## ANNOUNCEMENTS OF NEW RESULTS
(of authors having an address in Czech Republic)

### CHARACTERIZING CONGRUENCE LATTICES OF LATTICES

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It has been known since the fourties that the congruence lattice of a lattice $\mathbf{L}$ is algebraic and satisfies the distributive law. In this paper we announce a proof of the converse:

**Theorem.** *Every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice.*

Let $\mathbf{S}$ be the semilattice of compact elements of a distributive algebraic lattice $\mathbf{L}$. We take an upward directed system $\mathcal{S} = \{\mathbf{S}_A : A \in I\}$ of finite distributive subsemilattices of $\mathbf{S}$ such that $S = \bigcup_{A\in I} S_A$. Thus $\mathbf{S}$ is isomorphic to the colimit of the system $\mathcal{S}$ together with the inclusion embeddings $\phi_{A,B} : \mathbf{S}_A \to \mathbf{S}_B$, where $A \leq B$ in $I$. Then we construct a system $\mathcal{L} = \{\mathbf{L}_A : A \in I\}$ of finite lattices, a corresponding system $\{\chi_{A,B} : \mathbf{L}_A \to \mathbf{L}_B : A \leq B, A, B \in I\}$ of lattice embeddings, and for every $A \in I$ an isomorphism $\iota_A$ from $\mathbf{S}_A$ to the semilattice $\mathbf{Con}\,\mathbf{L}_A$ of compact congruences of $\mathbf{L}_A$ satisfying the commuting identity

$$\text{con}(\chi_{A,B}) \circ \iota_A = \iota_B \circ \phi_{A,B}. \tag{1}$$

Here $\text{con}(\chi_{A,B})$ is the mapping from $\mathbf{Con}\,\mathbf{L}_A$ to $\mathbf{Con}\,\mathbf{L}_B$ assigning to every compact congruence $\theta$ of $\mathbf{L}_A$ the congruence of $\mathbf{L}_B$ generated by $\{(\chi_{A,B}(a), \chi_{A,B}(b)) : (a,b) \in \theta\}$. We refer to a system $\mathcal{L}$ satisfying these conditions as a simultaneous representation of $\mathcal{S}$. The following result is crucial for the method (originally proposed by P. Pudlák).

**Lemma 1.** *If $\mathcal{L} = \{\mathbf{L}_A : A \in I\}$ is a simultaneous representation of $\mathcal{S} = \{\mathbf{S}_A : A \in I\}$ then the semilattice of compact congruences of the colimit of $\mathcal{L}$ is isomorphic to the colimit of $\mathcal{S}$, i.e. to $\mathbf{S}$.*

We start by defining the limit system $\mathcal{S}$. As the index set $I$ we choose the set of all finite subsets of nonzero elements of $S$ ordered by inclusion. We set $S_\emptyset = \{0\}$ and $S_{\{z\}} = \{0, z\}$ for every $z \in S$. If $B$ is a finite subset of $S$ containing more than one element and if $\mathbf{S}_A$ has already been defined for every proper subset $A$ of $B$, then we choose $\mathbf{S}_B$ as an arbitrary finite distributive subsemilattice of $\mathbf{S}$ containing $\bigcup_{A\subset B} S_A$.

Next we construct a simultaneous representation of the limit system $\mathcal{S}$. First of all we define finite atomistic lattices $\mathbf{L}_B$, for $B \in I$. By $J(\mathbf{S}_A)$ we denote the set of join-irreducible elements of $\mathbf{S}_A$.

We choose $\mathbf{L}_\emptyset = \{0\}$. If $B \neq \emptyset$, then the atoms of $\mathbf{L}_B$ are of the form

$$\langle (A_0, p_0, i_0), \ldots, (A_n, p_n, i_n) \rangle,$$

where

(i) $\emptyset \neq A_0 \prec A_1 \prec \ldots \prec A_n = B$,

(ii) $p_k \in \{0, 1, 2\}$,

(iii) $i_k \in J(\mathbf{S}_{A_k})$ and $i_{k+1} \leq i_k$ for $k < n$.

For a sequence $b$ as above we define $\mu_B(b) = i_n$. Note also that the initial part of length $k+1$ of any atom is an atom of the corresponding $\mathbf{L}_{A_k}$.

Next we specify the defining inequalities $\Phi_B$ for the lattices $\mathbf{L}_B$. We shall proceed by induction on $B$. If $B = \{z\}$, then we set $\mathbf{L}_{\{z\}}$ to be isomorphic to $\mathbf{M}_3$. Suppose now that $|B| \geq 2$ and that for every proper subset $A$ of $B$, the lattice $\mathbf{L}_A$ has already been defined by a set of minimal inequalities $\Phi_A$. If $a \leq \sum_{w \in W} b_w$ is an inequality of $\Phi_A$ and $i \in J(\mathbf{S}_B)$ satisfying $i \leq \mu_A(a)$, then we add to $\Phi_B$ the inequality

$$\langle a, (B, p, i) \rangle \leq \sum_{w,q} \langle b_w, (B, q, i) \rangle. \tag{2}$$

Suppose now $A_1$, $A_2$ are distinct proper maximal subsets of $B$ and $a \in At(\mathbf{L}_{A_1 \cap A_2})$. For every $j \leq \mu_{A_1 \cap A_2}(a)$ we add to $\Phi_B$ the inequalities

$$\sum \{\langle a, (A_1, p_1, i_1), (B, p, j) \rangle\} = \sum \{\langle a, (A_2, p_2, i_2), (B, q, j) \rangle\} \tag{3}$$

To assure that $\mathbf{Con}\,\mathbf{L}_B$ is isomorphic to $\mathbf{S}_B$ we add new inequalities

$$\langle (B, p, i) \rangle \leq \langle (B, q, i) \rangle + \langle (B, r, j) \rangle, \tag{4}$$

where $i, j \in J(\mathbf{S}_B)$, $i \leq j$ and $p \neq q \neq r \neq p$, and

$$\begin{aligned} \langle a, (B, p, i) \rangle &\leq \langle a, (B, q, i) \rangle + \langle (B, r, i) \rangle, \\ \langle (B, p, i) \rangle &\leq \langle a, (B, q, i) \rangle + \langle a, (B, r, i) \rangle. \end{aligned} \tag{5}$$

Now we define $\mathbf{L}_B$ as the atomistic lattice defined on $At(\mathbf{L}_B)$ by inequalities (2)–(5). It is straightforward to prove that the mapping $\iota_B : \mathbf{S}_B \to \mathbf{Con}\,\mathbf{L}_B$ assigning to every order ideal $J$ of $J(\mathbf{S}_B)$ the least congruence of $\mathbf{L}_B$ identifying with 0 every atom $b \in L_B$ such that $\mu_B(b) \in J$ is an isomorphism. It is considerably more difficult to prove that the formula $\chi_{A,B}(a) = \sum\{b \in At(\mathbf{L}_B) : b = \langle a, (B, p, i) \rangle\}$ determines a lattice embedding $\chi_{A,B} : \mathbf{L}_A \to \mathbf{L}_B$ for every maximal proper subset $A$ of $B$.

If $A \subset B$ is not maximal, then we compose an embedding $\chi_{A,B} : \mathbf{L}_A \to \mathbf{L}_B$ from the embeddings of the previous paragraph. The inequalities (3) imply that this definition is correct. A straightforward verification of the commuting identity leads to the following lemma. Combined with Lemma 1 it proves the theorem.

**Lemma 2.** *The family $\mathcal{L} = \{L_A : A \in I\}$ with the embeddings $\{\chi_{A,B} : A, B \in I, A \subseteq B\}$ and isomorphisms $\{\iota_A : A \in I\}$ is a simultaneous representation of $\mathcal{S}$.*

Complete proofs are given in

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A PERTURBATION THEOREM FOR LINEAR EQUATIONS

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We describe here explicit formulae for componentwise bounds on solution of a system of linear equations

$$Ax = b$$

($A$ square) under perturbation of all data. To make the result numerically tractable, we avoid the use of exact inverses, using instead some matrices $R$ and $M$ required only to satisfy certain inequalities. Hansen's optimality result [1], [2] is a special case of our theorem. Notations used: $I$ is the unit matrix, $\varrho$ denotes the spectral radius, for $A = (a_{ij})$ we denote $|A| = (|a_{ij}|)$ and inequalities are understood componentwise.

**Theorem.** *Let $A, \Delta \in R^{n\times n}$, $b, \delta \in R^n$, $\Delta \geq 0$, $\delta \geq 0$ and let $R$ and $M$ be arbitrary matrices satisfying*

$$\begin{aligned} MG + I &\leq M, \\ M &\geq 0, \end{aligned} \tag{1}$$

*where*

$$G = |I - RA| + |R|\Delta.$$

*Then for each $A'$ and $b'$ such that*

$$\begin{aligned} |A' - A| &\leq \Delta, \\ |b' - b| &\leq \delta, \end{aligned}$$

*$A'$ is nonsingular and the solution of the system*

$$A'x' = b'$$

*for each $i \in \{1, \ldots, n\}$ satisfies*

$$\min\left\{\frac{\underset{\sim}{x}_i}{\alpha_i}, \frac{\underset{\sim}{x}_i}{\beta_i}\right\} \leq x'_i \leq \max\left\{\frac{\widetilde{x}_i}{\alpha_i}, \frac{\widetilde{x}_i}{\beta_i}\right\}, \tag{2}$$

*where*

$$\begin{aligned} \underset{\sim}{x}_i &= -(M(|Rb| + |R|\delta))_i + m_i(Rb + |Rb|)_i \\ \widetilde{x}_i &= (M(|Rb| + |R|\delta))_i + m_i(Rb - |Rb|)_i \\ \alpha_i &= 1 + (|r_i| - r_i)m_i + h_i \\ \beta_i &= 2m_i - 1 - (|r_i| + r_i)m_i - h_i \\ m_i &= M_{ii} \\ r_i &= (I - RA)_{ii} \\ h_i &= (M - MG - I)_{ii} \end{aligned}$$

*and*

$$\beta_i \geq \alpha_i \geq 1.$$

*Moreover, if* $A = I$ *and* $\varrho(\Delta) < 1$, *and if we take* $R := I$ *and* $M := (I - \Delta)^{-1}$, *then the bounds* (2) *are exact* (*i.e. achieved*).

The *proof* employs the ideas of the proofs of Theorems 1 and 3 in [2]; details are omitted here.

*Comments.* The quantities $r_i$ and $h_i$ correct the influence of the approximate inverses $R$ and $M$; they vanish if $R = A^{-1}$ and $M = (I - G)^{-1} \geq 0$ are used. The last statement of the theorem is Hansen's optimality result [1] as reformulated in [2]. It can be shown that matrices $R$ and $M \geq 0$ satisfying (1) exist if and only if

$$\varrho\left(|A^{-1}|\Delta\right) < 1$$

holds. In this case, if $R$ is chosen sufficiently close to $A^{-1}$ to achieve $\varrho(G) < 1$, then $M$ can be computed by the following *finite* algorithm:

$F :=$ a (small) positive matrix; $M' := 0$;
**repeat** $M := M'$; $M' := MG + I + F$ **until** $|M' - M| < F$;
{then the last $M$ is positive and satisfies (1)}.

## References


[1] Hansen E.R., *Bounding the solution of interval linear equations*, SIAM J. Numer. Anal. **29** (1992), 1493–1503.

[2] Rohn J., *Cheap and tight bounds: the recent result by E. Hansen can be made more efficient*, to appear in Interval Computations.