$. Hence $t_\pi$ is also a bijection between two subsets of $\Omega/\pi$ and $\{t_\pi : t \in \mathcal{T}\}$ is a twisting structure on $\Omega/\pi$. Thus $\mathcal{T}/\pi$ is simply the quotient of the partial unary algebra $(\Omega, \mathcal{T})$ by the congruence $\pi$.

It is also useful to mention that the graph of $\mathcal{T}/\pi$ is naturally isomorphic to a quotient of the graph of $\mathcal{T}$. The vertices of $\mathbf{G}(\mathcal{T}_\pi)$ are blocks of the partition $\pi$ on $\Omega$. Whenever $a, b \in \mathrm{Dom}(t)$ are such that $(a, b) \in \pi$, then the two edges $(a, t)$

and $(b,t)$ of $\mathbf{G}(\mathcal{T})$ are identified into a single edge $([a],[a]t_\pi)$ of $\mathbf{G}(\mathcal{T}/\pi)$. If we assign to each edge $([a],[a]t_\pi)$ of $\mathbf{G}(\mathcal{T}/\pi)$ the value $t \in \mathbf{F}$, then we see that the values of $\pi$-loops at $v$ in the graph $\mathbf{G}(\mathcal{T})$ are in one-to-one correspondence with the values of ordinary loops at $[v]$ in the graph of $\mathbf{G}(\mathcal{T}/\pi)$.

**Lemma 7.** *If $\mathcal{T}$ is a connected twisting structure on $\Omega$ and $\pi < \rho$ two congruences of $\mathcal{T}$, then $\Phi_{\mathbf{F}}(\pi) \neq \Phi_{\mathbf{F}}(\rho)$.*

Proof: Let $(a,b) \in \rho \setminus \pi$. Since $\mathcal{T}$ is connected, there exist a reduced path $p = e_1 e_2 \cdots e_k$ in $\mathbf{G}(\mathcal{T})$ from $v$ to $a$ and a reduced path $q = f_1 f_2 \cdots f_l$ in $\mathbf{G}(\mathcal{T})$ from $v$ to $b$. Then $pq^{-1}$ is a $\rho$-loop in $\mathbf{G}(\mathcal{T})$ at $v$ but it is not a $\pi$-loop at $v$. If $pq^{-1}$ is not reduced, then there exists a twist $t \in \mathcal{T}$ such that $e_k = (at^{-1}, t)$ and $f_l = (bt^{-1}, t)$. Then also $(at^{-1}, bt^{-1}) \in \rho \setminus \pi$. So we can replace $p$ by $p' = e_1 e_2 \cdots e_{k-1}$ and $q$ by $q' = f_1 f_2 \cdots f_{l-1}$ to get a shorter $\rho$-loop $p'q'^{-1}$ at $v$ that is not a $\pi$-loop at $v$. Hence we may assume that the $\rho$-loop $r = pq^{-1}$ at $v$ is already reduced and it is not a $\pi$-loop. But then $\nu(r) \in \Phi_{\mathbf{F}}(\rho) \setminus \Phi_{\mathbf{F}}(\pi)$. □

Putting together previous results we get the following theorem.

**Theorem 8.** *Every lattice $L$ can be embedded into the subgroup lattice of a free group $\mathbf{F}$. If the lattice $L$ is finite, then the group $\mathbf{F}$ and all the subgroups of $\mathbf{F}$ representing elements of $L$ can be taken finitely generated.*

Proof: It remains to prove the second assertion. However, if the lattice $L$ is finite, then the twisting structure $\mathcal{T}$ of Lemma 4 can be taken finite by [Tů1]. But then the group $\Phi_{\mathbf{F}}(\pi)$ is isomorphic to the homotopy group of the graph $\mathbf{G}(\mathcal{T}/\pi)$, by the remarks preceding Lemma 7. Since the graph $\mathbf{G}(\mathcal{T}/\pi)$ is finite, its homotopy group is a finitely free group. □

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