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# Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular boundary points

Jan Malý*

*Abstract.* Let $u$ be a weak solution of a quasilinear elliptic equation of the growth $p$ with a measure right hand term $\mu$. We estimate $u(z)$ at an interior point $z$ of the domain $\Omega$, or an irregular boundary point $z \in \partial\Omega$, in terms of a norm of $u$, a nonlinear potential of $\mu$ and the Wiener integral of $\mathbf{R}^n \setminus \Omega$. This quantifies the result on necessity of the Wiener criterion.

*Keywords:* elliptic equations, Wiener criterion, nonlinear potentials, measure data

*Classification:* 35J67, 35J70, 35J65

## 1. Introduction

We study quasilinear elliptic equations of type

$$-\operatorname{div}\mathbf{A}(x,u,\nabla u)+\mathbf{B}(x,u,\nabla u)=\mu\,, \tag{1.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are Carathéodory functions (precise conditions depending on a growth exponent $p\in(1,\infty)$ will be given later) and $\mu\in(W_0^{1,p}(\Omega))^*$ is a nonnegative Radon measure. We refer to $(1.1_0)$ if $\mu=0$.

The model equation for (1.1) is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=\mu, \tag{1.2}$$

with $\lambda\in\mathbf{R}$. Sometimes we mention monotone type equations, by this we will understand equations satisfying the structure conditions of [13] (unweighted case). These equations satisfy additional assumptions which guarantee existence and uniqueness results.

We will work with the integrals

$$\mathbf{w}_p(x,E)=\int_0^{r_0}\Big(\frac{\operatorname{cap}_p(E\cap B(x,r),r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r} \tag{1.3}$$

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and

$$\mathbf{W}_p^\mu(x) = \int_0^{r_0} \Big(\frac{\mu(B(x,r))}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r}. \tag{1.4}$$

The function $\mathbf{W}_p^\mu$ is a kind of nonlinear potential of the measure $\mu$. These potentials were introduced by Adams and Meyers [3], Hedberg [9] and Hedberg and Wolff [10]. For more information on $\mathbf{W}_p$ potentials, we refer to the recent monograph by Adams and Hedberg [2].

We present pointwise estimates for subsolutions of (1.1) in terms of $\mathbf{W}_p^\mu$ and $\mathbf{w}_p(\cdot, \mathbf{R}^n \setminus \Omega)$.

In the interior case, and with $\mu = 0$, the presented estimate is a version of the Trudinger's Harnack inequality for subsolutions [27]. The interior estimate with a nontrivial $\mu$ has been proved for monotone type equations by Kilpeläinen and Malý [16]. Notice that lower interior estimates for supersolutions of (1.1) in terms of $\mathbf{W}_p^\mu$, generalizing Trudinger's Harnack inequality for supersolutions, are also valid, see Kilpeläinen and Malý [14] (for monotone type equations), Malý [20], and Malý and Ziemer [23]. Related, but different results are due to Rakotoson and Ziemer [25], Lieberman [17] and Adams [1].

Let $u_0 \in W^{1,p}(\Omega)$ and $u$ be a solution of $(1.1_0)$. We say that $u$ solves the Dirichlet problem with the boundary data $u_0$ if $u - u_0 \in W_0^{1,p}(\Omega)$. A point $z \in \partial\Omega$ is said to be regular for the equation $(1.1_0)$ if

$$\lim_{x\to z,\, x\in\Omega} u(x) = u_0(z)$$

whenever $u \in \mathcal{C}(\Omega)$ is a solution of the Dirichlet problem with boundary data $u_0 \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Wiener [28] showed that $z$ is regular for the Laplace equation if and only if the classical Wiener criterion is satisfied. This more or less says that $z$ is regular for the Laplace equation if and only if the Wiener integral $\mathbf{w}_2(z, \mathbf{R}^n \setminus \Omega)$ diverges. Littman, Stampacchia, Weinberger [19] proved that the same condition applies to linear elliptic divergence form equations with discontinuous bounded measurable coefficients. If $p \neq 2$, we say that the Wiener condition is satisfied at $z$ if $\mathbf{w}_p(z, \mathbf{R}^n \setminus \Omega)$ diverges, i.e. if $\mathbf{R}^n \setminus \Omega$ is not $p$-thin at $\Omega$. Maz'ya [21] established the sufficiency of the Wiener criterion under simpler structure assumptions. Gariepy and Ziemer [8] proved the sufficiency in the general case of equation $(1.1_0)$.

The Wiener criteria established by Wiener [28] and Littman, Stampacchia, Weinberger [19] were presented as both necessary and sufficient. On the other hand, the sufficient condition by Maz'ya [21] waited a longer time for its necessity counterpart. For a special class of equations, some necessary conditions differing in an exponent from the sufficient conditions were proved by Skrypnik [26]. The necessity of the Wiener condition for equations of the monotone type was shown by Lindqvist and Martio [18] and Heinonen and Kilpeläinen [11] with the restriction $p > n - 1$. For all $p \in (1, \infty)$, it was proved by Kilpeläinen and Malý in [16].

The estimate given in the present paper implies in some sense the necessity of the Wiener criterion for equations of type $(1.1_0)$ and quantifies the pointwise behavior of solutions at irregular points.

For a wider information about the topic we refer to the prepared monograph [23] by Malý and Ziemer. For consequences and relations to A-superharmonic functions in nonlinear potential theory we refer also to the papers by Kilpeläinen and Malý [16], Heinonen, Kilpeläinen and Martio [12] and to the monograph [13] by Heinonen, Kilpeläinen and Martio.

## 2. Preliminaries

In what follows, $\Omega$ is an open subset of $\mathbf{R}^n$ and $p$ is an exponent in $(1,n]$. We write $C, C'$ etc. for various constants (they may differ from line to line). We denote by $B(z,r)$ the open ball in $\mathbf{R}^n$ with center at $z$ and radius $r$. If $B = B(z,r)$, then $2B$ means the ball $B(z,2r)$. We denote by $\mathcal{C}_c^\infty(\Omega)$ the set of all infinitely differentiable functions with a compact support in $\Omega$. The norm in the Lebesgue space $L^p(\Omega)$, resp. in the Sobolev space $W^{1,p}(\Omega)$ is denoted by $\|...\|_p$, resp. $\|...\|_{1,p}$. We use $|E|$ for the Lebesgue measure of the set $E$.

We define the *p-capacity* of a set $E \subset \mathbf{R}^n$ by $\operatorname{cap}_p E = \operatorname{cap}_p(E,1)$, where

$$\operatorname{cap}_p(E,r) = \inf\Big\{ \int_{\mathbf{R}^n} \big(|\nabla\varphi|^p + r^{-p}|\varphi|^p\big)\,dx \colon \varphi \in W^{1,p}(\mathbf{R}^n),$$
$$\varphi \geq 1 \text{ on an open set containing } E\Big\}$$

This scale of capacities is natural in connection with the Wiener criterion; for $E \subset B$ it is equivalent to the "condenser capacity" of $E$ w.r.t. $2B$, cf. [13].

A set $U \subset \mathbf{R}^n$ is said to be *p-quasiopen* if for each $\varepsilon > 0$ there is an open set $G \subset \mathbf{R}^n$ such that $\operatorname{cap}_p G < \varepsilon$ and $U \cup G$ is open. Similarly, a function $u$ is said to be *p-quasicontinuous* on $\Omega$ if for each $\varepsilon > 0$ there is an open set $G \subset \mathbf{R}^n$ such that $\operatorname{cap}_p G < \varepsilon$ and $u|\Omega \setminus G$ is continuous.

We use the abbreviation $p$-q.e. (*p-quasi everywhere*) for the phrase "except a set of $p$-capacity zero". We say that a set $E \subset \mathbf{R}^n$ is *p-thin* at a point $z \in \mathbf{R}^n$ if the *Wiener integral* $\mathbf{w}_p(z,E)$ converges. The *p-fine closure* adds to every set $E$ the set of all points where $E$ is not $p$-thin. This introduces the *p-fine topology.*

Notice that every $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ has a $p$-quasicontinuous representative (see Federer and Ziemer [5], Maz'ya and Khavin [22], Meyers [24], Frehse [6] and that a function $u$ on $\Omega$ is $p$-quasicontinuous if and only if it is $p$-finely continuous $p$-q.e. (Fuglede [7], Brelot [4], Hedberg and Wolff [10]).

Due to Poincaré's inequality and approximation arguments,

$$\operatorname{cap}_p(E,r) \leq C \int_{B(x_0,2r)} |\nabla\psi|^p\,dx$$

holds whenever $E \subset B(x_0,r)$, $\psi \in W_0^{1,p}(B(x_0,2r))$, $\psi$ is $p$-quasicontinuous and $\psi \geq 1$ $p$-q.e. on $E$.

Now, let us state our assumptions concerning the equation (1.1). We suppose that the functions $\mathbf{A}\colon \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ and $\mathbf{B}\colon \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ are Borel measurable and satisfy the following structure conditions:

$$\begin{aligned}
&|\mathbf{A}(x,\zeta,\xi)| \le a_1|\xi|^{p-1} + a_2|\zeta|^{p-1} + a_3,\\
&|\mathbf{B}(x,\zeta,\xi)| \le b_1|\xi|^{p-1} + b_2|\zeta|^{p-1} + b_3 + b_0|\xi|^p,\\
&\mathbf{A}(x,\zeta,\xi)\cdot\xi \ge c_1|\xi|^p - c_2|\zeta|^p - c_3, \quad c_1>0,
\end{aligned}\tag{2.1}$$

where $a_i$, $b_i$, $c_i$ are nonnegative constants. We write $b = b_0/c_1$. The model example $\mathbf{A}(x,\zeta,\xi) = |\xi|^{p-2}\xi$, $\mathbf{B}(x,\zeta,\xi) = \lambda|\zeta|^{p-2}\zeta$ leads to (1.2).

We say that $u$ is a *subsolution* (frequently termed a "weak subsolution") of (1.1) in $\Omega$ if $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$, $u$ is $p$-quasicontinuous (i.e. we admit $p$-quasicontinuous representatives only) and

$$\int_\Omega \Big(\mathbf{A}(x,u,\nabla u)\cdot\nabla\varphi + \mathbf{B}(x,u,\nabla u)\,\varphi\Big)\,dx \le \int_\Omega \varphi\,d\mu \tag{2.2}$$

holds for all nonnegative "test functions" $\varphi\in\mathcal{C}^\infty_c(\Omega)$. Similarly we define *solutions* using the equality sign.

## 3. Main estimate

We consider an exponent

$$\gamma\in(p-1, n(p-1)/(n-p+1))$$

and write

$$\tau = \frac{\gamma}{p-1}, \qquad q = \frac{p\gamma}{p-\tau}.$$

Notice that $\tau>1$ and $q>p$. Let $\Omega$ be an open set and $R_0>0$ a fixed radius. We consider a fixed equation of type (1.1). We will denote by $C$ a general constant (not necessarily the same at different occurrences) depending only on $n,p,\gamma,R_0$, on the upper bound of $b_0u$ and on the structure constants.

**3.1 Lemma.** *Let $u\in W^{1,p}(\Omega)$ be a subsolution of $-\operatorname{div}\mathbf{A}+\mathbf{B}=\mu$ in $\Omega$. Suppose that either $u$ is upper bounded or $b_0=0$. Let $\ell\in[0,\infty)$, $\Phi$ be a nonnegative bounded Borel measurable function on $\mathbf{R}$ which vanishes on $(-\infty,\ell)$ and $\lambda$ be the $L^1$-norm of $\Phi$. Let $\omega\in W^{1,p}_0(\Omega)$, $0\le\omega\le1$. Then*

$$\begin{aligned}
&\int_\Omega \Phi(u)\,|\nabla u|^{\,p}\,\omega^p\,dx\\
&\quad\le C\int_{\Omega\cap\{u>\ell\}} \Phi(u)(1+u^p)\,\omega^p\,dx\\
&\qquad + C\lambda\,\Big(\int_{\Omega\cap\{u>\ell\}}\big(|\nabla u|^{\,p-1}+u^{p-1}+1\big)\,\omega^{p-1}(\omega+|\nabla\omega|)\,dx + \mu(\{\omega>0\})\Big).
\end{aligned}$$

Proof: We write

$$\Psi(t) = \int_0^t \Phi(s)\, ds,$$
$$L = \Omega \cap \{u > \ell\}.$$

Using the test function

$$\varphi = \Psi(u)\, e^{bu}\, \omega^p$$

with

$$\begin{aligned}\nabla\varphi = {}& \Phi(u)\, \nabla u\, e^{bu}\, \omega^p \\ & + b\, \Psi(u)\, \nabla u\, e^{bu}\, \omega^p \\ & + p\, \Psi(u)\, e^{bu}\, \omega^{p-1} \nabla\omega\end{aligned}$$

we obtain

$$\begin{aligned} & \int_L \mathbf{A}(x,u,\nabla u)\cdot \nabla u\, \Phi(u)\ e^{bu}\, \omega^p\, dx \\ & + b \int_L \mathbf{A}(x,u,\nabla u)\cdot \nabla u\, \Psi(u)\ e^{bu}\, \omega^p\, dx \\ & + p \int_L \mathbf{A}(x,u,\nabla u)\cdot \Psi(u)\, e^{bu}\, \omega^{p-1}\nabla\omega\, dx \\ & + \int_L \mathbf{B}(x,u,\nabla u)\, \Psi(u)\, e^{bu}\, \omega^p\, dx \\ & \quad \le \int_L \Psi(u)\, e^{bu}\, \omega^p\, d\mu. \end{aligned} \tag{3.1}$$

Taking the structure into account, we get

$$\begin{aligned} & \int_L \mathbf{A}(x,u,\nabla u)\cdot \nabla u\, \Phi(u)\, e^{bu}\, \omega^p\, dx \\ & \quad \ge \int_L \big(c_1\, |\nabla u|^{\,p} - c_2 u^p - c_3\big)\, \Phi(u)\, e^{bu}\, \omega^p\, dx\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} & - b \int_L \mathbf{A}(x,u,\nabla u)\cdot \nabla u\, \Psi(u)\, e^{bu}\, \omega^p\, dx \\ & \quad \le -bc_1 \int_L |\nabla u|^{\,p}\, \Psi(u)\, e^{bu}\, \psi^p \eta^p\, dx \\ & \qquad + b \int_L \big(c_2 u^p + c_3\big) \Psi(u)\, e^{bu}\, \omega^p\, dx\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & - \int_L \mathbf{A}(x,u,\nabla u)\cdot \Psi(u)\, e^{bu}\, \omega^{p-1} \nabla\omega\, dx \\ & \le \int_L \Big(a_1\, |\nabla u|^{\,p-1} + a_2 u^{p-1} + a_3\Big)\, \Psi(u)\, e^{bu}\, \omega^{p-1} \nabla\omega\, dx\,, \end{aligned} \tag{3.4}$$

and

$$
(3.5)\qquad
\begin{aligned}
&-\int_L \mathbf{B}(x,u,\nabla u)\,\Psi(u)\,e^{bu}\,\omega^p\,dx\\
&\le \int_L \Big(b_1\,|\nabla u|^{\,p-1}+b_2u^{p-1}+b_3\Big)\,\Psi(u)\,e^{bu}\,\omega^p\,dx\\
&\quad + b_0\int_L |\nabla u|^{\,p}\,\Psi(u)\,e^{bu}\,\omega^p\,dx.
\end{aligned}
$$

From (3.1)–(3.5) we obtain

$$
(3.6)\qquad
\begin{aligned}
&c_1\int_L \Phi(u)\,|\nabla u|^{\,p}\,e^{bu}\,\omega^p\,dx\\
&\qquad + bc_1\int_L \Psi(u)\,|\nabla u|^{\,p}\,e^{bu}\,\omega^p\,dx\\
&\le \int_L \Phi(u)\Big(c_2u^p+c_3\Big)\,e^{bu}\,\omega^p\,dx\\
&\qquad + \int_L \Psi(u)\Big(p\,\big(a_1\,|\nabla u|^{\,p-1}+a_2u^{p-1}+a_3\big)\,|\nabla\omega|\\
&\qquad + \big(b_1\,|\nabla u|^{\,p-1}+(c_2bu+b_2)u^{p-1}+c_3b+b_3\big)\,\omega\Big)\,e^{bu}\,\omega^{p-1}\,dx\\
&\qquad + b_0\int_L \Psi(u)\,|\nabla u|^{\,p}\,e^{bu}\,\omega^p\,dx\\
&\le \int_L \Psi(u)\omega^p\,d\mu.
\end{aligned}
$$

Since $b_0=bc_1$, $bu\le C$, $\omega\le 1$ and $\Psi\le\lambda$, it follows

$$
\begin{aligned}
&\int_L \Phi(u)\,|\nabla u|^{\,p}\,\omega^p\,dx\\
&\quad\le C\int_L \Phi(u)(1+u^p)\,\omega^p\,dx\\
&\qquad + C\lambda\,\Big(\int_{\Omega\cap\{u>\ell\}}\big(\,|\nabla u|^{\,p-1}+u^{p-1}+1\big)\,\omega^{p-1}(\omega+|\nabla\omega|)\,dx+\mu(\{\omega>0\})\Big)
\end{aligned}
$$

as required. □

**3.2 Lemma.** *Let $u\in W^{1,p}(\Omega)$ be a subsolution of $-\operatorname{div}\mathbf{A}+\mathbf{B}=\mu$ in $\Omega$. Suppose that either $u$ is upper bounded or $b_0=0$. Let $B=B(x_0,r)$, where $0<r<R_0$, be an open ball in $\mathbf{R}^n$. Let $\eta,\varphi,\psi\in W^{1,p}(B)$. Suppose that $0\le\eta\le 1$, $0\le\varphi\le 1$, $0\le\psi\le 1$, $\eta\psi\in W^{1,p}(B\cap\Omega)$, $(1-\varphi)(1-\psi)=0$ and $\nabla\eta\le 5/r$. Suppose that $\ell\ge 0$.*

(a) *If* $\delta > 0$, *then*

$$\int_L |\nabla w_\delta|^p \, dx \le C\Big(\delta^{-p} r^n (1+\ell^p)$$
$$+ r^{-p} \int_{B\cap\{u>\ell\}\cap\{\varphi<1\}} \Big(1 + \frac{u-\ell}{\delta}\Big)^\gamma \, dx$$
$$+ \delta^{1-p} \, \mu(B(x_0, r))$$
$$+ \delta^{1-p} (1 + \|u\|_\infty)^{p-1} \int_B (r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p) \, dx\Big),$$

*where*

$$w_\delta = \Big(\Big(1 + \frac{(u-\ell)^+}{\delta}\Big)^{\gamma/q} - 1\Big)\psi\eta.$$

(b) *There is a constant* $\kappa > 0$, *depending only on* $n, p, \gamma, R_0$, *on the upper bound of* $b_0 u$ *and on the structure constants, such that*

$$\Big(r^{-n} \int_{B\cap\Omega\cap\{u>\ell\}} (u-\ell)^\gamma \psi^q \eta^q \, dx\Big)^{(p-1)/\gamma}$$
$$\le C\Big( r^{p-1}(1+\ell)^{p-1}$$
$$+ r^{p-n} \, \mu(B(x_0, r))$$
$$+ (1 + \|u\|_\infty)^{p-1} \, r^{p-n} \int_B (r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p) \, dx\Big),$$

*provided that*

$$|B \cap \{u > \ell\} \cap \{\varphi < 1\}| \le (2r)^n \kappa \tag{3.7}$$

*and*

$$\int_{B\cap\{u>\ell\}\cap\{\varphi<1\}} (u-\ell)^\gamma \, dx \le 2^{n+\gamma} \int_{B\cap\Omega\cap\{u>\ell\}} (u-\ell)^\gamma \psi^q \eta^q \, dx. \tag{3.8}$$

Proof: (a) We write

$$\begin{aligned}
&\omega = \psi\eta, \\
&\sigma = \omega\varphi, \\
&v = \frac{(u-\ell)^+}{\delta}, \\
&M = 1 + \|u\|_\infty, \\
&L = B \cap \Omega \cap \{u > \ell\}, \\
&E = L \cap \{\varphi < 1\}, \\
&F = L \cap \{\varphi = 1\}.
\end{aligned}$$

Note that

$$w_\delta = \big((1+v)^{\gamma/q}-1\big)\omega,$$

$$\nabla w_\delta = \frac{\gamma}{q}(1+v)^{-\tau/p}\nabla v\ \omega + \big((1+v)^{\gamma/q}-1\big)\,\nabla\omega.$$

Since

$$\begin{aligned}
&\big((1+v)^{\gamma/q}-1\big)^p \le C\min(v^{p-\tau},v^p) \le C\min\big((1+v)^\gamma, v^{p-1}\big),\\
&v^{p-1} \le \delta^{1-p}u^{p-1} \le \delta^{1-p}M^{p-1},\\
&\omega = \eta \text{ on } E,\\
&\omega = \sigma \text{ on } F,
\end{aligned} \tag{3.9}$$

it follows

$$\begin{aligned}
&\int_L |\nabla w_\delta|^p\,dx\\
&\quad\le C\Big(\int_E (1+v)^\gamma|\nabla\eta|^p\,dx + M^{p-1}\delta^{1-p}\int_F|\nabla\sigma|^p\,dx\Big)\\
&\qquad + \delta^{-p}\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx.
\end{aligned} \tag{3.10}$$

We use Lemma 3.1 with

$$\Phi(t) = \begin{cases} (1+\frac{(t-\ell)^+}{\delta})^{-\tau}, & t>\ell,\\ 0, & t\le\ell.\end{cases}$$

Then the $L^1$-norm of $\Phi$ is bounded by $(\tau-1)^{-1}\,\delta$. We get

$$\begin{aligned}
&\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx\\
&\quad\le C\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\,dx\\
&\qquad + C\delta\Big(\int_L\big(\,|\nabla u|^{\,p-1}+u^{p-1}+1\big)\ \omega^{p-1}(\omega+|\nabla\omega|)\,dx + \mu(B)\Big).
\end{aligned} \tag{3.11}$$

We estimate

$$\begin{aligned}
(1+u^p)(1+v)^{-\tau} &\le (1+u^p)(1+v)^{-1} \le C(1+\ell^p+\delta^p v^p)(1+v)^{-1}\\
&\le C(1+\ell^p+\delta^p v^{p-1})
\end{aligned}$$

Using (3.9) it follows

$$\begin{aligned}
&\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\,dx\\
&\quad\le Cr^n(1+\ell^p) + \delta^p\int_L v^{p-1}\omega^p\,dx\\
&\quad\le C\Big(r^n(1+\ell^p) + \delta M^{p-1}\int_F \sigma^p\,dx + \delta^p\int_E (1+v)^\gamma\omega^p\,dx\Big).
\end{aligned} \tag{3.12}$$

Choose $\varepsilon > 0$. Young's inequality yields

$$
\begin{aligned}
&(1+u^{p-1}+|\nabla u|^{p-1})\omega^{p-1}(\omega+|\nabla\omega|) \\
(3.13)\qquad &\leq C\frac{\varepsilon}{\delta}(1+v)^{-\tau}(1+u^p+|\nabla u|^p)\omega^p + C\Big(\frac{\varepsilon}{\delta}\Big)^{1-p}(1+v)^{\gamma}(\omega^p+|\nabla\omega|^p).
\end{aligned}
$$

Recall that $\omega=\eta$ on $E$. We infer from (3.13) that

$$
\begin{aligned}
&\int_E \big(|\nabla u|^{p-1}+u^{p-1}+1\big)\ \omega^{p-1}(\omega+|\nabla\omega|)\,dx \\
(3.14)\qquad &\leq C\frac{\varepsilon}{\delta}\int_L (1+v)^{-\tau}(1+u^p+|\nabla u|^p)\omega^p\,dx \\
&\quad + C\Big(\frac{\varepsilon}{\delta}\Big)^{1-p}\int_E (1+v)^{\gamma}(\eta^p+|\nabla\eta|^p)\,dx.
\end{aligned}
$$

Now, we will estimate the integration on $F$. We use Lemma 3.1 again with $\Phi$ being the characteristic function of the interval $[\ell, M]$ and with $\sigma$ instead of $\omega$. Then the $L^1$-norm of $\Phi$ is bounded by $M$ and we get

$$
\begin{aligned}
&\int_L |\nabla u|^p\,\sigma^p\,dx \\
&\leq CM\Big(\int_L \big(|\nabla u|^{p-1}+u^{p-1}+1\big)\ \sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx+\mu(B)\Big) \\
(3.15)\qquad &\quad + C\int_L (1+u^p)\,\sigma^p\,dx \\
&\leq CM^p\int_B (\sigma^p+|\nabla\sigma|^p)\,dx \\
&\quad + CM\Big(\int_L |\nabla u|^{p-1}\sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx+\mu(B)\Big).
\end{aligned}
$$

Choose $\varepsilon_1 > 0$. A use of Young's inequality yields

$$
\begin{aligned}
&|\nabla u|^{p-1}\,\sigma^{p-1}(\sigma+|\nabla\sigma|) \\
(3.16)\qquad &\leq \frac{\varepsilon_1}{M}\ |\nabla u|^p\sigma^p + C\Big(\frac{\varepsilon_1}{M}\Big)^{1-p}(\sigma^p+|\nabla\sigma|^p).
\end{aligned}
$$

From (3.15) and (3.16) we get

$$
\begin{aligned}
&\int_L \big(|\nabla u|^{p-1}+u^{p-1}+1\big)\ \sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx \\
&\leq CM^{p-1}\int_B (\sigma^p+|\nabla\sigma|^p)+\int_L |\nabla u|^{p-1}\sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx \\
(3.17)\qquad &\leq C(1+\varepsilon_1^{1-p})M^{p-1}\int_B (\sigma^p+|\nabla\sigma|^p)\,dx + C\frac{\varepsilon_1}{M}\int_L |\nabla u|^p\sigma^p\,dx \\
&\leq C(1+\varepsilon_1+\varepsilon_1^{1-p})M^{p-1}\int_B (\sigma^p+|\nabla\sigma|^p)\,dx \\
&\quad + C\varepsilon_1\int_L |\nabla u|^{p-1}\sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx + C\varepsilon_1\mu(B).
\end{aligned}
$$

Using $\varepsilon_1$ small enough, by a cancellation we obtain

$$\begin{aligned}\int_L \big( |\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \sigma^{p-1}(\sigma + |\nabla\sigma|)\,dx \\ \le C\Big(M^{p-1}\int_B (\sigma^p + \ |\nabla\sigma|^{\,p})\,dx + \mu(B)\Big).\end{aligned}$$

As $\sigma = \omega$ on $F$, it follows

$$\begin{aligned}\int_F \big( |\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \omega^{p-1}(\omega + |\nabla\omega|)\,dx \\ \le C\Big(M^{p-1}\int_B (\sigma^p + \ |\nabla\sigma|^{\,p})\,dx + \mu(B)\Big).\end{aligned}\tag{3.18}$$

From (3.11), (3.12), (3.13), (3.14) and (3.18) we deduce that

$$\begin{aligned}&\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx \\ &\quad\le C\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\,dx \\ &\qquad + C\delta\Big(\int_L \big( |\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \omega^{p-1}(\omega + |\nabla\omega|)\,dx + \mu(B)\Big) \\ &\quad\le C\varepsilon\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx \\ &\qquad + C(1+\varepsilon)\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\,dx \\ &\qquad + C\delta^p\varepsilon^{1-p}\int_E (1+v)^{\gamma}(\eta^p + |\nabla\eta|^p)\,dx \\ &\qquad + C\delta\mu(B) + \delta M^{p-1}\int_L (\sigma^p + |\nabla\sigma|^p)\,dx \\ &\quad\le C\varepsilon\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx \\ &\qquad + C(1+\varepsilon+\varepsilon^{1-p})\Big(r^n(1+\ell^p) + \delta\mu(B) + \delta M^{p-1}\int_L (\sigma^p + |\nabla\sigma|^p)\,dx \\ &\qquad + \delta^p\int_E (1+v)^{\gamma}(\eta^p + |\nabla\eta|^p)\,dx\Big).\end{aligned}$$

Choosing $\varepsilon$ small enough it follows

$$\begin{aligned}\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx \\ \le C\Big(r^n(1+\ell^p) + \delta^p\int_E (1+v)^{\gamma}(\eta^p + |\nabla\eta|^p)\,dx \\ + \delta M^{p-1}\int_L (\sigma^p + |\nabla\sigma|^p) + \delta\mu(B)\Big).\end{aligned}\tag{3.19}$$

From (3.10) and (3.19) we get

$$\begin{aligned}\int_L |\nabla w_\delta|^p\,dx &\le C\delta^{-p}r^n(1+\ell^p) + C\int_E (1+v)^\gamma(\eta^p+|\nabla\eta|^p)\,dx\\ &\quad + C\delta^{1-p}\Big(M^{p-1}\int_L(\sigma^p+|\nabla\sigma|^p)+\mu(B)\Big).\end{aligned}\tag{3.20}$$

Since

$$\int_E (1+v)^\gamma(\eta^p+|\nabla\eta|^p)\,dx \le Cr^{-p}\int_E (1+v)^\gamma\,dx$$

and

$$\sigma^p+|\nabla\sigma|^p \le Cr^{-p}\varphi^p+|\nabla\varphi|^p+|\nabla\psi|^p,$$

it follows that

$$\begin{aligned}\int_L |\nabla w_\delta|^p\,dx &\le Cr^{-p}\int_E (1+v)^\gamma\,dx + C\delta^{-p}r^n(1+\ell)^p\\ &\quad + C\delta^{1-p}\Big(M^{p-1}\int_L(r^{-p}\varphi^p+|\nabla\varphi|^p+|\nabla\psi|^p)\,dx+\mu(B)\Big).\end{aligned}$$

This proves the part (a).

(b) We consider $\kappa>0$; its choice will be specified latter. We continue to use the notation introduced in the course of the proof of (a) with the choice

$$\delta := \Big(\frac{1}{\kappa r^n}\int_L (u-\ell)^\gamma\omega^q\,dx\Big)^{1/\gamma}.$$

Notice that

$$\kappa = r^{-n}\int_L v^\gamma\omega^q\,dx.\tag{3.21}$$

By (3.7) and (3.21),

$$\begin{aligned}2\kappa r^n &= 2\int_L v^\gamma\omega^q\,dx\\ &\le 2^{-n}\int_L \omega^q\,dx + \int_{L\cap\{v^\gamma\ge 2^{-n-1}\}} v^\gamma\omega^q\,dx\\ &\le 2^{-n}\big(|E|+\int_F\sigma^q\,dx\big) + \int_{L\cap\{v^\gamma\ge 2^{-n-1}\}} v^\gamma\omega^q\,dx\\ &\le \kappa r^n + \int_{L\cap\{v^\gamma\ge 2^{-n-1}\}} v^\gamma\omega^q\,dx + \int_B \sigma^q\,dx\end{aligned}$$

and thus

$$\begin{aligned}\kappa r^n &\le \int_{B\cap\{v^\gamma\ge 2^{-n-1}\}\}} v^\gamma\omega^q\,dx + \int_B\sigma^q\,dx\\ &\le C\Big(\int_L w_\delta^q\,dx + \int_B \sigma^q\,dx\Big).\end{aligned}$$

We apply the Sobolev inequality to the functions $w_\delta$ and $\sigma$ and obtain

$$\begin{aligned}\kappa^{p/q} &\le \big(r^{-n}\int_{B\cap\Omega} w_\delta^q\,dx + r^{-n}\int_B \sigma^q\,dx\big)^{p/q}\\ &\le Cr^{p-n}\Big(\int_{B\cap\Omega} |\nabla w_\delta|^{\,p}\,dx + \int_B |\nabla\sigma|^p\,dx\Big).\end{aligned}\tag{3.22}$$

From (a) we obtain

$$\begin{aligned}r^{n-p}\kappa^{p/q} &\le C\Big(\int_L |\nabla w_\delta|^{\,p}\,dx + \int_B |\nabla\sigma|^p\,dx\Big)\\ &\le Cr^{-p}\int_E (1+v)^\gamma\,dx + C\delta^{-p}r^n(1+\ell)^p\\ &\quad + C\delta^{1-p}\Big((\delta+M)^{p-1}\int_L(\sigma^p+|\nabla\sigma|^p)\,dx + \mu(B)\Big).\end{aligned}\tag{3.23}$$

By (3.7) and (3.8),

$$\begin{aligned}\int_E (1+v)^\gamma\,dx &\le C(|E| + \int_E v^\gamma\,dx)\\ &\le C(|E| + \int_L v^\gamma\,\omega^q\,dx)\\ &\le C\kappa r^n.\end{aligned}\tag{3.24}$$

We infer from (3.23) and (3.24) that

$$\begin{aligned}\kappa^{p/q} &\le C_1\kappa + C\delta^{-p}r^p(1+\ell)^p\\ &\quad + C\delta^{1-p}r^{p-n}\Big((\delta+M)^{p-1}\int_L(r^{-p}\sigma^p+|\nabla\sigma|^p)\,dx\\ &\quad + \mu(B)\Big)\end{aligned}$$

holds for some constant $C_1$. If we specify $\kappa$ to be so small that $\kappa^{p/q} - C_1\kappa > 0$, we obtain

$$\begin{aligned}1 &\le C\delta^{-p}r^p(1+\ell)^p\\ &\quad + C\delta^{1-p}r^{p-n}\Big((\delta+M)^{p-1}\int_L(r^{-p}\sigma^p+|\nabla\sigma|^p)\,dx + \mu(B)\Big).\end{aligned}$$

It follows that either

$$1 \le C\delta^{-p}r^p(1+\ell)^p$$

or

$$1 \le C\delta^{1-p}r^{p-n}\Big((\delta+M)^{p-1}\int_L(r^{-p}\sigma^p+|\nabla\sigma|^p)\,dx + \mu(B)\Big).$$

Anyway we deduce

$$\begin{aligned}
&\Big(\frac{1}{\kappa r^n}\int_L (u-\ell)^\gamma \psi^q \eta^q\,dx\Big)^{(p-1)/\gamma} \\
&\quad = \delta^{p-1} \le Cr^{p-1}(1+\ell)^{p-1} \\
&\qquad + Cr^{p-n}\Big((\delta+M)^{p-1}\int_B (r^{-p}\sigma^p + |\nabla\sigma|^p)\,dx + \mu(B)\Big).
\end{aligned}$$

Taking into account the estimates

$$r^{-p}\sigma^p + |\nabla\sigma|^p \le C(r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p)$$

and

$$\delta \le CM,$$

we conclude the proof. □

**3.3 Theorem.** *Let $u$ be a subsolution of $-\operatorname{div}\mathbf{A}+\mathbf{B}=\mu$ in $\Omega$. Suppose that either $u$ is upper bounded or $b_0=0$. Then*

$$\begin{aligned}
&p\text{-fine-}\limsup_{x\to z} u(x) \le C\Bigg(\Big(r_0^{-n}\int_{B(x_0,r_0)\cap\Omega\cap\{u>0\}} u^\gamma\,dx\Big)^{1/\gamma} \\
&\quad + \int_0^{r_0}\Big(\frac{\mu B(x_0,r)}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r} \\
&\quad + (1+\|u\|_\infty)\int_0^{2r_0}\Big(\frac{\operatorname{cap}_p(B(x_0,r)\setminus\Omega,\,r)}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r}\Bigg)
\end{aligned} \tag{3.25}$$

*for all $x_0\in\overline{\Omega}$ and $r_0\le R_0$.*

Proof: We denote $M=1+\|u\|_\infty$ and set $\kappa\in(0,1)$ to be the constant from Lemma 3.2. We set $r_j=2^{-j}r_0$ and pick cutoff functions $\eta_j$ such that $0\le\eta_j\le 1$, $\eta_j=0$ outside $B(x_0,r_j)$, $\eta_j=1$ on $B(x_0,r_{j+1})$ and $|\nabla\eta_j|\le 5/r_j$. Further, we find functions $g_j\in W^{1,p}(\mathbf{R}^n)$ such that $0\le g_j\le 1$, the interior of $\{g_j=1\}$ contains $B(x_0,r_j)\setminus\Omega$ and

$$\int_{\mathbf{R}^n}(r_j^{-p}g_j^p + |\nabla g_j|^p)\,dx \le C\operatorname{cap}_p(B(x_0,r_j)\setminus\Omega,\,r_j)\,. \tag{3.26}$$

We denote

$$\begin{aligned}
\psi_j &= \min(1,\,(2-3g_j)^+),\\
\varphi_j &= \min(1,\,3g_j+3g_{j-1}),\quad j\ge 1,\\
B_j &= B(x_0,r_j),\\
L_j &= B_j\cap\Omega\cap\{u\ge\ell_j\}\\
E_j &= L_j\cap\{\varphi_j<1\},\\
F_j &= L_j\cap\{\varphi_j=1\}.
\end{aligned}$$

Then by (3.26),

$$
\begin{aligned}
&\int_{B_j} \big(r_j^{-p}\varphi_j^p + |\nabla\varphi|_j^{\,p}\big)\,dx \le C\,\operatorname{cap}_p(B_{j-1}\setminus\Omega,\, r_j),\\
&\int_{B_j} |\nabla\psi|_j^{\,p}\,dx \le C\,\operatorname{cap}_p(B_j\setminus\Omega,\, r_j).
\end{aligned} \tag{3.27}
$$

We define recursively $\ell_0 = 0$,

$$
\ell_{j+1} = \ell_j + \Big(\frac{1}{\kappa r_j^n}\int_{L_j} (u-\ell_j)^\gamma \psi_j^q \eta_j^q\,dx\Big)^{1/\gamma}, \qquad j = 0,1,2,\dots
$$

We write

$$
\delta_j = \ell_{j+1} - \ell_j.
$$

We claim that, for $j \ge 1$,

$$
\begin{aligned}
\delta_j \le \frac{1}{2}\,\delta_{j-1} + C\Bigg( & r_j(1+\ell_j) + \Big(\frac{\mu B_j}{r_j^{n-p}}\Big)^{1/(p-1)}\\
&+ M\Big(\frac{\operatorname{cap}_p(B_{j-1}\setminus\Omega,\, r_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\Bigg).
\end{aligned} \tag{3.28}
$$

This is trivial when $\delta_j \le \frac{1}{2}\,\delta_{j-1}$, so assume that $\delta_{j-1} \le 2\delta_j$. In this case, since $\psi_{j-1}\eta_{j-1} = 1$ on $E_j$, we have

$$
\begin{aligned}
|E_j| &\le \delta_{j-1}^{-\gamma}\int_{L_{j-1}} (u-\ell_{j-1})^\gamma\,\psi_{j-1}\eta_{j-1}\,dx\\
&= \kappa r_{j-1}^n \le 2^n\kappa r_j^n
\end{aligned} \tag{3.29}
$$

and

$$
\begin{aligned}
&\int_{E_j} (u-\ell_j)^\gamma\,dx\\
&\le \int_{L_{j-1}} (u-\ell_{j-1})^\gamma \psi_{j-1}^q \eta_{j-1}^q\,dx = \delta_{j-1}^\gamma \kappa r_{j-1}^n = 2^{n+\gamma}\delta_j^\gamma \kappa r_j^n\\
&= 2^{n+\gamma}\int_{L_j} (u-\ell_j)^\gamma \psi_j^q \eta_j^q\,dx.
\end{aligned} \tag{3.30}
$$

Thus (3.7) and (3.8) are verified and Lemma 3.2 yields

$$
\begin{aligned}
\delta_j \le C\Bigg( & r_j(1+\ell_j) + \Big(\frac{\mu B_j}{r_j^{n-p}}\Big)^{1/(p-1)}\\
&+ M\Big(\frac{\operatorname{cap}_p(B_{j-1}\setminus\Omega,\, r_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\Bigg)
\end{aligned}
$$

which proves (3.28). Summing up (3.28) for $j = 1, \dots, k$ we get

$$\begin{aligned}
\frac{1}{2}\ell_{k+1} &= \frac{1}{2}(\delta_0 + \dots + \delta_k) \le \delta_k + \frac{1}{2}(\delta_0 + \dots + \delta_{k-1}) \\
&\le \delta_0 + C\Bigg(\sum_{j=1}^{k} r_j(1+\ell_{j+1}) + \sum_{j=1}^{k}\Big(\frac{\mu B_j}{r_j^{n-p}}\Big)^{1/(p-1)} \\
&\quad + M\sum_{j=1}^{k}\Big(\frac{\operatorname{cap}_p(B_{j-1}\setminus\Omega, r)}{r_j^{n-p}}\Big)^{1/(p-1)}\Bigg) \\
&\le C\, r_0\ell_{k+1} + C\Bigg(\Big(r_0^{-n}\int_{E_0} u^\gamma\,dx\Big)^{1/\gamma} \\
&\quad + \sum_{j=1}^{k}\int_{r_j}^{r_{j-1}}\Big(\frac{\mu B(x_0,r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r} \\
&\quad + M\sum_{j=1}^{k}\int_{r_j}^{r_{j-1}}\Big(\frac{\operatorname{cap}_p(B(x_0,2r)\setminus\Omega, r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r}\Bigg).
\end{aligned}$$

If $r_0 \le R_1 := \frac{1}{2C_2}$, we obtain

$$\begin{aligned}
\lim_j \ell_j \le C\Bigg(&\Big(r_0^{-n}\int_{B(x_0,r_0)\cap\Omega\cap\{u>0\}} u^\gamma\,dx\Big)^{1/\gamma} \\
&+ \int_0^{r_0}\Big(\frac{\mu B(x_0,r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r} \\
&+ M\int_0^{2r_0}\Big(\frac{\operatorname{cap}_p(B(x_0,r)\setminus\Omega, r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r}\Bigg).
\end{aligned} \tag{3.31}$$

If $R_1 < r_0 < R_0$, then (3.31) holds as well, because then

$$r_0/R_1 \le R_0/R_1 \le C.$$

It remains to prove that

$$p\text{-fine-}\limsup_{x\to z} u(x) \le \lim_j \ell_j\,. \tag{3.32}$$

We may assume that the right hand part of (3.25) is finite, otherwise the assertion of the theorem is trivial. We choose $\varepsilon > 0$ and denote $\ell = \lim_j \ell_j$. Set

$$w_j = (2^{\gamma/q} - 1)^{-1}\Big(\Big(1 + \frac{(u-\ell-\varepsilon)^+}{\varepsilon}\Big)^{\gamma/q} - 1\Big)\psi_j\eta_j$$

on $\Omega$ and $w_j = 0$ elsewhere. Then $w_j \in W_0^{1,p}(B_j)$, $w_j + \varphi_j\eta_j \ge 1$ on $B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}$ and thus

$$\operatorname{cap}_p(B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}, r_j) \le C \int_{B_j} (|\nabla w_j|^p + |\nabla(\varphi_j\eta_j)|^p)\, dx.$$

Denote

$$E'_j = B_j \cap \Omega \cap \{u > \ell + \varepsilon\} \cap \{\varphi_j < 1\}.$$

Using Lemma 3.2.a we obtain

$$\begin{aligned}
&\operatorname{cap}_p(B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}, r_j)\\
&\quad \le C \int_{B_j} (|\nabla w_j|^p + |\nabla(\varphi_j\eta_j)|^p)\, dx \le C\Big(\varepsilon^{-p} r_j^n (1 + (\ell+\varepsilon)^p)\\
&\qquad + r_j^{-p} \int_{E'_j} \Big(1 + \frac{u - \ell - \varepsilon}{\varepsilon}\Big)^\gamma dx\\
&\qquad + \varepsilon^{1-p}\, \mu(B_j)\\
&\qquad + (1 + \varepsilon^{1-p})(1 + \|u\|_\infty)^{p-1} \int_{B_j} (r_j^{-p}\varphi_j^p + |\nabla\varphi_j|^p + |\nabla\psi_j|^p)\, dx\Big).
\end{aligned}$$

It follows

$$\begin{aligned}
&\sum_j \Big(\frac{\operatorname{cap}_p(B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}, r_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\\
&\quad \le C\Bigg(\varepsilon^{-p/(p-1)} r_0^{p/(p-1)} (1 + \ell^p)^{1/(p-1)}\\
&\qquad + \sum_j \Big(r_j^{-n} \int_{E'_j} \Big(1 + \frac{u - \ell - \varepsilon}{\varepsilon}\Big)^\gamma dx\Big)^{1/(p-1)}\\
&\qquad + \varepsilon^{-1} \sum_j \Big(\frac{\mu(B_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\\
&\qquad + (1 + \varepsilon^{-1})(1 + \|u\|_\infty) \sum_j \Big(\frac{\operatorname{cap}_p(B_j \setminus \Omega, r_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\Bigg).
\end{aligned} \tag{3.33}$$

Note that

$$\sum_{j=0}^{\infty} (\delta_j/\ell)^{\gamma/(p-1)} \le \sum_{j=0}^{\infty} (\delta_j/\ell) = 1. \tag{3.34}$$

Using (3.27) and (3.34) we estimate

$$\begin{aligned}
&\sum_j \Big(r_j^{-n} \int_{E'_j} \Big(1 + \frac{u-\ell-\varepsilon}{\varepsilon}\Big)^\gamma dx\Big)^{1/(p-1)} \\
&\quad \le C \sum_j \Big(r_j^{-n} \int_{E_j} \varepsilon^{-\gamma}(u-\ell_{j-1})^\gamma dx\Big)^{1/(p-1)} \\
&\quad \le C \sum_j \Big(r_j^{-n} \int_{L_{j-1}} \varepsilon^{-\gamma}(u-\ell_{j-1})^\gamma \psi_{j-1}\varphi_{j-1} dx\Big)^{1/(p-1)} \\
&\quad \le C \sum_j (\kappa\varepsilon^{-\gamma}\delta_{j-1}^\gamma)^{1/(p-1)} < \infty.
\end{aligned}$$

If the right hand part of (3.32) is finite, then the remaining sums on the right hand part of (3.33) also converge (we assumed this), so that the set

$$\Omega \cap \{u > \ell + 2\varepsilon\}$$

is $p$-thin at $x_0$ for any $\varepsilon > 0$. We proved (3.32), which concludes the proof. □

## 4. Necessity of the Wiener condition

**4.1 Example.** Let $\Omega$ be a bounded open set and let $u_0 \in W^{1,p}(\Omega)$. Consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0, \\ u - u_0 \in W_0^{1,p}(\Omega). \end{cases} \tag{4.1}$$

Then we obtain a unique solution $u$ of (4.1) by minimizing

$$\int_\Omega |\nabla v|^p \, dx$$

in the closed convex set

$$\{v \in W^{1,p}(\Omega) \colon v - u_0 \in W_0^{1,p}(\Omega)\}.$$

Since

$$\int_\Omega |\nabla u|^p \, dx \le \int_\Omega |\nabla u_0|^p \, dx,$$

using Poincaré's inequality we get

$$\begin{aligned}
\int_\Omega |u|^p \, dx &\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |u-u_0|^p \, dx\Big) \\
&\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |\nabla u - \nabla u_0|^p \, dx\Big) \\
&\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |\nabla u|^p + |\nabla u_0|^p \, dx\Big) \\
&\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |\nabla u_0|^p \, dx\Big).
\end{aligned}$$

Let $M = \|u_0\|_\infty < \infty$. If we test the minimizing property by

$$v(x) = \begin{cases} u, & |u| \le M, \\ M, & u > M, \\ -M, & u < M, \end{cases}$$

then we get that $u \le M$ a.e. Similar estimates hold for all equations of the monotone type.

**4.2 Theorem.** *In addition to* (2.1), *suppose that for any* $u_0 \in \mathcal{C}_c^1(\mathbf{R}^n)$ *there is* $u \in W^{1,p}(\Omega)$ *such that*

$$\text{(4.2)} \qquad \begin{cases} -\operatorname{div}\mathbf{A} + \mathbf{B} = 0, \\ u - u_0 \in W_0^{1,p}(\Omega), \end{cases}$$

*and*

$$\text{(4.3)} \qquad \int_\Omega |u|^p\,dx \le C \int_\Omega \big(|u_0|^p + |\nabla u_0|^p\big)\,dx, \ \ \|u\|_\infty \le C\|u_0\|_\infty$$

*with a constant* $C$ *independent of* $u_0$. *Let* $z \in \partial\Omega$ *and suppose that*

$$\mathbf{w}_p(z, \mathbf{R}^n \setminus \Omega) < \infty.$$

*Then* $z$ *is irregular for the equation*

$$-\operatorname{div}\mathbf{A} + \mathbf{B} = 0.$$

Proof: Choose $\varepsilon > 0, \rho \in (0,1)$ to be specified later. The singleton $\{z\}$ has zero $p$-capacity. Hence, we find a $\mathcal{C}^1$-function $u_0$ on $\mathbf{R}^n$ supported in $B(z,1)$ such that $u_0(z) = 1$ and $\int_{\mathbf{R}^n}\big(|u_0|^p + |\nabla u_0|^p\big)\,dx < \varepsilon$. Let $u$ be a continuous solution of (4.2), (4.3). By Theorem 3.3,

$$p\text{-fine-}\limsup_{x\to z} u(x) \le C_1\Big(\rho^{-n}\int_{B(z,\rho)} u^\gamma\,dx\Big)^{1/\gamma} + C_2\int_0^\rho \Big(\frac{\operatorname{cap}_p(B(z,r)\setminus\Omega)}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r}.$$

Hölder's inequality yields

$$\Big(\rho^{-n}\int_{B(x,\rho)} |u^\gamma|\,dx\Big)^{1/\gamma} \le C\rho^{-n/p}\Big(\int_{B(x,\rho)} |u|^p\,dx\Big)^{1/p} \le C_3\rho^{-n/p}\varepsilon^{1/p}.$$

Since $\mathbf{R}^n \setminus \Omega$ is $p$-thin at $z$, we can find $\rho \in (0,1)$ such that

$$C_2\int_0^\rho \Big(\frac{\operatorname{cap}_p(B(z,r)\setminus\Omega)}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r} < \frac{1}{3}.$$

Then we can specify the choice of $\varepsilon$ so that

$$C_1C_3\rho^{-n/p}\varepsilon^{1/p} \le \frac{1}{3}.$$

We obtain that

$$p\text{-fine-}\limsup_{x\to z} u(x) < 1 = u_0(z),$$

hence $z$ is not regular. □

**Note added in proof.** In a new preprint Gianazza, Marchi and Villani prove Wiener criteria for a related class of equations which is neither a subclass, nor a superclass of the class of equations investigated here.

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