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Abstract. We prove some closed mapping theorems on k-spaces with point-countable k-networks. One of them generalizes Lašnev's theorem. We also construct an example of a Hausdorff space Ur with a countable base that admits a closed map onto metric space which is not compact-covering. Another our result says that a k-space X with a point-countable k-network admitting a closed surjection which is not compact-covering contains a closed copy of Ur.

Keywords: k-space, k-network, closed map, compact-covering map
Classification: 54B10, 54A20

1. Introduction and preliminary results

A collection γ of subsets of X is called point-countable if every point $x \in X$ belongs to at most countably many $\xi \in \gamma$. This concept plays an important role in the theory of generalized metric spaces (see [GMT], [T] and references there). Usually point-countable collections with some additional properties on topological spaces are investigated. One of the most useful is the property to be a k-network. Recall that a family γ of subsets of a topological space X is *a k-network* if for every compact $K \subseteq X$ and any open $U \supseteq K$ there is finite $\gamma_K \subseteq \gamma$ such that $K \subseteq \cup\gamma_K \subseteq U$. On the other hand, in some interesting classes of spaces (for example in the class of k-spaces), spaces with special point-countable collections turn out to be related to spaces having point-countable k-networks. In several papers it was shown that k-spaces with point-countable k-networks borrow many basic features from metrizable spaces. Thus every such space is sequential (see e.g. [GMT]); in the case when a point-countable k-network consists of closed sets, a k-space with such network is an image of a metric space under a quotient s-map (see [GMT]; recall that s-map is a map with separable fibres). The papers [GMT] and [H] contain characterizations of quotient s-images of metric spaces as spaces having a special type of k-networks. In addition to these results, there is a description of k-spaces with point-countable k-networks as images of metric spaces under special quotient maps in the paper [V1]. Hence k-spaces with point-countable k-networks may be considered to be a nontrivial generalization of metric spaces.

In this paper we try to show that classes of metric spaces and k-spaces with point-countable k-networks not only have the common origin, not only 'genetically' close but also have some 'categorical' likenesses. We prove generalizations of two well-known theorems from the metrizable spaces theory: Arkhangel'skii's theorem ([A]) saying that every closed mapping of metrizable space is compact-covering (see below for the definition of a compact-covering map) and Lašnev's theorem ([L]) on the set of all points at which a given closed map with a metrizable domain is compact (i.e. the preimage of the point is compact). It seems worthwhile to note that the theorems obtained by replacing metrizable space in the theorems of Arkhangel'skii and Lašnev by k-spaces with point-countable k-networks are not true. So Proposition 2.3 says that if a closed map with the domain being a k-space with a point-countable k-network is not compact-covering, then the space contains a closed copy of a canonical space. The canonical space is not regular so this corollary generalizes Arkhangel'skii's theorem. Likewise Theorem 3.6 which generalizes Lašnev's theorem imposes some restrictions on a range of a closed map. For technical purposes we introduce a class of point-countable networks which is closely connected with the class of k-networks (see Lemma 1.2).

We use standard notation for main topological operations and invariants. By $[A]_X$ or $[A]$, if X is clear, we denote a closure of $A \subseteq X$ in the topological space X. By $\operatorname{card}(X)$ we denote the cardinality of X, $\exp(X) = \{F \mid F \subseteq X\}$. We use the following cardinal invariants: $d(X) = \min\{\tau \mid [A] = X, \operatorname{card}(A) = \tau\}$ (density of X), $t(X) = \min\{\tau \mid x \in [A] \subseteq X \Rightarrow x \in [T], \operatorname{card}(T) \leq \tau, T \subseteq A \subseteq X\}$ (tightness of X). A map $f : X \to Y$ is called *compact-covering* or simply *k-covering* if every compact $K \subseteq Y$ is a subset of an image of some compact $K' \subseteq X$. Analogously f is *sequence-covering* (see [GMT]) if any convergent sequence in Y is a subset of an image of some compact $K \subseteq X$, and f is called *cc-covering* if every countably-compact subset of Y is a subset of an image of a countably compact subset of X. All spaces are assumed to be Hausdorff and all maps to be continuous and onto.

Definition 1.1. Let γ be a network for a topological space X. Let us say that γ is *an s-network for X* if for any $A \neq [A] \subseteq X$ there exists a point $x \in X$ with the following property:

$(*_{\gamma,A})$ for any open neighborhood U of x there is $\xi \in \gamma$ such that $\xi \subseteq U$, $\operatorname{card}(\xi \cap A) \geq \aleph_0$ (it follows that $x \in [A]$).

Let $Y \subseteq X$ and γ be a network for Y. We say that γ is *an exterior s-network for Y in X* if for every $A \subseteq Y$ such that $[A]_X \neq A$ there exists a point $x \in X$ with the property $*_{\gamma,A}$ (a neighborhood U appearing in this property is supposed to be open in X). We call a space X *nearmonolite* if every countable subset of X has a countable exterior s-network in X. The class of nearmonolite spaces contains classes of spaces with point-countable s-networks and strongly-$\aleph_0$-monolite spaces (i.e. spaces in which a closure of any countable set has a countable base).

The following simple lemma shows some connections between k-networks and s-networks.

Lemma 1.2. (a) *Every point-countable s-network is a k-network. In k-spaces every k-network is an s-network.*

(b) *If γ is an s-network for X and $\mathcal{U}$ is an open cover of X, then $\gamma' = \{\xi \in \gamma \mid \xi \subseteq U \in \mathcal{U}\}$ is an s-network.* □

Proposition 1.3. *Let $f : X \to Y$ be a closed map and X have a point-countable s-network. Then Y has a point-countable s-network.*

Proof: For every point $y \in Y$ we choose a point $x_y \in f^{-1}(y)$. Let $X_f = \{x_y \mid y \in Y\}$. If γ is a point-countable s-network for X, then we have written $\gamma_f = \{\xi \cap X_f \mid \xi \in \gamma\}$, $\delta = \{f(\lambda) \mid \lambda \in \gamma_f\}$. Now δ is point-countable and using the closedness of f it may be easily proved that δ is an s-network. □

Next two lemmas will be used in the proof of Theorem 2.2.

Let $t : \omega^2 \to \Gamma \subseteq X$ be a bijection. Then we say that a pair (t, Γ) is *a table in X*. A set $\theta \subseteq \omega^2$ will be called *thin* (*thick*) if $\mathrm{card}(\theta) \geq \aleph_0$ and for any $n_0 \in \omega$ the following holds: $\mathrm{card}(\{m \in \omega \mid (n_0, m) \in \theta\}) < \aleph_0$ $(\mathrm{card}(\{n \in \omega \mid \mathrm{card}((\{n\} \times \omega) \cap \theta) \geq \aleph_0\}) \geq \aleph_0$, respectively). We say that a set $\Gamma_\theta = t(\theta) \subseteq X$ is thin (thick) if $\theta \subseteq \omega^2$ is a thin (thick) set.

Lemma 1.4. *Let (t, Γ) be a table in X and Γ has a countable exterior s-network. Then there exists a thin set $\Gamma_{\theta'} = \Gamma \backslash \Gamma_\theta$ for some $\Gamma_\theta \subseteq \Gamma$ such that for any thin set $\kappa \subseteq \theta$ there is a thick set σ such that for any open neighborhood U of a nonempty set of all cluster points of an arbitrary thin set $\Gamma_{\kappa'}$ where $\kappa' \subseteq \sigma$, we have $U \cap \Gamma_{\kappa \backslash f} \neq \emptyset$, where $f \subseteq \omega^2$ is finite.*

Proof: Let γ be a countable exterior s-network for Γ. Without loss of generality we may assume that γ is closed under finite unions and intersections and that $\Gamma \backslash f \in \gamma$ for any finite $f \subseteq \Gamma$. Let $\Xi = \{\xi_i \mid i \in \omega\} \subseteq \gamma$ be the set of all such $\xi_i \in \gamma$ that the set $\Gamma \backslash (\xi_i \cup \Gamma_{\{0,\dots,n_i\} \times \omega})$ is thin for some $n_i \in \omega$. Without loss of generality we may assume that $n_0 = 0$ and that $n_{i+1} > n_i$. We let

$$\Gamma_\theta = \bigcup_{i \in \omega} ((\bigcap_{k \leq i} \xi_k) \cap \Gamma_{\{n_i+1,\dots,n_{i+1}\} \times \omega}).$$

Then $\Gamma \backslash \Gamma_\theta$ is a thin set by the way ξ_i's and n_i's were chosen. Let now $\kappa \subseteq \theta$ be an arbitrary thin set. Let $M = \{\mu_i \mid i \in \omega\} \subseteq \gamma$ be the set of all $\mu_i \in \gamma$ such that $\mu_i \notin \Xi$ and $\mu_i \cap \Gamma_\kappa = \emptyset$. We will show that M is closed under finite unions. Let $M' \subseteq M$ be finite. Then $\mu = \cup M' \in \gamma$, $\mu \cap \Gamma_\kappa = \emptyset$. Let us prove that $\mu \notin \Xi$. Suppose not, then $\mu = \xi_k$ for some $k \in \omega$. Since κ is infinite and thin, then there exists $n' > n_k$ such that $(n', m) \in \kappa$ for some $m \in \omega$. Since $\kappa \subseteq \theta$, then $t(n', m) \in \bigcap_{i \leq k} \xi_i \subseteq \xi_k = \mu$. But $\mu \cap \Gamma_\kappa = \emptyset$. A contradiction. Hence $\mu \notin \Xi$ and $\mu \in M$. For every $i \in \omega$ choose $m_i \in \omega$ such that $\mathrm{card}(t(\{m_i\} \times \omega) \backslash \bigcup_{j \leq i} \mu_j) \geq \aleph_0, m_{i+1} > m_i$. It is possible since otherwise there is $j \in \omega$ such that $\Gamma \backslash (\bigcup_{i \leq j} \mu_i \cap (\Gamma \backslash \Gamma_{\{0,\dots,m_j\} \times \omega}))$ is thin for some $m_j \in \omega$. Then $\bigcup_{i \leq j} \mu_i \in \Xi$ contradicting $\bigcup_{i \leq j} \mu_i \in M$ and $M \cap \Xi = \emptyset$. We let

$$\Gamma_\sigma = \bigcup_{i \in \omega} (t(\{m_i\} \times \omega) \backslash \bigcup_{j \leq i} \mu_j).$$

By the way μ_i's and m_i's were chosen σ is a thick set. Let $\kappa' \subseteq \sigma$ be a thin set, $B \neq \emptyset$ be the set of all cluster points of $\Gamma_{\kappa'}$. Suppose that there is an open neighborhood $U \supseteq B$ such that $U \cap \Gamma_{\kappa\setminus f} = \emptyset$ for some finite $f \subseteq \omega^2$. Then $T = (U \cap \Gamma_{\kappa'})\setminus\{x\}$, $x \in B$ is not closed and (since $[T] \subseteq U$) $[T] \subseteq X\setminus[\Gamma_{\kappa\setminus f}]$. Then there exists $\xi \in \gamma$ such that $\xi \cap [\Gamma_{\kappa\setminus f}] = \emptyset$ and $\mathrm{card}(\xi \cap T) \geq \aleph_0$. Since $\xi\setminus\Gamma_f \in \gamma$ we may assume that $\xi \cap [\Gamma_\kappa] = \emptyset$. Then there are two possibilities:

1. $\xi \notin M$. Then it follows from $\xi \in \gamma$, $\xi \cap \Gamma_\kappa = \emptyset$ that $\xi \in \Xi$. But then $\xi \cap \Gamma_\kappa \neq \emptyset$. A contradiction.

2. $\xi \in M$. Then $\xi = \mu_i$ for some $i \in \omega$. Since $\xi \cap \Gamma_{\kappa'}$ is infinite and thin, there is $n \in \omega$, $n \geq i$ such that $t(m_n, k) \in \xi = \mu_i$ and $(m_n, k) \in \kappa'$ for some $k \in \omega$. But $t(m_n, k) \in t(\{m_n\} \times \omega)\setminus \bigcup_{j\leq n} \mu_j \subseteq X\setminus\mu_i$. A contradiction. □

Lemma 1.5. *Let $K \subseteq X$ be relatively countably compact in X and $x \in X$ be a cluster point for K. Let K have an exterior point-countable s-network in X. Then there exists a sequence $S = \{x_n \mid n \in \omega\} \subseteq K$ such that $x_n \to x$ as $n \to \infty$.*

Proof: Consider the subfamily $\delta \subseteq \gamma$ consisting of all $\xi \in \gamma$ such that $[\xi] \not\ni x$. For any $y \in K$ let $\{\xi_i(y) \mid i \in \omega\} = \{\xi \in \delta \mid y \in \xi\}$. Choose by induction x_i's such that $x_i \neq x$ and $x_i \notin \cup\{\xi_k(x_n) \mid k, n < i\}$. Suppose that at one moment we have $K\setminus \cup \{\xi_k(x_n) \mid k, n < i\} = \{x\}$. Then $K\setminus[\cup\{\xi_k(x_n) \mid k, n < i\}] = \{x\}$ contradicting x is a cluster point for K. Hence we may assume that $x_i \neq x_j$ if $i \neq j$. We will prove that $x_i \to x$ as $i \to \infty$.

Since K is relatively countably compact, the set $S = \{x_n \mid n \in \omega\}$ has a cluster point x'. Let us prove that $x = x'$. Suppose not. Then we can find two open neighborhoods $U \ni x$ and $V \ni x'$ such that $U \cap V = \emptyset$. The set $R = (S \cap V)\setminus\{x'\}$ is not closed. Then there is a point $x'' \in X$ having the property $*_{\gamma,R}$; $x'' \in [R]$, so $x'' \neq x$. Choose U' and V'-open in X such that $x \in U'$, $x'' \in V'$ and $U' \cap V' = \emptyset$. Using $*_{\gamma,R}$ find $\xi \in \gamma$ such that $\xi \in V'$, $\mathrm{card}(\xi \cap S) \geq \aleph_0$. Then $\xi \in \delta$. But then $\xi = \xi_i(x_j)$ for some $i, j \in \omega$ and $x_k \notin \xi_i(x_j) = \xi$ if $k > \max\{i, j\}$ contradicting $\mathrm{card}(\xi \cap S) \geq \aleph_0$. Now S is relatively countably compact and has unique cluster point. Hence S is a convergent sequence. □

2. Closed and compact-covering mappings and spaces with certain point-countable collections

In this section we prove two theorems on closed mappings with the domain having certain networks. First of them (Theorem 2.2) deals with the property to be a *cc*-covering map and the second one (Theorem 2.4) gives a sufficient condition for a closed map from a k-space with a point-countable k-network to be compact-covering.

In our following example we construct a k-space with point-countable s-network (with countable base indeed) and a closed map from it onto a convergent sequence which is not k-covering. The theorem after the example shows that a map with such properties is in obvious sense unique in the class of k-spaces with point-countable k-networks.

Example 2.1. Consider the set $L = \omega^2 \cup \{p\} \cup \{p_i \mid i \in \omega\}$. Let L have the topology defined as follows:

1. All points $x \in \omega^2$ are isolated.

2. The typical neighborhood of point p_i is defined to be $B_k(p_i) = \{p_i\} \cup (\{i\} \times (\omega \backslash \{0, \dots, k\}))$.

3. The typical neighborhood of p is $B^k(p) = \{p\} \cup (\omega^2 \backslash (\{0, \dots, k\} \times \omega))$.

We will denote the obtained space as Ur. It is easy to check that the quotient space Sq obtained from Ur by identifying all points $x \in \{p\} \cup \{p_i \mid i \in \omega\}$ is homeomorphic to a convergent sequence. Besides, the corresponding quotient map $p : Ur \to Sq$ is closed and is not compact-covering. □

Theorem 2.2. *Let X be a nearmonolite space, $f : X \to Y$ be a closed map which is not cc-covering. Then there exist closed embeddings h and h' such that the diagram*

$$(*)\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h'} \\ Ur & \xrightarrow{p} & Sq \end{array}$$

commutes.

Proof: Let $C \subseteq Y$ be an arbitrary countably compact subspace of Y. For every point $y \in C$ choose a point x_y such that $f(x_y) = y$. Let us consider the set $C' = \{x_y \mid y \in C\} \subseteq X$. Since f is closed and C is countably compact, C' is relatively countably compact. We let $K = \bigcup\{[S] \mid S \subseteq C', \operatorname{card}(S) \leq \aleph_0\}$. It follows from the fact that f is not *cc*-covering that K is not countably compact. Then there is closed in X discrete subset $D = \{d_i \mid i \in \omega\} \subseteq K$. Since C' is relatively countably compact and $f(C') \subseteq C$, we may assume that $D \subseteq K \backslash C'$ and $\operatorname{card}(f(D)) = 1$. By the definition of K for every d_i there is a countable set $S_i \subseteq C'$ such that $[S_i] \ni d_i$. Since X is nearmonolite, every S_i has a countable exterior s-network. By virtue of Lemma 1.5 there is $\Gamma = \{x_n^i \mid i, n \in \omega\} \subseteq \bigcup_{i \in \omega} S_i \subseteq C'$ such that $x_n^i \to d_i$ as $n \to \infty$. We may assume that $f(x_n^i) \neq f(x_m^j)$ if $(i, n) \neq (j, m)$. Let $t(n, m) = x_m^n$. Then (t, Γ) is a table in X. We will use the notation of Lemma 1.4. Using the fact that C' is relatively countably compact and Lemma 1.5, we obtain that there is a sequence $\Gamma_\kappa = \{x_{l_i}^{k_i} \mid i \in \omega\} \subseteq \Gamma_\theta$, $x_{l_i}^{k_i} \to x$ as $i \to \infty$, $k_{i+1} > k_i$; obviously Γ_κ is thin. Choose σ as in Lemma 1.4. We may assume without loss of generality that

$$(1)\qquad \Gamma_\sigma = \bigcup_{i \in \omega} \bigcup_{j \in \omega} t(n_i, m_j), \quad (n_i, m_j) \neq (n_{i'}, m_{j'}) \text{ if } (i, j) \neq (i', j')$$

and $x \neq d_i$ for every $i \in \omega$. Let $U \ni x$ be an arbitrary open set. We shall prove that for some $n \in \omega$

$$(2)\qquad \Gamma_\sigma \backslash (U \cup \Gamma_{\{0, \dots, n\} \times \omega}) = \emptyset.$$

Suppose the contrary. Then it follows from Lemma 1.5 and the properties of C' that there is a thin set $\Gamma_{\kappa'} = \{x'_i \mid i \in \omega\}$, $x'_i \to x'$ as $i \to \infty$, $\kappa' \subseteq \sigma$ such that $\Gamma_{\kappa'} \subseteq \Gamma \backslash U$. $x \in U \not\ni x'$ implies that $x \neq x'$. Then there exist $W \ni x'$ and $V \ni x$ such that $W \cap V = \emptyset$. This contradicts Lemma 1.4 since x is the unique cluster point of Γ_κ and $\Gamma_{\kappa'} \backslash W$ is finite. Let now $h(n,m) = t(n_n, m_m)$, $h(p) = x$, $h(p_i) = d_{n_i}$. It is easy to check now that $h : Ur \to X$ is a closed embedding (it easily follows from (1) and (2); closedness is checked as (2) using Lemma 1.4). Since Y is Hausdorff we have that $f(D) = \{f(x)\}$ so h' automatically exists. □

Now Lemma 1.2 gives:

Proposition 2.3. *Let X be a k-space with a point-countable k-network which admits a closed onto map that is not compact-covering. Then X contains a closed copy of Ur.* □

Using the fact that Ur is not regular and has no closed point-countable k-network (see [V1], the proof for the last example there works for Ur too) we come to the following proposition which generalizes Arkhangel'skii's theorem ([A]):

Theorem 2.4. *Let X be a k-space with a point-countable k-network. If X is regular or has a closed point-countable k-network, then any closed map $f : X \to Y$ is k-covering.* □

Proposition 2.5. *Let X be a k-space with a point-countable k-network. Then the following are equivalent for Hausdorff Y:*

1. *Every closed map $f : X \to Y$ is sequence-covering.*
2. *Every closed map $f : X \to Y$ is k-covering.*

Proof: The implication $2 \Rightarrow 1$ is obvious. Let us prove that $1 \Rightarrow 2$. If there is a map $f : X \to Y$ which is not k-covering then by Theorem 2.2 and Lemma 1.2 (a) we have the commutative diagram $(*)$. Let $q : X \to Z$ be the map which shrinks the set $f^{-1}f(h(p)), p \in Ur$ to a point. It is easy to check that Z is Hausdorff and that q is not sequence-covering. □

3. Around Lašnev's theorem

To obtain a generalization of Lašnev's theorem (Theorem 3.6) we need some preliminary results on closed mappings.

Let us recall the well-known construction of Alexandroff double which assigns to an arbitrary topological space X and its subspace Y a space $AD(X,Y) = X \times \{0\} \cup Y \times \{1\}$ with the following topology:

1. Every point $y \in Y \times \{1\}$ is isolated.
2. Every point $(x,0) \in X \times \{0\}$ has the following neighborhoods: $U \times \{0\} \cup (U \cap Y) \times \{1\} \backslash \{(x,1)\}$.

It is easily seen that the natural projection $\pi_{AD} : AD(X,Y) \to X$, where $\pi_{AD}((x,i)) = x$, $i \in \{0,1\}$ is a perfect map.

Lemma 3.1. *Let $Y \subseteq X$. Then a closed map $f : Z \to X$ such that Z is a space with a point-countable s-network and $f^{-1}(y)$ is not compact for every $y \in Y$ exists if and only if $AD(X, Y)$ has a point-countable s-network.*

Proof: Let $f : Z \to X$ be a closed map such that Z has a point-countable s-network and $f^{-1}(y)$ is not compact for every $y \in Y \subseteq X$. Then for every $y \in Y$ there is $A_y = \{a_y^i \mid i = 1, 2, \dots\} \subseteq f^{-1}(y)$ — a countable discrete subset of Z. We denote $A^i = \{a_y^j \mid j \geq i,\ y \in Y\}$, $B^i = \{a_y^i \mid y \in Y\}$. Let γ be a point-countable s-network for Z. By Proposition 1.3 X has a point-countable s-network, say γ'. We shall show that

$$\delta = \bigcup_{i=1,2,\dots} \{f(\xi \cap A^i) \times \{1\} \mid \xi \in \gamma\} \cup \{f(\xi \cap B^i) \times \{1\} \mid \xi \in \gamma\} \cup \{\xi' \times \{0\} \mid \xi' \in \gamma'\}$$

is an s-network for $AD(X, Y)$; obviously δ is point-countable.

Let $A \subseteq AD(X, Y)$ and $A \neq [A]$. Then either $A \cap (X \times \{0\}) \neq [A \cap (X \times \{0\})]$ or $A \cap (Y \times \{1\}) \neq [A \cap (Y \times \{1\})]$. Let us consider the second case (the first one is trivial). First assume that $\operatorname{card}(A) \leq \aleph_0$. Let $B = \pi_{AD}(A \cap Y \times \{1\}) \subseteq X$, $B = \{b_i \mid i = 1, 2, \dots\}$. Removing a cluster point from B if necessary we may assume that $B \neq [B]$. Since f is a closed map, the set $C = \{a_{b_i}^i \mid i = 1, 2, \dots\}$ is not closed in Z. Let $z \in Z$ be a point with property $*_{\gamma,C}$. Then $z \notin A^n$ for some $n = 1, 2, \dots$. We will prove that $x = (f(z), 0)$ has property $(*_{\delta,A})$. Indeed, let $U \in AD(X, Y)$ be an arbitrary neighborhood of x. We may assume that $U = U' \times \{0\} \cup ((U' \cap Y) \times \{1\} \setminus \{(f(z), 1)\})$, where U' is some open neighborhood of $f(z)$. Consider the following neighborhood of z: $V = f^{-1}(U') \setminus (f^{-1}(f(z)) \cap A^n)$. Since z has property $*_{\gamma,C}$ there is $\xi \in \gamma$ such that $\xi \subseteq V$ and $\operatorname{card}(\xi \cap C) \geq \aleph_0$. Then $\eta = f(\xi \cap A^n) \times \{1\} \in \delta$ and $\eta \subseteq U$, $\operatorname{card}(\eta \cap A) = \operatorname{card}(\eta \cap f(C)) \geq \aleph_0$. So x has property $*_{\delta,A}$.

Let now $\operatorname{card}(A) > \aleph_0$. Then there are two possibilities:

1. There is $E \subseteq A$ such that $E \neq [E] \subseteq AD(X, Y)$ and $\operatorname{card}(E) \leq \aleph_0$.
2. For any $E \subseteq A$ such that $\operatorname{card}(E) \leq \aleph_0$ E is closed in $AD(X, Y)$.

Case 1 may be easily reduced to the case when $\operatorname{card}(A) \leq \aleph_0$. Consider Case 2. Let $B = \pi(A \cap Y \times \{1\}) \subseteq X$. Again assume $B \neq [B]$. Put $C = f^{-1}(B) \cap B^1$. By the property of A and construction of B^1 every countable subset of C is closed. Since f is a closed map, C is not closed in Z. Let $z \in Z$ be a point with property $*_{\gamma,C}$. Suppose $f(z) \in Y$. Since γ is point-countable, the set

$$\gamma_z = \{\xi \subseteq \gamma \mid a_{f(z)}^1 \in \xi, z' \in \xi \cap C, z' \neq z \ \text{ for some } z' \in Z\}$$

is countable. Now $\gamma_z = \{\xi_i \mid i = 1, 2, \dots\}$. Choose z_i so that $z_i \in (\xi_i \cap C) \setminus \{z\}$ for $i = 1, 2, \dots$ then $C \supset [\{z_i \mid i = 1, 2, \dots\}] = \{z_i \mid i = 1, 2, \dots\} \not\ni z$. We will show that $(f(z), 0) = x$ has property $*_{\delta,A}$. Let $U = U' \times \{0\} \cup (U' \cap Y) \times \{1\} \setminus \{(f(z), 1)\}$, where U' is open in X, be a neighborhood of x. Then there exists $\xi \in \gamma$ such that $\xi \subseteq f^{-1}(U')\backslash[\{z_i \mid i = 1, 2, \dots\}]$, $\operatorname{card}(\xi \cap C) \geq \aleph_0$. If $a^1_{f(z)} \in \xi$ then $\xi \in \gamma_z$, $z_i \in \xi$ for some $i \in \omega$ contradicting $\xi \subseteq Z\backslash[\{z_i \mid i = 1, 2, \dots\}]$. Now $f(z) \notin f(\xi \cap B^1)$ and thus $\eta = f(\xi \cap B^1) \times \{1\} \in \delta, \eta \subseteq U$. The case $f(z) \notin Y$ is similar.

Let now $AD(X, Y) = X \times \{0\} \cup Y \times \{1\}$ have a point-countable s-network γ. Consider the space $Z = X \times \{0\} \cup Y \times \{1, 2, \dots\}$ with the topology defined as follows:

1. Every point $y \in Y \times \{1, 2, \dots\}$ is isolated.
2. Every point $(x, 0) \in X \times \{0\}$ has the following neighborhoods:

$$U \times \{0\} \cup ((U\backslash\{x\}) \times \{1, 2, \dots\}).$$

It is easily seen that the natural projection $\pi : Z \to X$, $\pi((x, i)) = x$ is a closed map and $\pi^{-1}(y), y \in Y$ is a countable discrete space. Now if $\gamma \subseteq \exp(AD(X, Y))$ is a point-countable s-network then

$$\gamma' = \{\xi' \mid \xi' = (\xi_0 \times \{0\} \cup \xi_1 \times \{1, 2, \dots\}), (\xi_0 \times \{0\} \cup \xi_1 \times \{1\}) \in \gamma\} \cup \{\{z\} \mid z \in Z\}$$

is a point-countable s-network for Z. □

Definition 3.2. Let X be a topological space, γ be a cover of X. We say that γ is *a q-cover of X* if for any point $x \in X$ there is a countable family $\Xi_x = \{\xi_i \in \gamma \mid x \in \xi_i,\ i \in \omega\}$ such that every sequence $S = \{x_i\}_{i \in \omega}$, $x_i \in \xi_i$, $i \in \omega$ is either trivial (*i.e.* $\operatorname{card}(S) < \aleph_0$) or has a cluster point.

Obviously every point-countable network is a q-cover and every cover consisting of compacts is a q-cover.

Lemma 3.3. *Let X be a space with a point-countable s-network δ, $f : X \to Y$ be a perfect map and Y have a q-cover γ. Then for every $\xi \in \gamma$ we can choose a countable family $\{\delta^i_\xi\}_{i \in \omega} \subseteq \delta$ such that $\alpha = \{\delta^i_\xi \cap f^{-1}(\xi) \mid \xi \in \gamma, i \in \omega\}$ is a network for X.*

Proof: Let $\delta_\xi \subseteq \delta$ for some $\xi \in \gamma$ consist of all $\delta'_\xi \in \delta$ such that there is $\delta^\xi \subseteq \delta$ such that $\delta^\xi \cup \{\delta'_\xi\}$ is a finite minimal cover of $f^{-1}(\xi)$. By virtue of Miščenko lemma ([Miš]) δ_ξ is countable. So $\delta_\xi = \{\delta^i_\xi\}_{i \in \omega}$. We shall prove that $\alpha = \{\delta^i_\xi \cap f^{-1}(\xi) \mid \xi \in \gamma,\ i \in \omega\}$ is a network for X. Let $x \in X$ be an arbitrary point and $U \ni x$ be an arbitrary open neighborhood of x. Consider the following family:

$$\beta = \{\eta \in \delta \mid \eta \not\ni x \ \text{ or } \ \eta \subseteq U\} \subseteq \delta.$$

By Lemma 1.2 (b) β is an s-network. For every point $z \in X$ we have written $\{\beta_i(z)\}_{i \in \omega} = \{\eta \in \beta \mid z \in \eta\}$. Obviously it is enough to prove that for some $\xi \in \gamma$, $\xi \ni f(x)$ there is finite $\beta^\xi \subseteq \beta$ that covers $f^{-1}(\xi)$. Suppose the contrary. Using the fact that γ is a q-cover we choose a countable family $\Xi_{f(x)} \subseteq \gamma$ with the property announced in Definition 3.2. Then by induction choose $x_i \in X$, $i \in \omega$ such that $x_i \in f^{-1}(\xi_i) \backslash (\bigcup\{\beta_j(x_k) \mid j, k \leq i\})$. By the property of $\Xi_{f(x)}$ the sequence $S = \{y_i\}_{i \in \omega}$ provided $y_i = f(x_i)$ is either trivial or has a cluster point. Since f is perfect, and $x_i \neq x_j$ when $i \neq j$ by the way x_i's were chosen there is $x' \in f^{-1}(y)$ for some $y \in Y$ which is a cluster point for $T = \{x_i\}_{i \in \omega}$. Then β being s-network provides existence of $\eta' \subseteq \beta$ such that $\mathrm{card}(\eta' \cap T) \geq \aleph_0$. Then $\eta' = \beta_m(x_n)$ for some $n, m \in \omega$ and $x_l \notin \eta'$ provided $l > \max\{m, n\}$. But $\mathrm{card}(\eta' \cap T) \geq \aleph_0$. This contradiction concludes the proof. □

Theorem 3.4. *Let X have a point-countable s-network, $f : X \to Y$ be a closed mapping, γ be a q-cover of Y. Let also $Z \subseteq Y$ be a set of points at which f is not compact, i.e. $f^{-1}(z)$ is not compact for every $z \in Z$. Then for every $\xi \in \gamma$ there is countable $S_\xi \subseteq \xi$ such that $Z = \bigcup_{\xi \in \gamma} S_\xi$.*

Proof: By Lemma 3.1 $AD(Y, Z)$ has a point-countable s-network. Then using the fact that the natural projection $\pi_{AD} : AD(Y, Z) \to Y$ is perfect and Lemma 3.3, we obtain that there exists a network α such that for any $\xi \in \gamma$ there is countable $\alpha_\xi \subseteq \alpha$, $\eta \subseteq \pi_{AD}^{-1}(\xi)$ for every $\eta \in \alpha_\xi$ and $\bigcup_{\xi \in \gamma} \alpha_\xi = \alpha$. But any network for $AD(Y, Z)$ has to contain the family $\mathcal{Z} = \{\{x\} \mid x \in Z \times \{1\} \subseteq AD(Y, Z)\}$. Then letting $S_\xi = \pi_{AD}(\bigcup(\alpha_\xi \cap \mathcal{Z}))$ completes the proof. □

The following definition may be found in [V2].

Definition 3.5 ([V2])**.** Let X be a topological space, γ be a family of subsets of X. Then γ is called *a d-family* if every subset of the form $D = \{x_\xi \mid x_\xi \in X,\ x_\xi \in \xi \in \gamma\}$ is a closed discrete subspace of X.

It is easy to see that a closed image of a d-family is a d-family and every locally-finite collection is a d-family. Now σ-d-family (σ-d-network) is a family (network) which may be represented as a union of countably many d-families. In particular, every σ-locally finite network is a σ-d-network.

Corollary 3.4.1. *Let X have a point-countable s-network, $f : X \to Y$ be a closed map and Y have a σ-d-network γ. Then $f^{-1}(y)$ is compact for every $y \in Y \backslash Q$, where $Q = \bigcup_{n \in \omega} Q_n$ and every $D \subseteq Q_n$, $n \in \omega$ such that* $\mathrm{card}(D) \leq \aleph_1$ *is a closed discrete subspace of Y.*

Hint of proof. We have that $\gamma = \bigcup_{n \in \omega} \gamma_n$ where every $\gamma_n, n \in \omega$ is a d-family. First we let:

$$\gamma_n' = \gamma_n \cup \{\xi_x^n \mid \xi_x^n = \bigcap_{x \in \xi \in \gamma_n} \xi, x \in X, \mathrm{card}(\{\xi \mid x \in \xi\}) \geq \aleph_1\}, \gamma' = \bigcup_{n \in \omega} \gamma_n'.$$

Then it may be easily shown that γ' is a q-cover. Finally Theorem 3.4 and the property of γ_n to be a d-family are used to complete the proof. □

Corollary 3.4.2. *Let X have a point-countable s-network, $f : X \to Y$ be a closed map and Y have a σ-d-cover γ consisting of compacts. Then $f^{-1}(y)$ is compact for every $y \in Y \backslash Q$, where $Q = \bigcup_{n\in\omega} Q_n$ and every Q_n, $n \in \omega$ is a closed discrete subspace of Y.*

Proof: Obviously γ is a q-cover. Then the use of Theorem 3.4 concludes the proof. $\square$

The following theorem generalizes Lašnev's theorem ([L]):

Theorem 3.6. *Let X be a k-space having a point-countable k-network, $f : X \to Y$ be a closed map and Y have a σ-d-network. Then $f^{-1}(y)$ is compact for every $y \in Y \backslash Q$, where $Q = \bigcup_{n\in\omega} Q_n$ and every Q_n, $n \in \omega$ is a closed discrete subspace of Y.*

Proof: This obviously follows from Corollary 3.4.1 and Lemma 1.2 (a) for in this situation Y has a point countable k-network (Lemma 1.2 (a) and Proposition 1.3) and being a k-space has countable tightness ($t(Y) = \aleph_0$) (in fact Y is sequential; see [GMT]). $\square$

Proposition 3.7. *Let X have a point-countable base, $f : X \to Y$ be a closed map, Y be regular and $d(Y) = \tau$. Then $f^{-1}(y)$ is compact for every $y \in Y \backslash Q$ for some Q, $\mathrm{card}(Q) \leq \tau$.*

Hint of proof. First we prove the following lemma:

Lemma 3.8. *Let Y be a regular Fréchet-Urysohn space with a point-countable k-network and $d(Y) = \tau$. Then Y has a point-countable k-network γ such that $\mathrm{card}(\gamma) \leq \tau$.* $\square$

Then we apply Lemma 1.2 and Proposition 1.3 to see that Y satisfies the conditions of Lemma 3.8. Now we may use Theorem 3.4 which completes the proof. $\square$

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