Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 35 (1994), No. 2, 283--289

Persistent URL: http://dml.cz/dmlcz/118667

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Opial's property and James' quasi-reflexive spaces

Tadeusz Kuczumow, Simeon Reich

Abstract. Two of James' three quasi-reflexive spaces, as well as the James Tree, have the uniform w^*-Opial property.

Keywords: fixed points, James' quasi-reflexive spaces, James Tree, nonexpansive mappings, Opial's property, the demiclosedness principle
Classification: Primary 46B20, 47H10

Introduction

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder finite dimensional decomposition (FDD) [1], [20]. Define $\beta_p((X,\|\cdot\|))$ for $p \in [1,\infty)$ to be the infimum of the set of numbers λ such that

$$(\|x\|^p + \|y\|^p)^{1/p} \le \lambda \|x+y\| \tag{1}$$

for every x and y in X with $\operatorname{supp}(x) < \operatorname{supp}(y)$ (we use here the notation in [1], [15]). In [15] M.A. Khamsi proved the following result.

Theorem A. *Let $(X,\|\cdot\|)$ be a Banach space with a finite codimensional subspace Y such that $\beta_p((Y,\|\cdot\|)) < 2^{1/p}$ for some $p \in [0,\infty)$. Then X has weak normal structure.*

He then used this theorem to deduce that James quasi-reflexive space, which consists of all null sequences $x = \{x^i\} = \sum_{i=1}^{\infty} e_i$ ($\{e_i\}$ is the standard basis in c_0) for which the squared variation

$$\sup_{\substack{m \\ p_1 < \cdots < p_m}} \left[\sum_{j=2}^{m} |x^{p_j} - x^{p_{j-1}}|^2 \right]^{1/2} \tag{2}$$

is finite, with the norm $\|\cdot\|$ given by (2), has weak normal structure by claiming that $\beta_2((J\|\cdot\|_1)) = 1$. As a matter of fact, $\beta_2((J,\|\cdot\|_1)) \ge 2^{1/2}$, which can be easily seen by taking $x = e_2$ and $y = e_3$. Fortunately, Theorem A remains true with a slight modification of the definition of $\beta_p((X,\|\cdot\|))$, namely

$$\widetilde{\beta}((X,\|\cdot\|)) = \\ = \inf_{k=0,1,2,\dots} \left\{ \inf[\lambda : (1) \text{ is valid for } x,y \in X \text{ with } \operatorname{supp}(x) + k < \operatorname{supp}(y)] \right\},$$

and $\widetilde{\beta}_2((J,\|\cdot\|)) = 1$. Thus $(J,\|\cdot\|_1)$ does indeed possess weak normal structure. In the present paper we will prove that $(J,\|\cdot\|_1)$ has, in fact, the uniform w^*-Opial property [23], which, of course, also implies weak normal structure [1], [3], [4], [8].

Definitions and notations

We recall that a (dual) Banach space $(X, \|\cdot\|)$ has the (w^*-) Opial property if whenever a sequence $\{x_n\}$ in X converges weakly (weakly*) to x_0, then for $x \neq x_0$

$$\liminf_n \|x_0 - x_n\| < \liminf \|x - x_n\|$$

[21], [22]. Opial's property plays an important role in the study of weak convergence of iterates and random products of nonexpansive mappings and of the asymptotic behavior of nonlinear semigroups [4], [5], [7], [13], [18], [21], [22]. Moreover, it can be introduced in the open unit ball of a complex Hilbert space, equipped with the hyperbolic metric, where it is useful in proving the existence of fixed points of holomorphic self-mappings of B [5], [6].

The (dual) Banach space $(X, \|\cdot\|)$ is said to have the uniform (w^*-) Opial property [23] if for every $c > 0$ there exists an $r > 0$ such that

$$1 + r \leq \liminf_n \|x - x_n\| \tag{3}$$

for each $x \in X$ with $\|x\| \geq c$ and every sequence $\{x_n\}$ with $w\text{-}\lim_n x_n = 0$ ($w^*\text{-}\lim_n x_n = 0$) and $\liminf_n \|x_n\| \geq 1$.

In the linear space J defined by (2) one uses three different, but equivalent norms, $\|\cdot\|_1$ (defined by (2)), $\|\cdot\|_2$, and $\|\cdot\|_3$, introduced by R.C. James [9], [10], [11]:

$$\|x\|_2 = \sup_{\substack{k \\ p_1 \cdots < p_{2k}}} \left[\sum_{j=1}^{k} |x^{p_{2j}} - x^{p_{2j-1}}|^2 \right]^{1/2},$$

$$\|x\|_3 = \sup_{\substack{m \\ p_1 \cdots < p_m}} \left[\sum_{j=2}^{m} |x^{p_j} - x^{p_{j-1}}|^2 + |x^{p_m} - x^{p_1}|^2 \right]^{1/2}.$$

The choice of norms depends on one's goals [2], [9], [10], [11], [20].

In [14] M.A. Khamsi used the ultraproduct method to prove that $(J, \|\cdot\|_3)$ has the fixed point property for nonexpansive mappings (FPP), i.e. for every nonempty weakly compact convex subset C of $(J, \|\cdot\|_3)$ any nonexpansive self-mapping $T : C \to C$ has a fixed point. D. Tingley [24] has recently shown that $(J, \|\cdot\|_3)$ has, in fact, weak normal structure ([3]): every nonempty weakly compact convex subset C of $(J, \|\cdot\|_3)$ with $\operatorname{diam} C > 0$ has a nondiametral point y, i.e.

$$\sup_{x \in C} \|y - x\|_3 < \operatorname{diam} C.$$

This property immediately guarantees the FPP [17]. The proof of weak normal structure is based on the following property of weakly convergent sequences in $(J, \|\cdot\|_3)$: if $\{x_n\}$ converges weakly to 0 and $\operatorname{diam}\{x_n\} > 0$, then

$$\sup_m \left(\limsup_n \|x_m - x_n\|_3 \right) > \liminf_n \|x_n\|_3 .$$

But it is easy to observe that the sequence $\{-e_n + e_{n+1}\}$ tends weakly to 0 in $(J, \|\cdot\|_3)$ and

$$\|\frac{1}{3}e_1 + e_n - e_{n+1}\|_3 = \| - e_n + e_{n+1}\|_3 = \sqrt{8}$$

for $n \geq 3$. Therefore $(J, \|\cdot\|_3)$ does not have Opial's property.

Main result

In this section we are concerned with the spaces $(J, \|\cdot\|_1)$ and $(J, \|\cdot\|_2)$.

The predual Banach space I to $(J, \|\cdot\|_j)$, $j = 1, 2$, is generated by the biorthogonal functionals $\{f_n\}$ to the basis $\{u_n\} = \{e_1 + \cdots + e_n\}$ [12], [19]. Throughout this paper we will always treat J as I^*.

Theorem. *For $j = 1, 2$ the space $(J, \|\cdot\|_j)$ has the uniform w^*-Opial property.*

PROOF: Let $k \in \mathbb{N}$ and let P_k and Q_k be the natural projections in J associated with the basis $\{u_n\}$:

$$P_k x = \sum_{n=1}^{k} \xi^n u_n$$

and

$$Q_k x = \sum_{n=k+1}^{\infty} \xi^n u_n$$

for each $x = \sum_{n=1}^{\infty} \xi^n u_n \in J$. Note that if $x = \sum_{n=1}^{\infty} \xi^n u_n$, then

$$\|x\|_1 = \sup_{\substack{m \\ p_1 < \cdots < p_m}} \left\{ \sum_{j=2}^{m} \left[\sum_{n=p_{j-1}}^{p_j - 1} \xi^n \right]^2 \right\}^{1/2}$$

and

$$\|x\|_2 = \sup_{\substack{k \\ p_1 < \cdots < p_{2k}}} \left\{ \sum_{j=1}^{k} \left[\sum_{n=p_{2j-1}}^{p_{2j} - 1} \xi^n \right]^2 \right\}^{1/2}$$

[11], [12]. Directly from these formulas we obtain

$$\|x\|_j = \lim_k \|P_k x\|_j$$

and

$$\lim_k \|Q_k x\|_j = 0$$

for all $x \in J$ and $j = 1, 2$. Assume that a sequence $\{x_n\}$ in $(J, \|\cdot\|_j)$ converges weakly* to 0 and let $x \in J$. Then we have

$$\lim_n \|P_k x_n\|_j = 0,$$
$$\liminf_n \|Q_k x_n\|_j = \liminf_n \|x_n\|_j,$$

and

$$\begin{aligned}
\liminf_n \|x - x_n\| &\geq \liminf_n \Big[\|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j - \|P_{k+1} x_n\|_j \Big] \\
&= \liminf_n \Big[\|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j \Big] \\
&\geq \liminf_n \Big[\|P_k x\|_j^2 + \|Q_{k+1} x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j \\
&= \Big[\|P_k x\|_j^2 + \liminf_n \|Q_{k+1} x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j \\
&= \Big[\|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j
\end{aligned}$$

for $k = 1, 2, \ldots$. Hence we obtain the following inequality

$$(*) \qquad \begin{aligned}
\liminf_n \|x - x_n\|_j &\geq \lim_k \left\{ \Big[\|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \Big]^{1/2} - \|Q_k x\|_j \right\} \\
&= \Big[\|x\|_j^2 + \liminf_n \|x_n\|_j^2 \Big]^{1/2}
\end{aligned}$$

which leads to (3). In other words, $(J, \|\cdot\|_j)$ has the uniform w^*-Opial property for $j = 1, 2$. □

Corollary 1. *For $j = 1, 2$ the space $(J, \|\cdot\|_j)$ has the uniform Opial property.*

Remark 1. The uniform w^*-Opial property of $(J, \|\cdot\|_j)$, $j = 1, 2$, implies the following important property of these spaces. The (w^*-) modulus of noncompact convexity of a (dual) Banach space $(X, \|\cdot\|)$ is the function $\Delta_x : [0, 1] \to [0, 1]$ ($\Delta_x^* : [0, 1] \to [0, 1]$) defined by

$$\Delta_x(\varepsilon) = \inf\{1 - \mathrm{dist}\,(0, A)\}$$
$$(\Delta_x^*(\varepsilon) = \inf\{1 - \mathrm{dist}\,(0, A)\}),$$

where the infimum is taken over all convex (weak* compact convex) subsets A of the closed unit ball with $\chi(A) \geq \varepsilon$, and χ is the Hausdorff measure of noncompactness [4]. In the case of $(J, \|\cdot\|_j)$, $j = 1, 2$, the inequality $(*)$ implies $\Delta_j^*(\varepsilon) > 0$

for all $\varepsilon > 0$. This means that these spaces are Δ^*-uniformly convex and every weakly* compact convex subset C of $(J, \|\cdot\|_j)$ $(j = 1, 2)$ has a compact asymptotic center [4]. Taking $A = \operatorname{conv}\{u_n\}$, where $u_n = \sum_{i=1}^n$, we see that

$$\chi(A) = 1$$

and

$$\forall_{x \in A} \ \|x_j\| = 1$$

$(j = 1, 2)$. Therefore $\Delta_x \equiv 0$ for $X = (J, \|\cdot\|_j)$, $j = 1, 2$.

Here we have to mention that generally the uniform Opial property does not imply the Δ-uniform convexity as the following example shows.

Example ([23])**.** For $\lambda > 1$ let X be the space l_2 with the norm

$$\|(\alpha_n)\| = \max\{\lambda|\alpha_1|, \ \|(\alpha_n)\|_2\}$$

where $\|\cdot\|_2$ is the norm in l_2. Then

$$\begin{aligned} \liminf_n \|x_n - x\| &= \max\left\{\lambda|\alpha_1|, \ \left(\liminf_n \|x_n\|_2^2 + \|x\|_2^2\right)^{1/2}\right\} \\ &\geq \left(1 + \|x\|_2^2\right)^{1/2} \geq \left(1 + \lambda^{-2}\|x\|_2^2\right)^{1/2} \end{aligned}$$

for each $x \in X$ and each sequence $\{x_n\}$ with $w\text{-}\lim_n x_n = 0$ and $\liminf \|x_n\| \geq 1$. This inequality guarantees the uniform Opial property of X, but $\Delta_x(\varepsilon) = 0$ for all $\varepsilon \leq (1 - \lambda^{-2})^{1/2}$.

Remark 2. It is easy to observe that James Tree JT constructed by R.C. James [11] also has the w^*-uniform Opial property, where JT is the dual space to the Banach space B generated by the biorthogonal functionals $\{f_{n,i}\}$ to the basis $\{e_{n,i}\}$ (this basis is analogous to the basis $\{u_n\}$ in J) given in [19]. The proof of this fact is a slight modification of the proof of the Theorem. Corollary 1 and Remark 1 are also valid for JT. (See also [13] for the w^*-Opial property.)

Remark 3. The uniform (w^*-) Opial property of $(J, \|\cdot\|_j)$ with $j = 1, 2$ and JT implies that these spaces satisfy the weak (weak*) uniform Kadec-Klee property [16].

We conclude our paper with three corollaries.

Corollary 2. $(J, \|\cdot\|_j)$, $j = 1, 2$, *and* JT *have weak and weak* normal structure.*

Corollary 3. $(J, \|\cdot\|_j)$, $j = 1, 2$, *and* JT *have the* FPP *for weakly* compact convex subsets.*

Recall that a Banach space $(X, \|\cdot\|)$ is said to satisfy the (w^*-) demiclosedness principle [1], [4] if whenever C is a nonempty weakly (weakly*) compact convex subset of X and $T : C \to X$ is nonexpansive, then the mapping $I - T$, where I is the identity operator, is (w^*-) demiclosed, i.e. if $\{x_n\}$ is weakly (weakly*) convergent to x and $\{x_n - Tx_n\}$ converges strongly to y, then $x - Tx = y$. It is known that every Banach space with the (w^*-) Opial property satisfies the (w^*-) demiclosedness principle.

Corollary 4. *$(J, \|\cdot\|_j)$, $j = 1, 2$, and JT satisfy the (w^*-) demiclosedness principle.*

Acknowledgement. This paper was prepared during a visit of the first author to the Department of Mathematics at the University of Southern California. He is grateful to his colleagues at USC for their hospitality.

References

[1] Aksoy A.G., Khamsi M.A., *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York, 1990.

[2] Andrew A., *Spreading basic sequences and subspaces of James' quasi-reflexive space*, Math. Scan. **48** (1981), 109–118.

[3] Brodskii M.S., Milman D.P., *On the center of a convex set*, Dokl. Akad. Nauk SSSR **59** (1948), 837–840.

[4] Goebel K., Kirk W.A., *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[5] Goebel K., Reich S., *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[6] Goebel K., Sekowski T., Stachura A., *Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball*, Nonlinear Analysis **4** (1980), 1011–1021.

[7] Gọrnicki J., *Some remarks on almost convergence of the Picard iterates for nonexpansive mappings in Banach spaces which satisfy the Opial condition*, Comment. Math. **29** (1988), 59–68.

[8] Gossez J.P., Lami Dozo E., *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40** (1972), 565–573.

[9] James R.C., *Bases and reflexivity of Banach spaces*, Ann. of Math. **52** (1950), 518–527.

[10] ______, *A non-reflexive Banach space isometric with its second conjugate space*, Proc. Nat. Acad. Sci. USA **37** (1951), 134–137.

[11] ______, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Amer. Math. Soc. **80** (1974), 738–743.

[12] ______, *Banach spaces quasi-reflexive of order one*, Studia Math. **60** (1977), 157–177.

[13] Karlovitz L.A., *On nonexpansive mappings*, Proc. Amer. Math. Soc. **55** (1976), 321–325.

[14] Khamsi M.A., *James' quasi-reflexive space has the fixed point property*, Bull. Austral. Math. Soc. **39** (1989), 25–30.

[15] ______, *Normal structure for Banach spaces with Schauder decomposition*, Canad. Math. Bull. **32** (1989), 344–351.

[16] ______, *On uniform Opial condition and uniform Kadec-Klee property in Banach and metric spaces*, preprint.

[17] Kirk W.A., *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

[18] Kuczumow T., *Weak convergence theorems for nonexpansive mappings and semigroups in Banach spaces with Opial's property*, Proc. Amer. Math. Soc. **93** (1985), 430–432.

[19] Lindenstrauss J., Stegall C., *Examples of separable spaces which do not contain l_1 and whose duals are non-separable*, Studia Math. **54** (1975), 81–105.

[20] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces, Vol. I and II*, Springer-Verlag, BerlinHeidelberg-New York, 1977 and 1979.

[21] Opial Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[22] ______, *Nonexpansive and Monotone Mappings in Banach Spaces*, Lecture Notes 61-1, Center for Dynamical Systems, Brown University, Providence, R.I., 1967.

[23] Prus S., *Banach spaces with the uniform Opial property*, Nonlinear Analysis **18** (1992), 697–704.
[24] Tingley D., *The normal structure of James' quasi-reflexive space*, Bull. Austral. Math. Soc. **42** (1990), 95–100.

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(*Received* May 17, 1993)