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Support prices for weakly maximal programs of a growth model with uncertainty

Nikolaos S. Papageorgiou*

Abstract. We consider an infinite dimensional, nonstationary growth model with uncertainty. Using techniques from functional analysis and the subdifferentiation theory of concave functions, we establish the existence of a supporting price system for a weakly maximal program.

Keywords: weakly maximal program, support prices, utility function, value function, Yosida-Hewitt decomposition, concave subdifferential

Classification: Primary 90A15; Secondary 49B20

1. Introduction

One of the main problems in growth theory is that of characterization of optimal programs by a system of competitive prices. In this paper, we address this problem for a general growth model with uncertainty. There is uncertainty in both the production technologies and the utility functions. Suppose that a certain program (growth path) is optimal according to a given utility criterion. We ask the question: "does there exist a price system such that for the optimal program producers maximize their profits, while consumers maximize their utility?" The first to consider this problem for growth models with uncertainty were Radner [9] and Jeanjean [6]. Jeanjean's model is a stationary Markov model with a given transition probability and the "Lagrange multipliers" that he produces do not have an immediate economic interpretation. Radner's model is stationary (i.e. the probabilistic structure is unaffected by economic decisions). His model was extended further by the fundamental works of Dana [3] and Zilha [15]. All these works generate "Lagrange multipliers" which have interpretation as prices, but the uncertainty in all these models is stationary and the commodity space is $\mathbb{R}^n$. Here we consider a nonstationary model with an infinite dimensional commodity space. For such a general model, using techniques for the subdifferentiation theory of concave functions, we show that the optimal program can be sustained by a system of prices such that: (i) we have maximization of the expected intertemporal profit; (ii) also we have minimization of the expected cost among all programs producing no less utility; (iii) the expected value of the difference between a program with finite future gains and the optimal program has a nonnegative limit superior (weak transversality condition).

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Our approach is based on an induction argument and the extension to the Lebesgue-Bochner space $L^{\infty}(X)$ of the Yosida-Hewitt theorem, due to Levin [7].

2. The model

Let (Ω, Σ, μ) be a complete probability space. Each $\omega \in \Omega$ represents a possible state of the environment, Σ is the collection of all possible events and $\mu(\cdot)$ is the probability distribution of the states. Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ be our time horizon. So our model is a discrete-time, infinite horizon model. The uncertainty about the states is described by an increasing sequence $\{\Sigma_n\}_{n\geq 1}$ of complete sub-σ-fields of Σ. As usual Σ_n represents the information about the states available up until time n.

The commodity space is modelled by a separable reflexive Banach space X which is ordered by a nonempty, closed and convex cone X_+. In the recent years several mathematical economists, in particular those working on equilibrium theory, have considered models with an infinite dimensional commodity space (see the book of Aliprantis-Brown-Burkinshaw [1] and the references therein).

The production technology available at time n is described by a multifunction $P_n : \Omega \to 2^{X\times X} \setminus \{\emptyset\}$, which is Σ_n-graph measurable; i.e. $GrP_n = \{(\omega, x, y) \in \Omega \times X \times X : (x, y) \in P_n(\omega)\} \in \Sigma_n \times B(X) \times B(X)$, with $B(X)$ being the Borel σ-field of X. So $P_n(\omega)$ describes all possible transformations of capital at time n, when the state of the environment is $\omega \in \Omega$. Thus $(x, y) \in P_n(\omega)$ means that with the technology available at time n, we can transform a capital input x at time $n-1$ into a capital output y at time n. Note that the uncertainty in the production technology is manifested in the hypothesis that $GrP_n \in \Sigma_n \times B(X) \times B(X)$.

Also at every time instant $n \geq 1$, we are given a function $u_n : \Omega \times X \times X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, describing the utility (social satisfaction) achieved by the economy at time n when the state of the environment is $\omega \in \Omega$ and the input-output pair is (x, y). Again the uncertainty in the utility function is embedded in the hypothesis that $u_n(\cdot, \cdot, \cdot)$ is $\Sigma_n \times B(X) \times B(X)$-measurable.

A "program" is a discrete-time stochastic process $k_n : \Omega \to X$ such that for all $n \geq 0$, $k_n \in L^{\infty}(\Sigma_n, X)$. We say that a program $k = (k_n)_{n\geq 0}$ is "feasible", if $(k_{n-1}(\omega), k_n(\omega)) \in P_n(\omega)$ μ-a.e. and $k_0(\omega) = \overline{x}_0(\omega)$, where $\overline{x}_0(\cdot) \in L^{\infty}(\Sigma_0, X)$ is the initial capital stock.

Since we are not discounting future utilities, their sum over time may diverge and so we realize that the standard strong optimality criterion is not appropriate here. Instead, we use the weak maximality criterion, first introduced by Brock [2] in the context of a deterministic growth model. According to this criterion program $\{k_n\}_{n\geq 0}$ is "weakly maximal", if it is feasible and for any other feasible program $y = \{y_n\}_{n\geq 0}$ we have

$$\varliminf_{N\to\infty} \int_{\Omega} \sum_{n=1}^{N} (u_n(\omega, y_{n-1}(\omega), y_n(\omega)) - u_n(\omega, k_{n-1}(\omega), k_n(\omega)))\, d\mu(\omega) \leq 0,$$

i.e. $(k_n)_{n\geq 0}$ is weakly maximal, if no other feasible trajectory weakly overtakes it.

By a "price system", we understand a discrete-time stochastic process $p_n : \Omega \to X^*$ such that $p_n \in L^1(\Sigma, X^*)$, $p_n \geq 0$ (i.e. $p_n(\omega) \in X^*_+$ μ-a.e. with $X^*_+ = \{x^* \in X^* : (x^*, x) \geq 0$ for all $x \in X_+\}$, the positive dual cone of X_+). Our goal is to characterize weakly maximal programs using a system of support prices. We will do this using a normalized value (Bellman) function and techniques for the subdifferential theory of concave functions. Recall that if $g : x \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ is concave, then its subdifferential at $x \in X$ is defined to be the set $\partial g(x) = \{x^* \in X^* : g(y) - g(x) \leq (x^*, y - x)$ for all $x \in X\}$.

Now let us state the precise mathematical hypotheses on the data of our model.

<u>H_1</u>: $P_n : \Omega \to 2^{X\times X} \setminus \{\emptyset\}$ is a multifunction with closed and convex values, such that $GrP_n = \{(\omega, x, y) \in \Omega \times X \times x : (x, y) \in P_n(\omega)\} \in \Sigma_n \times B(X) \times B(X)$ and if $(x, y) \in P_n(\omega)$ and $x \leq x'$ (i.e. $x' - x \in X_+$), then $(x', y) \in P_n(\omega)$.

Hypothesis H_1 is very common in models of economic growth. Note that the multivaluedness of the map $P_n(\cdot)$ implies that the input does not uniquely determine the technologically possible output, a feature consistent with the nature of most economic process. Also the convexity requirement follows from the well-known "law of diminishing returns". The last requirement in hypothesis H_1 is a free disposability assumption.

<u>H_2</u>: $u_n : \Omega \times X \times X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, $n \geq 1$, is an integrand such that

(α) $u_n(\cdot, \cdot, \cdot)$ is $\Sigma_n \times B(X) \times B(X)$-measurable,
(β) $u_n(\omega, \cdot, \cdot)$ is u.s.c. and concave,
(γ) for every $(x, y) \in L^\infty(\Sigma_{n-1}, X) \times L^\infty(\Sigma_n, X)$, $J_n(x, y) = \int_\Omega u_n(\omega, x(\omega), y(\omega))\, d\mu(\omega)$ is finite,
(δ) if $(x, y) \in P_n(\omega)$ and $x \leq x'$ (i.e. $x' - x \in X_+$), then $u_n(\omega, x, y) \leq u_n(\omega, x', y)$.

Since we have undiscounted utilities, we have to normalize them to get an appropriate value function, when the expected value of the sum of utilities does not converge. In what follows by $S_n(v)$ we denote the set of all feasible programs that originate at time n from the capital stock $v \in L^\infty(\Sigma_n, X)$. Let $\{b_n\}_{n\geq 1}$ be a sequence of real numbers and $x \in S_n(v)$. We define

$$\xi_n^b(x) = \varliminf_{N\to\infty} \sum_{k=n+1}^{N} (J_k(x_{k-1}, x_k) - b_k)$$

where recall that $J_k(x_{k-1}, x_k) = \int_\Omega u_k(\omega, x_{k-1}(\omega), x_k(\omega))\, d\mu(\omega)$. Using $\xi_n^b(\cdot)$, we can define the following value function:

$$B_n^b(v) = \sup\left[\xi_n^b(x) : x \in S_n(v)\right] .$$

We consider only sequences $\{b_n\}_{n\geq 1}$ for which $B_0^b(\overline{x}_0)$ is finite. Brock [2] calls such sequences "good", while Takayama [11] uses the term "eligible". If $\{k_n\}_{n\geq 0}$ is a weakly maximal program, then the natural choice of a good sequence is $\{b_n = J_n(k_{n-1}, k_n)\}_{n\geq 1}$. In that case, we denote the value function by $B_n(\cdot)$. It is this function that we will be using in the proof of our main theorem (see Section 4). However to define a value function we do not need an a priori knowledge of a weakly maximal program.

From the above definitions we see that if $\{b_n\}_{n\geq 1}$ is good and $(v, w) \in L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X)$ are such that $(v(\omega), w(\omega)) \in P_{n+1}(\omega)$ μ-a.e., then

$$J_{n+1}(v, w) - b_{n+1} + B_{n+1}^b(w) \leq B_n^b(v)$$

and in fact, we have

$$B_n^b(v) = \sup\Big[J_{n+1}(v, w) - b_{n+1} + B_{n+1}^b(w) : w \in H_{n+1}(v)\Big]$$

with $H_{n+1}(v) = \{w \in L^\infty(\Sigma_{n+1}, X) : (v(\omega), w(\omega)) \in P_{n+1}(\omega)\ \mu\text{-a.e.}\}$. This is the well-known "Bellman dynamic programming equation".

3. Some auxiliary results

In this section, we have collected some auxiliary results that we will need in the proof of our main theorem in Section 4.

The first result actually tells us that every weakly maximal program solves the dynamic programming equation.

Proposition 3.1. *__If__ $\{z_n\}_{n\geq 0}$ is a feasible program and $\{b_n\}_{n\geq 1}$ is a good sequence and $\xi_0^b(z) = B_0^b(\overline{x}_0)$, __then__ $B_n^b(z_n) = J_{n+1}(z_n, z_{n+1}) - b_{n+1} + B_{n+1}^b(z_{n+1})$ for all $n \geq 0$.*

Proof: Since by hypothesis $\xi_0^b(z) = B_0^b(\overline{x}_0)$, we have

$$\begin{aligned}
B_0^b(\overline{x}_0) &= \varliminf_{N\to\infty} \sum_{k=1}^{N} (J_k(z_{k-1}, z_k) - b_k) \\
&= \sum_{k=1}^{n+1} (J_k(z_{k-1}, z_k) - b_k) + \varliminf_{N\to\infty} \sum_{k=n+2}^{N} (J_k(z_{k-1}, z_k) - b_k) \\
&= \sum_{k=1}^{n+1} (J_k(z_{k-1}, z_k) - b_k) + \xi_{n+1}^b(z) \\
&\leq \sum_{k=1}^{n+1} (J_k(z_{k-1}, z_k) - b_k) + B_{n+1}^b(z_{n+1}).
\end{aligned}$$

Also for any feasible program $\{v_n\}_{n\geq 0}$ such that $v_k = z_k$ for $k = 0, 1, 2, \ldots, n$, we have

$$\xi_0^b(v) = B_0^b(\overline{x}_0) \leq \sum_{k=1}^{n+1}(J_k(z_{k-1}, z_k) - b_k) + B_{n+1}^b(z_{n+1})$$

$$\Rightarrow \sum_{k=1}^{n}(J_k(v_{k-1}, v_k) - b_k) + \xi_n^b(v) \leq \sum_{k=1}^{n+1}(J_k(z_{k-1}, z_k) - b_k) + B_{n+1}^b(z_{n+1})$$

$$\Rightarrow \sum_{k=1}^{n}(J_k(z_{k-1}, z_k) - b_k) + \xi_n^b(v) \leq \sum_{k=1}^{n+1}(J_k(z_{k-1}, z_k) - b_k) + B_{n+1}^b(z_{n+1})$$

$$\Rightarrow \xi_n^b(v) \leq J_{n+1}(z_n, z_{n+1}) - b_{n+1} + B_{n+1}^b(z_{n+1}).$$

Since $\{v_n\}_{n\geq 0}$ was an arbitrary feasible program such that $v_k = z_k$ for $k = 0, 1, 2, \ldots, n$, we get

$$B_n^b(z_n) \leq J_{n+1}(z_m, z_{n+1}) - b_{n+1} + B_{n+1}^b(z_{n+1}).$$

Now recall that the opposite inequality is always true (dynamic programming equation). So we conclude that

$$B_n^b(z_n) = J_{n+1}(z_n, z_{n+1}) - b_{n+1} + B_{n+1}^b(z_{n+1}).$$

□

Remarks.

(i) Note that if $\{k_n\}_{n\geq 0}$ is a weakly maximal program and $\{b_n = J_n(k_{n-1}, k_n)\}_{n\geq 1}$ then we automatically have $\xi_0^b(k) = B_0^b(\overline{x}_0)$. Also recall that from the definition of weak maximality, we have that $\{b_n\}_{n\geq 1}$ is good.

(ii) If for a feasible program $\{z_n\}_{n\geq 0}$ and for a good sequence $\{b_n\}_{n\geq 1}$ we have $\theta_0^b(z) = B_0^b(\overline{x}_0)$, then $\{z_n\}_{n\geq 0}$ is weakly maximal. Indeed let $v \in S_0(\overline{x}_0)$. We have $\underline{\lim}_{N\to\infty} \sum_{k=1}^{N}(J_k(v_{k-1}, v_k) - J_k(z_{k-1}, z_k)) \leq \underline{\lim} \sum_{k=1}^{N}(J_k(v_{k-1}, v_k) - b_k) - \underline{\lim}_{N\to\infty} \sum_{k=1}^{N}(J_k(z_{k-1}, z_k) - b_k) \leq B_0^b(\overline{x}_0) - \xi_0^b(z) = B_0^b(\overline{x}_0) - B_0^b(\overline{x}_0) = 0$. Since $v \in S_0(\overline{x}_0)$ was arbitrary, we conclude that $\{z_n\}_{n\geq 0}$ is weakly maximal.

Using hypotheses H_1 and H_2, we can easily see that for any good sequence $\{b_n\}_{n\geq 0}$ the value function $B_n^b(\cdot)$ is concave for every $n \geq 0$. In particular, $B_n(\cdot)$ is concave for every $n \geq 0$ (recall that $B_n = B_n^b$ with $b_n = J_n(z_{n-1}, z_n)$, $n \geq 1$, for a weakly maximal program $\{z_n\}_{n\geq 0}$). It should be noted here that to get the concavity property of the value function, it is crucial that in its definition we use the limit inferior. By $\partial B_n(\cdot)$ we denote the concave subdifferential of $B_n(\cdot)$; i.e. for any $v \in L^\infty(\Sigma_n, X)$ we have

$$\partial B_n(v) = \{p \in L^\infty(\Sigma_n, X)^* : B_n(z) - B_n(v) \\ \leq p(z-v) \text{ for all } z \in L^\infty(\Sigma_n, X)\}.$$

We will make the following hypothesis concerning the above subdifferential:

$\underline{H_3}$: for every $n \geq 0$, the value function $B_n(\cdot)$ is continuous at some point w in $L^\infty(\Sigma_n, X)$ for which there exists $v \in L^\infty(\Sigma_{n-1}, X)$ such that $(v(\omega), w(\omega)) \in P_n(\omega)$ μ-a.e. (or equivalently $\operatorname{int}\operatorname{dom} B_n \cap \Gamma_n \neq \emptyset$, where $\operatorname{dom} B_n(\cdot) = \{v \in L^\infty(\Sigma_n, X) : -\infty < B_n(v)\}$ and $\Gamma_n = \{(v,w) \in L^\infty(\Sigma_{n-1}, X) \times L^\infty(\Sigma_n, X) : (v(\omega), w(\omega)) \in P_n(\omega)\ \mu\text{-a.e.}\}$ (see Section 4)) and $\partial B_0(\overline{x}_0) \neq \emptyset$.

From the definition of $\partial B_n(\cdot)$, we see that we have a problem. The dual space $L^\infty(\Sigma_n, X)^*$ is strictly bigger that $L^1(\Sigma_n, X^*)$, while by definition prices belong in $L^1(\Sigma_n, X^*)$. To remedy this, we will use a decomposition result due to Levin [7], which in turn extends a classical theorem of Yosida-Hewitt [14].

A functional $p \in L^\infty(\Sigma_n, X)^*$ is said to be "absolutely continuous" with respect to $\mu(\cdot)$, if there exists $q \in L^1(\Sigma_n, X^*)$ such that

$$p(x) = \int_\Omega (q(\omega), x(\omega))\, d\mu(\omega)$$

for all $x \in L^\infty(\Sigma_n, X)$. In the sequel, by $\langle \cdot, \cdot \rangle$ we will denote the duality brackets for the pair $(L^1(\Sigma_n, X^*), L^\infty(\Sigma_n, X))$. So for every $q \in L^1(\Sigma_n, X^*)$ and for every $x \in L^\infty(\Sigma_n, X)$, we have $\langle q, x \rangle = \int_\Omega (q(\omega), x(\omega))\, d\mu(\omega)$.

On the other hand, a functional $p \in L^\infty(\Sigma_n, X)^*$ is said to be "singular" with respect to $\mu(\cdot)$, if there exists a sequence $\{C_m\}_{m\geq 1} \subseteq \Sigma_n$ such that

(α) $C_{m+1} \subseteq C_m$ for all $m \geq 1$,
(β) $\mu(C_m) \downarrow 0$ as $m \to \infty$,
(γ) $p(x) = p(\chi_{C_m} x)$ for all $m \geq 1$ and all $x \in L^\infty(\Sigma_n, X)$; i.e. the sets C_m, $m \geq 1$, support the singular functional $p(\cdot)$.

The decomposition theorem of Levin [7] is the following:

Proposition 3.2. *Every functional $p(\cdot) \in L^\infty(\Sigma_n, X)^*$ admits a unique decomposition $p = p^a + p^s$, with $p^a(\cdot)$ being absolutely continuous with respect to $\mu(\cdot)$ and $p^s(\cdot)$ being singular with respect to $\mu(\cdot)$. In addition, we have $\|p\| = \|p^a\| + \|p^s\|$.*

Remark. The result is true for any Banach space X not necessarily separable and/or reflexive. In this case, the function $q(\cdot)$ corresponding to the absolutely continuous part $p^a(\cdot)$, belongs in $L^1(\Sigma_n, X^*_{w^*})$; i.e. for every $x \in X$, $\omega \to (q(\omega), x)$ is measurable (w^*-measurable) and $\|q(\cdot)\| \in L^1(\Sigma_n)_+$. For further details we refer to the paper of Levin [7]. In the sequel for simplicity, we will identify the absolutely continuous part with the $L^1(\Sigma_n, X^*)$-function corresponding to it.

The next auxiliary result is also interesting by itself as a general functional analytic result. Let τ denote the Mackey topology on $L^\infty(\Sigma_n, X)$ induced by the pair $(L^\infty(\Sigma_n, X), L^1(\Sigma_n, X^*))$. From the Mackey-Arens theorem (see for example Wilansky [13, p. 133]) we know that τ is the topology of uniform convergence on the weakly compact, convex subsets of $L^1(\Sigma_n, X^*)$. Finally by $\xrightarrow{\mu}$, we will denote the convergence in μ-measure.

Proposition 3.3. *If* $\{g_m, g\}_{m\geq 1} \subseteq L^\infty(\Sigma_n, X)$, $\|g_m\|_\infty \leq \theta$ *and* $g_m \xrightarrow{\mu} g$ *as* $m \to \infty$, *then* $g_m \xrightarrow{\tau} g$ *in* $L^\infty(\Sigma_n, X)$ *as* $m \to \infty$.

Proof: Let W be a nonempty, weakly compact and convex subset of the Lebesgue-Bochner $L^1(\Sigma_n, X^*)$. From Theorem 4, p. 104 of Diestel-Uhl [4], we have that W is uniformly integrable. Hence the set $\{v = \|f\| \cdot \|w\| \in L^1(\Sigma_n) : \|f\|_\infty \leq \theta,\ w \in W\}$ is uniformly integrable in $L^1(\Sigma_n)$ and so given $\varepsilon > 0$, we can find $\varphi \in L^1(\Sigma_n)$, $\varphi > 0$, such that for all $\|f\|_\infty \leq \theta$ and all $w \in W$, we have

$$\int\limits_{\{\|f\|\cdot\|w\|\geq\varphi\}} \|f(\omega)\| \cdot \|w(\omega)\|\, d\mu(\omega) \leq \varepsilon.$$

Without any loss of generality, we will assume that $g = 0$ and that for all $w \in W$, $\|w\|_1 = \int_\Omega \|w(\omega)\|\, d\mu(\omega) \leq 1$. We need to show that

$$\lim_{m\to\infty} \sup_{w\in W} \int\limits_{\{\|g_m\|\cdot\|w\|\leq\varphi\}} \|g_m(\omega)\| \cdot \|w(\omega)\|\, d\mu(\omega) = 0.$$

Note that $\{\omega \in \Omega : \varphi(\omega) = 0\} = \bigcap_{n>0}\{\omega \in \Omega : \varphi(\omega) < \eta\}$. Thus we can find $\gamma > 0$ such that $\int_{\{\varphi<\gamma\}} \varphi(\omega)\, d\mu(\omega) \leq \varepsilon$. Choose $\delta > 0$ so that if $A \in \Sigma_n$ and $\mu(A) \leq \delta$, then $\int_A \varphi(\omega)\, d\mu(\omega) \leq \varepsilon$. Let $0 < \beta < \min(\varepsilon, \delta)$. Since by hypothesis $g_m \xrightarrow{\mu} g$, there exists $m_0 \geq 1$ such that for all $m \geq m_0$ we have $\mu\{\omega \in \Omega : \varphi(\omega) \geq \gamma,\ \|g_m(\omega)\| \geq \beta\} \leq \beta$. Hence for $m \geq m_0$ and for all $w \in W$, we have

$$\int\limits_{\{\|g_m\|\cdot\|w\|<\varphi\}\cap\{\varphi\geq\gamma\}\cap\{\|g_m\|\geq\beta\}} \|g_m(\omega)\| \cdot \|w(\omega)\|\, d\mu(\omega)$$

$$\leq \int\limits_{\{\varphi\geq\gamma\}\cap\{\|g_m\|\geq\beta\}} \varphi(\omega)\, d\mu(\omega) \ \leq\ \varepsilon.$$

Note that for all $m \geq m_0$ and all $w \in W$, we have

$$\int\limits_{\{\|g_m\|\leq\beta\}} \|g_m(\omega)\| \cdot \|w(\omega)\|\, d\mu(\omega) \leq \beta.$$

So for all $m \geq m_0$ and all $w \in W$, we can write

$$\int_{\{\|g_m\|\cdot\|w\|<\varphi\}} \|g_m(\omega)\| \cdot \|w(\omega)\| \, d\mu(\omega)$$

$$= \int_{\{\|g_m\|\cdot\|w\|<\varphi\}\cap\{\varphi\geq\gamma\}\cap\{\|g_m\|\geq\beta\}} \|g_m(\omega)\| \cdot \|w(\omega)\| \, d\mu(\omega)$$

$$+ \int_{\{\|g_m\|\cdot\|w\|<\varphi\}\cap\{\varphi<\gamma\}\cup\{\|g_m\|<\beta\}} \|g_m(\omega)\| \cdot \|w(\omega)\| \, d\mu(\omega)$$

$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

$$\Rightarrow \lim_{m\to\infty} \sup_{w\in W} \int_{\{\|g_m\|\cdot\|w\|<\varphi\}} \|g_m(\omega)\| \cdot \|w(\omega)\| \, d\mu(\omega) = 0$$

$$\Rightarrow g_m \xrightarrow{\tau} g \quad \text{as} \quad m \to \infty.$$

□

4. Support prices

In this section we state and prove our main theorem, which establishes the existence of support process for a given weakly maximal program.

To this end, we define $\beta_1^n : L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X) \to L^\infty(\Sigma_n, X)$ by

$$\beta_1^n(f, g) = f \quad \text{(i.e. projection on the first component)}$$

and $\beta_2^n : L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X) \to L^\infty(\Sigma_{n+1}, X)$ by

$$\beta_2^n(f, g) = g \quad \text{(i.e. projection on the second component).}$$

Then we can easily check that $\beta_1^{n^*} : L^\infty(\Sigma_n, X)^* \to L^\infty(\Sigma_n, X)^* \times L^\infty(\Sigma_{n+1}, X)^*$ and $\beta_2^{n^*} : L^\infty(\Sigma_{n+1}, X)^* \to L^\infty(\Sigma_n, X)^* \times L^\infty(\Sigma_{n+1}, X)^*$ are defined by

$$\beta_1^{n^*}(v^*) = (v^*, 0) \quad \text{and} \quad \beta_2^{n^*}(w^*) = (0, w^*).$$

Also let $\Gamma_{n+1} = \{(f, g) \in L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X) : (f(\omega), g(\omega)) \in P_{n+1}(\omega)$ μ-a.e.$\}$. So Γ_{n+1} is the set of all possible transformations of capital stocks between times n and $n+1$. It is clear from hypothesis H_1 that Γ_{n+1} is closed and convex. By $\delta_{\Gamma_{n+1}}$ we will be denoting the indicator function of Γ_{n+1}; i.e. $\delta_{\Gamma_{n+1}}(y, z) = 0$ if $(y, z) \in \Gamma_{n+1}$ and $\delta_{\Gamma_{n+1}}(y, z) = -\infty$ otherwise. This is an u.s.c. and concave function.

Theorem 4.1. *If hypotheses H_1, H_2, H_3 hold, $\{k_n\}_{n\geq 0}$ is a weakly maximal program and for every $n \geq 0$ there exists some $\hat{e}_n \in L^\infty(\Sigma_n, X)$, $\hat{e}_n(\omega) \in X_+$ μ-a.e. such that $J_{n+1}(k_n, k_{n+1}) < J_{n+1}(k_n + \hat{e}_n, k_{n+1})$, then there exists a system of nonzero prices $p_n \in L^1(\Sigma_n, X^*)$ such that*

(1) $B_n(v) - \langle p_n, v\rangle \leq B_n(k_n) - \langle p_n, k_n\rangle$ *for all* $v \in \operatorname{proj}_1 \Gamma_{n+1} = \{y \in L^\infty(\Sigma_n, X) :$ *there exists* $z \in L^\infty(\Sigma_{n+1}, X)$ *such that* $(y, z) \in \Gamma_{n+1}\}$,

(2) $J_{n+1}(f, g) - \langle p_n, f\rangle + \langle p_{n+1}, g\rangle \leq J_{n+1}(k_n, k_{n+1}) - \langle p_n, k_n\rangle + \langle p_{n+1}, k_{n+1}\rangle$ *for all* $(f, g) \in \Gamma_{n+1}$,

(3) *if* $y \in S_0(\overline{x}_0)$ *and* $\underline{\lim}_{N\to\infty} \sum_{n=1}^{N} (J_n(y_{n-1}, y_n) - J_n(k_{n-1}, k_n)) > -\infty$, *then we have* $\overline{\lim}_{n\to\infty} \langle p_n, y_n - k_n\rangle \geq 0$.

Proof: Let $\gamma_1^n : L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ be defined by $\gamma_1^n = B_n \circ \beta_1^n$ and $\gamma_2^n : L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ be defined by $\gamma_2^n = J_{n+1} - b_{n+1} + B_{n+1} \circ \beta_2^n + \delta_{\Gamma_{n+1}}$ where $b_{n+1} = \int_\Omega u_{n+1}(\omega, k_n(\omega), k_{n+1}(\omega))\, d\mu(\omega) = J_{n+1}(k_n, k_{n+1}) \in \mathbb{R}$ (see hypothesis H_2). Clearly from the above definitions, for every $(f, g) \in L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X)$, we have

$$\gamma_2^n(f, g) \leq \gamma_1^n(f, g)$$

while from Proposition 3.1 (see also Remark (i) following the proof of that proposition), we have

$$\gamma_2^n(k_n, k_{n+1}) = \gamma_1^n(k_n, k_{n+1}).$$

From these facts and the definition of the concave subdifferential, we deduce that

$$\partial\gamma_1^n(k_n, k_{n+1}) \subseteq \partial\gamma_2^n(k_n, k_{n+1}).$$

But because of hypothesis H_3 and Theorem 2, p. 201 of Ioffe-Tichomirov [5], we have that

$$\partial\gamma_1^n(k_n, k_{n+1}) = \beta_1^{n^*} \partial B_n(k_n).$$

Also from hypothesis H_2 and Theorem 22, p. 61 of Rockafellar [10], we have that $J_{n+1}(\cdot, \cdot)$ is continuous (in fact τ-continuous) on $L^\infty(\Sigma_n, X) \times L^\infty(\Sigma_{n+1}, X)$. So using Theorem 1, p. 120 of Ioffe-Tichomirov [5], we get that

$$\partial\gamma_2^n(k_n, k_{n+1}) = \partial J_{n+1}(k_n, k_{n+1}) + \partial(B_{n+1} \circ \beta_2^n)(k_n, k_{n+1}) + \partial\delta_{\Gamma_{n+1}}(k_n, k_{n+1}).$$

A new application of Theorem 2, p. 201 of Ioffe-Tichomirov [5], gives us

$$\partial(B_{n+1} \circ \beta_2^n)(k_n, k_{n+1}) = \beta_2^{n^*} \partial B_{n+1}(k_{n+1}).$$

So finally we have

$$\partial\gamma_2^n(k_n, k_{n+1}) = \partial J_{n+1}(k_n, k_{n+1}) + \beta_2^{n^*} \partial B_{n+1}(k_{n+1}) + \partial\delta_{\Gamma_{n+1}}(k_n, k_{n+1}).$$

Recalling that $\partial\gamma_1^n(k_n,k_{n+1}) \subseteq \partial\gamma_2^n(k_n,k_{n+1})$, if $p_n \in \partial B_n(k_n)$, then we can find

$$(1) \qquad (z_n, z_{n+1}) \in \partial J_{n+1}(k_n,k_{n+1})$$
$$(2) \qquad p_{n+1} \in \partial B_{n+1}(k_{n+1})$$
$$(3) \qquad \text{and} \quad (y_n, y_{n+1}) \in \partial\delta_{\Gamma_{n+1}}(k_n,k_{n+1})$$

such that $\beta_1^{n^*}(p_n) = (z_n,z_{n+1}) + \beta_2^{n^*}(p_{n+1}) + (y_n,y_{n+1})$

$$\begin{aligned}
&\Rightarrow (p_n,0) = (z_n,z_{n+1}) + (0,p_{n+1}) + (y_n,y_{n+1})\\
&\Rightarrow (p_n,-p_{n+1}) = (z_n+y_n, z_{n+1}+y_{n+1})\\
&\Rightarrow p_n = z_n + y_n \quad \text{and} \quad -p_{n=1} = z_{n+1}+y_{n+1}.
\end{aligned}$$

First we will show that from (2) above, we have $p_{n+1}^a \in \partial B_{n+1}(k_{n+1})$. To this end note that from the definition of the concave subdifferential we have

$$B_{n+1}(w) - B_{n+1}(k_{n+1}) \le p_{n+1}(w-k_{n+1}) \quad \text{for all} \quad w \in L^\infty(\Sigma_{n+1},X).$$

Let $\{C_m\}_{m\ge1} \subseteq \Sigma_{n+1}$ be the decreasing sequence of Σ-sets such that $\mu(C_m)\downarrow 0$ and they support the singular part p_{n+1}^s of p_{n+1} (see Section 3). Set

$$w_m = \chi_{C_m^c} w + \chi_{C_m} k_{n+1} \in L^\infty(\Sigma_{n+1},X).$$

Then for all $m \ge 1$, we have using Proposition 3.2:

$$\begin{aligned}
&B_{n+1}(w_m) - B_{n+1}(k_{n+1}) \le p_{n+1}(w_m - k_{n+1})\\
&\quad = \langle p_{n+1}^a, w_m - k_{n+1}\rangle + p_{n+1}^s(w_m - k_{n+1})\\
&\quad = \langle p_{n+1}^a, w_m - k_{n+1}\rangle + p_{n+1}^s(\chi_{C_m}(w_m - k_{n+1}))\\
&\quad = \langle p_{n+1}^a, w_m - k_{m+1}\rangle.
\end{aligned}$$

Note that $w_m \xrightarrow{\mu} w$ and so by Proposition 3.3 we have that $w_m \xrightarrow{\tau} w$ as $m\to\infty$. Our claim is that $B_{n+1}(w) \le \underline{\lim}_{m\to\infty} B_{n+1}(w_m)$. To see this, let $\varepsilon > 0$ and choose $y \in S_{n+1}(w)$ such that for all $N \ge N_0(\varepsilon)$, we have

$$B_{n+1}(w) - \varepsilon \le \sum_{\tau=n+2}^{N} (J_\tau(y_{\tau-1},y_\tau) - b_\tau)$$

where $b_\tau = J_\tau(k_{\tau-1},k_\tau)$. Set $y_\tau^m = \chi_{C_m^c} y_\tau + \chi_{C_m} k_\tau$, $\tau \ge n+1$. Then clearly $y_\tau^m \xrightarrow{\mu} y_\tau$ as $m \to \infty$ and so by Proposition 3.3, $y_\tau^m \xrightarrow{\tau} y_\tau$ in $L^\infty(\Sigma_\tau,X)$ as $m\to\infty$. Recalling that $J_\tau(\cdot,\cdot)$ is τ-continuous on $L^\infty(\Sigma_{\tau-1},X)\times L^\infty(\Sigma_\tau,X)$ (see

Rockafellar [10, Theorem 2.2, p. 61]), we have that $J_\tau(y^m_{\tau-1}, y^m_\tau) \to J_\tau(y_{\tau-1}, y_\tau)$ as $m \to \infty$. So there exists $m_0(\varepsilon)$ such that for all $m \geq m_0(\varepsilon)$, we have

$$\left| \sum_{\tau=n+2}^{N} J_\tau(y^m_{\tau-1}, y^m_\tau) - J_\tau(y_{\tau-1}, y_\tau) \right| \leq \frac{\varepsilon}{2}.$$

Therefore for $m \geq m_0(\varepsilon)$ and $N \geq N_0(\varepsilon)$, we have

$$\begin{aligned} & B_{n+1}(w) - \frac{\varepsilon}{2} \leq \sum_{\tau=n+2}^{N} J_\tau(y^m_{\tau-1}, y^m_\tau) - b_\tau \\ & \Rightarrow B_{n+1}(w) - \frac{\varepsilon}{2} \leq B_{n+1}(w_m) \ \text{ for all } \ m \geq m_0 \\ & \Rightarrow B_{n+1}(w) \leq \underline{\lim}_{m\to\infty} B_{n+1}(w_m). \end{aligned}$$

Hence in the limit as $m \to \infty$, we get

$$\begin{gathered} B_{n+1}(w) - B_{n+1}(k_{n+1}) \leq \langle p^a_{n+1}, w - k_{n+1} \rangle, \ w \in L^\infty(\Sigma_{n+1}, X) \\ \Rightarrow p^a_{n+1} \in \partial B_{n+1}(k_{n+1}). \end{gathered}$$

In a similar fashion and using the τ-continuity of $J_{n+1}(\cdot,\cdot)$, we get

$$(z_n, z^a_{n+1}) \in \partial J_{n+1}(k_n, k_{n+1}) \ \text{ and } \ (y_n, y^a_{n+1}) \in \partial \delta_{\Gamma_{n+1}}(k_n, k_{n+1})$$

while from Proposition 3.2 and since $-p_{n+1} = z_{n+1} + y_{n+1}$, we get $-p^a_{n+1} = z^a_{n+1} + y^a_{n+1}$.

Therefore, so far we have:

$$(z_n, z^a_{n+1}) \in \partial J_{n+1}(k_n, k_{n+1}) \tag{1$'$}$$

$$p^a_{n+1} \in \partial B_{n+1}(k_{n+1}) \tag{2$'$}$$

$$(y_n, y^a_{n+1}) \in \partial \delta_{\Gamma_{n+1}}(k_n, k_{n+1}) \tag{3$'$}$$

and $p_n = z_n + y_n$, $-p^a_{n+1} = z^a_{n+1} + y^a_{n+1}$. From relation $(1)'$ above, we have

$$\begin{aligned} & J_{n+1}(v, w) - J_{n+1}(k_n, k_{n+1}) \leq z_n(v - k_n) + \langle z^a_{n+1}, w - k_{n+1} \rangle \\ & \Rightarrow J_{n+1}(v, w) - z_n(v) - \langle z^a_{n+1}, w \rangle \\ & \leq J_{n+1}(k_n, k_{n+1}) - z_n(k_n) - \langle z^a_{n+1}, k_{n+1} \rangle. \end{aligned} \tag{4}$$

Similarly from relation $(3)'$ above, we get

$$0 \leq y_n(v - k_n) + \langle y^a_{n+1}, w - k_{n+1} \rangle \ \text{ for all } \ (v, w) \in \Gamma_{n+1}. \tag{5}$$

Adding inequalities (4) and (5), we get

$$\begin{aligned}
&J_{n+1}(v,w)-(z_n+y_n)(v)-\langle z^a_{n+1}+y^a_{n+1},w\rangle\\
&\leq J_{n+1}(k_n,k_{n+1})-(z_n+y_n)(k_n)-\langle z^a_{n+1}+y^a_{n+1},k_{n+1}\rangle\\
&\Rightarrow J_{n+1}(v,w)-p_n(v)+\langle p^a_{n+1},w\rangle\\
&\leq J_{n+1}(k_n,k_{n+1})-p_n(k_n)+\langle p^a_{n+1},k_{n+a}\rangle \quad \text{for all} \quad (v,w)\in\Gamma_{n+1}.
\end{aligned}$$

We claim that in the above inequality we can replace $p_n(\cdot)$ by its absolutely continuous part $p^a_n(\cdot)$. So as before let $\{A_m\}_{m\geq 1}\subseteq\Sigma_n$, be a decreasing sequence of Σ_n-sets such that $\mu(A_m)\downarrow 0$ and they support the singular part $p^s_n(\cdot)$. Define

$$\begin{aligned}
v_m&=\chi_{A^c_m}v+\chi_{A_m}k_n\in L^\infty(\Sigma_n,X)\\
\text{and}\quad w_m&=\chi_{A^c_m}w+\chi_{A_m}k_{n+1}\in L^\infty(\Sigma_{n+1},X).
\end{aligned}$$

Note that $p_n(v_m-k_n)=\langle p^a_n,v_m-k_n\rangle+p^s_n(v_m-k_n)=\langle p^a_n,v_m-k_n\rangle+p^s_n(\chi_{A_m}(v_m-k_n))=\langle p^a_n,v_m-k_n\rangle\to\langle p^a_n,v-k_n\rangle$ as $m\to\infty$ since $v_m\xrightarrow{\tau}v$ (Proposition 3.3). Recalling that $J_{n+1}(\cdot,\cdot)$ is τ-continuous, in the limit as $m\to\infty$, we get

$$\begin{aligned}
&J_{n+1}(v,w)-\langle p^a_n,k_n\rangle+\langle p^a_{n+1},w\rangle\\
&\leq J_{n+1}(k_n,k_{n+1})-\langle p^a_n,k_n\rangle+\langle p^a_{n+1},k_{n+1}\rangle \quad \text{for all} \quad (v,w)\in\Gamma_{n+1},
\end{aligned}$$

which proves our claim. Since $p_n\in\partial B_n(k_n)$, as above we can have that $p^a_n\in\partial B_n(k_n)$. Therefore so far we have established the following: if $p_n\in\partial B(k_n)$, then $p^a_n\in\partial B_n(k_n)$ and we can find $p^a_{n+1}\in L^1(\Sigma_{n+1},X^*)$ such that $p^a_{n+1}\in\partial B_{n+1}(k_{n+1})$ and $J(v,w)-\langle p^a_n,v\rangle+\langle p^a_{n+1},w\rangle\leq J_{n+1}(k_n,k_{n+1})-\langle p^a_n,k_n\rangle+\langle p^a_{n+1},k_{n+1}\rangle$ for all $(v,w)\in\Gamma_{n+1}$.

From hypothesis H_0, we have $\partial B_0(k_0)=\partial B_0(\overline{x}_0)\neq\emptyset$ and so by induction we can generate a sequence $p_n\in L^1(\Sigma_n,X^*)$ satisfying the inequalities in conclusions (1) and (2) of the theorem.

Next we will show that for all $n\geq 0$, $p_n\geq 0$. To this end let $e\in L^\infty(\Sigma_{n+1},X)_+$ (i.e. $e\in L^\infty(\Sigma_{n+1},X)$, $e(\omega)\in X_+$ μ-a.e.). Let $z_n=k_n+e$. Then from the free disposability hypothesis, we have that $(z_n,k_{n+1})\in\Gamma_{n+1}$ and so from the inequality we just proved, we have

$$\begin{gathered}
J_{n+1}(z_n,k_{n+1})-\langle p^a_n,z_n\rangle+\langle p^a_{n+1},k_{n+1}\rangle\\
\leq J_{n+1}(k_n,k_{n+1})-\langle p^a_n,k_n\rangle+\langle p^a_{n+1},k_{n+1}\rangle\\
\Rightarrow J_{n+1}(z_n,k_{n+1})-J_{n+1}(k_n,k_{n+1})\leq\langle p^a_n,z_n-k_n\rangle=\langle p^a_n,e\rangle\\
\Rightarrow 0\leq\langle p^a_n,e\rangle \quad \text{(see hypothesis } H_2\,(\delta))\\
\Rightarrow p^a_n\geq 0 \quad \text{for all} \quad n\geq 0.
\end{gathered}$$

Also let $\hat{e}_n \in L^\infty(\Sigma_n, X)$, $\hat{e}_n \geq 0$ postulated by the hypothesis of the theorem. Then from the free-disposability assumption, we have that $(k_n + \hat{e}_n, k_{n+1}) \in \Gamma_{n+1}$ for all $n \geq 0$, and furthermore

$$
\begin{gathered}
0 < J_{n+1}(k_n + \hat{e}_n, k_{n+1}) - J_{n+1}(k_n, k_{n+1}) \leq \langle p_n^a, \hat{e}_n \rangle \\
\Rightarrow 0 < \langle p_n^a, \hat{e}_n \rangle; \quad \text{i.e.} \;\; p_n^a \neq 0 \;\; \text{for all} \;\; n \geq 0.
\end{gathered}
$$

So we have established that $\{p_n\}_{n\geq 0}$ is a nontrivial price system.

Finally let us check the transversality condition (3). So let $y \in S_0(\overline{x}_0)$ be such that $-\infty < \underline{\lim}_{N\to\infty} \sum_{\tau=1}^{N} (J_\tau(y_{\tau-1}, y_\tau) - J(k_{\tau-1}, k_\tau))$. From inequality (1) of the theorem, that we have already proved, we have

$$
B_n(y_n) \leq \langle p_n, y_n - k_n \rangle + B_n(k_n).
$$

Also from the definition of the value function, we have

$$
\begin{aligned}
\xi_0(y) - \sum_{\tau=1}^{n} (J_\tau(y_{\tau-1}, y_\tau) - J_\tau(k_{\tau-1}, k_\tau)) &\leq B_n(y_n) \\
\Rightarrow \xi_0(y) - \underline{\lim}_{n\to\infty} \sum_{\tau=1}^{n} (J_\tau(y_{\tau-1}, y_\tau) - J_\tau(k_{\tau-1}, k_\tau)) &\leq \overline{\lim} B(y_n) \\
\Rightarrow \xi_0(y) - \xi_0(y) = 0 \leq \overline{\lim} B(y_n) \;\; &(\text{from the choice of} \;\; y \in S_0(\overline{x}_0)).
\end{aligned}
$$

Since $\overline{\lim} B_0(k_n) = 0$, we conclude that

$$
0 \leq \overline{\lim}_{n\to\infty} \langle p_n, y_n - k_n \rangle.
$$

□

Remarks.

(i)′ Our hypothesis on the existence of $\hat{e}_n$'s is automatically satisfied if $\operatorname{int} X_+ \neq \emptyset$ and $u_n(\omega, x, y)$, $n \geq 1$, is strictly increasing in x (i.e. if $x' - x \in \operatorname{int} X_+$, then $u_n(\omega, x, y) < u_n(\omega, x', y)$). This is the case in Radner [9], Jeanjean [6], Dana [3] and Zilha [15].

(ii)′ A feasible program supported by a price system, is usually called in the economics literature "competitive". So we have proved that every weakly maximal program is competitive.

(iii)′ Inequality (1) of the theorem can be rewritten as follows

$$
B_n(v) - B_n(k_n) \leq \langle p_n, v - k_n \rangle \;\; \text{for all} \;\; v \in \operatorname{proj}_1 \Gamma_{n+1}, \; n \geq 0.
$$

This has the interpretation that the weakly maximal program $\{k_n\}_{n\geq 0}$ is cost minimizing for the price system $\{p_n\}_{n\geq 0}$ among all programs producing no less value. Also in inequality (2) the quantity $\langle p_{n+1}, w \rangle - \langle p_n, v \rangle$

is the expected value of the output w at time $n+1$ minus the expected cost of the input v at time n. Hence $\langle p_{n+1}, w\rangle - \langle p_n, v\rangle$ represents the expected net profit realized by the industrial process (v,w). So $J_{k+1}(v,w) - \langle p_n, v\rangle + \langle p_{n+1}, w\rangle$ is the expected total utility for the pair (v,w) and the inequality (2) tells us that for the price system $\{p_n\}_{n\geq 0}$, the weakly maximal program maximizes the expected total utility.

(iv)′ Condition (3) is sometimes known as the weak transversality condition, and it says that it is impossible to diminish the value of the weakly maximal program, except perhaps by incurring on infinite loss. From the previous works, Radner [9, Theorem 5.1], Jeanjean [6, Theorem 8], and Dana [3, Theorem VIII.I] do not get a transversality condition. Zilha [15, Theorem 1, p. 177] gets one, but he has discounted utilities and uses the strong optimality criterion. Also it seems to us that there is a gap in his proof. Namely in pp. 181–182, the set $\{C_n\}_{n\geq 1}$, which supports the singular part of v_{t+1} as a functional on $L^\infty(\Sigma_t, X)$, does not necessarily support its singular part as a functional on $L^\infty(\Sigma_{t+1}, X)$ (the notation is that of Zilha [15]).

(v)′ If we adopt the strong optimality criterion, the transversality condition becomes $\lim_{n\to\infty}\langle p_n, k_n\rangle = 0$, known as the strong transversality condition (see Zilha [15]).

(vi)′ The problem of existence of weakly maximal programs, for models with uncertainty was studied by Dana [3] and more recently by Pantelides-Papageorgiou [8].

(vii)′ It will be interesting to have a continuous time analog of this work. For the deterministic model this was done by Takekuma [12].

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