# Fixed points of asymptotically regular mappings in spaces with uniformly normal structure

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*Abstract.* It is proved that: for every Banach space $X$ which has uniformly normal structure there exists a $k > 1$ with the property: if $A$ is a nonempty bounded closed convex subset of $X$ and $T : A \to A$ is an asymptotically regular mapping such that

$$\liminf_{n\to\infty} |||T^n||| < k,$$

where $|||T|||$ is the Lipschitz constant (norm) of $T$, then $T$ has a fixed point in $A$.

*Keywords:* asymptotically regular mappings, uniformly normal structure, fixed points

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## 1. Introduction.

The concept of uniformly normal structure is due to A.A. Gillespie and B.B. Williams [7]. A Banach space $X$ has uniformly normal structure if

$$N(X) = \sup\{r_A(A) : A \subset X, \quad \text{convex}, \ \operatorname{diam} A = 1\} < 1,$$

where

$$r_A(A) = \inf \{\sup\{\|x - y\| : y \in A\} : x \in A\}.$$

It was proved in [4], [2] that $N(X) \le 1-\delta_X(1)$; thus $\varepsilon_0(X) < 1$ implies uniformly normal structure. In the paper [11] X.T. Yu proved that if $X$ is a uniformly smooth space (or more generally, $\lim_{t\downarrow 0} \rho_X(t)t^{-1} < \frac{1}{2}$), then $X$ has a uniformly normal structure. Also, in [12] it was proved that uniformly normal structure does not necessarily imply that the space has good geometric properties.

The concept of asymptotic regularity is due to F. Browder and V. Petryshyn [1]. A mapping $T : X \to X$ is said to be asymptotically regular if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

for all $x \in X$.

If $T$ is nonexpansive, then $T_\lambda := \lambda \cdot I + (1 - \lambda) \cdot T$ is asymptotically regular for all $0 < \lambda < 1$ (see [6]).

Recently P.K. Lin in [10] has constructed a uniformly asymptotically regular Lipschitzian mapping acting on a weakly compact subset of $l^2$ which has no fixed point.

E.A. Lifshitz (see [5]) associated with each metric space $(M, d)$ a constant $\kappa(M) \geq 1$. Define Lifshitz characteristic $\kappa_0(X)$ to be the infimum of $\kappa(C)$ where $C$ ranges over all nonempty closed bounded convex subsets of the Banach space $X$. D.J. Downing and B. Turett [5] proved the following

**Theorem 1.** *Let $X$ be a Banach space.*

(1) *Then $\varepsilon_0(X) < 1$ if and only if $\kappa_0(X) > 1$.*
(2) *If $\gamma > 1$ satisfies $\gamma(1 - \delta_X(\gamma^{-1})) = 1$, then $\gamma \leq \kappa_0(X)$.*

In [8] the present author proved the following

**Theorem 2.** *Let $X$ be a Banach space with the Lifshitz characteristic $\kappa_0(X) > 1$ and let $C$ be a nonempty bounded closed convex subset of $X$. If $T : C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} |||T^n||| < \kappa_0(X),$$

*then $T$ has a fixed point in $C$.*

## 2. Main result.

The main result of this paper is interesting in the Banach spaces $X$ which satisfy the conditions: $\varepsilon_0(X) \geq 1$ and $N(X) < 1$ (cf. [3]).

We start with the following

**Lemma 1** [3]**.** *Let $X$ be a Banach space with $N(X) < 1$. Then for every bounded sequence $\{x_n\}$ there exists a point $z \in \overline{\text{conv}}\{x_n\}$, such that:*

(i) $\limsup_{n\to\infty} \|z - x_n\| \leq N(X) \cdot \lim_{s\to\infty} \sup\{\|x_n - x_m\| : n, m \geq s\}$,
(ii) *for every $y \in X$, $\|z - y\| \leq \limsup_{n\to\infty} \|y - x_n\|$.*

**Lemma 2** [9]**.** *Let $A$ be a nonempty closed convex subset of a Banach space $X$ and let $\{n_i\}$ be an increasing sequence of natural numbers. Assume that $T : A \to A$ is an asymptotically regular mapping such that for some $m \in \mathbb{N}$, $T^m$ is continuous. If*

$$\hat{r}(x) = \limsup_{i\to\infty} \|x - T^{n_i}u\| = 0$$

*for some $u \in A$ and $x \in A$, then $Tx = x$.*

**Theorem 3.** *Let $A$ be a nonempty bounded closed convex subset of a Banach space $X$ which has uniformly normal structure, i.e. $N(X) < 1$. If $T : A \to A$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} |||T^n||| = k < [N(X)]^{-1/2},$$

*then $T$ has a fixed point in $A$.*

PROOF: Let $T : A \to A$ and let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n\to\infty} |||T^n||| = \lim_{i\to\infty} |||T^{n_i}||| = k < [N(X)]^{-1/2}.$$

Consider the sequence $\{T^{n_i}x\}$ for an $x \in A$. Let $z(x)$ be a point satisfying Lemma 1 for $\{T^{n_i}x\}$. Let $r(x) = \limsup_{i\to\infty} \|T^{n_i}x - x\|$. By the condition (i) of Lemma 1, we have

(1)

$$\begin{aligned}
&\limsup_{i\to\infty} \|T^{n_i}x - z\| \le N(X) \cdot \lim_{s\to\infty} \sup\{\|T^{n_i}x - T^{n_j}x\| : n_i, n_j \ge s\} \le \\
&\le N(X) \cdot \limsup_{i\to\infty}\big(\limsup_{j\to\infty} \|T^{n_i}x - T^{n_j}x\|\big) \le \\
&\le N(X) \cdot \limsup_{i\to\infty}\big(\limsup_{j\to\infty}(\|T^{n_i}x - T^{n_i+n_j}x\| + \|T^{n_i+n_j}x - T^{n_j}x\|)\big) \le \\
&\le N(X) \cdot \limsup_{i\to\infty}\big(\limsup_{j\to\infty}(|||T^{n_i}||| \cdot \|x - T^{n_j}x\| + \sum_{v=0}^{n_i-1} \|T^{n_j+v+1}x - T^{n_j+v}x\|)\big) \le \\
&\le N(X) \cdot \limsup_{i\to\infty} |||T^{n_i}||| \cdot \limsup_{j\to\infty} \|x - T^{n_j}x\| = \\
&= k \cdot N(X) \cdot \limsup_{j\to\infty} \|x - T^{n_j}x\|.
\end{aligned}$$

Moreover, for $i > 1$, we have

$$\begin{aligned}
\|T^{n_i}z - z\| &\le \limsup_{j\to\infty} \|T^{n_i}z - T^{n_j}x\| \le \\
&\le \limsup_{j\to\infty}\big(\|T^{n_i}z - T^{n_i+n_j}x\| + \|T^{n_i+n_j}x - T^{n_j}x\|\big) \le \\
&\le \limsup_{j\to\infty}\big(|||T^{n_i}||| \cdot \|z - T^{n_j}x\| + \sum_{v=0}^{n_i-1} \|T^{n_j+v+1}x - T^{n_j+v}x\|\big) \le \\
&\le |||T^{n_i}||| \cdot \limsup_{j\to\infty} \|z - T^{n_j}x\|.
\end{aligned} \tag{2}$$

By (1) and (2)

$$r(z) \le k^2 \cdot N(X) \cdot r(x) = a \cdot r(x), \quad \text{with} \quad a < 1. \tag{3}$$

Define a sequence $\{x_m\}$ in the following way: $x_1$ is an arbitrarily chosen point of $A$, $x_{m+1} = z(x_m)$. Then $\{x_m\}$ is a Cauchy sequence. In fact, we have

$$\begin{aligned}
\|x_{m+1} - x_m\| &\le \|x_{m+1} - T^{n_i}x_m\| + \|T^{n_i}x_m - x_m\| \le \\
&\le \|x_{m+1} - T^{n_i}x_m\| + r(x_m).
\end{aligned}$$

Taking the limit superior as $i \to +\infty$,

$$\begin{aligned}\|x_{m+1} - x_m\| &\le \limsup_{i\to\infty} \|x_{m+1} - T^{n_i}x_m\| + r(x_m) \le \\ &\le k \cdot N(X) \cdot r(x_m) + r(x_m) = [1 + k \cdot N(X)] \cdot r(x_m).\end{aligned}$$

Hence, by (3)

$$\|x_{m+1} - x_m\| \le [1 + k \cdot N(X)] \cdot r(x_m) \le [1 + k \cdot N(X)] \cdot a^m \cdot r(x_1) \to 0$$

as $m \to +\infty$. Let $x_0 = \lim_{m\to\infty} x_m$. Finally

$$\begin{aligned}\|x_0 - T^{n_i}x_0\| &\le \|x_0 - x_m\| + \|x_m - T^{n_i}x_m\| + \|T^{n_i}x_m - T^{n_i}x_0\| \le \\ &\le \big(1 + |||T^{n_i}|||\big) \cdot \|x_0 - x_m\| + \|x_m - T^{n_i}x_m\|.\end{aligned}$$

Taking the limit superior as $i \to +\infty$ on both sides we get

$$\begin{aligned}\limsup_{i\to\infty} \|x_0 - T^{n_i}x_0\| &\le (1 + k) \cdot \|x_0 - x_m\| + r(x_m) \le \\ &\le (1 + k) \cdot \|x_0 - x_m\| + a^m \cdot r(x_1) \to 0\end{aligned}$$

as $m \to +\infty$. Therefore, by Lemma 2, $Tx_0 = x_0$. □

For James spaces $X_M = \big(l^2, |\cdot|_M\big)$, where $|\cdot|_M = \max\{\|\cdot\|_2, M\cdot\|\cdot\|_\infty\}$, $(M \ge 1)$ we have

1)
$$\varepsilon_0(X_M) = \begin{cases} 2\cdot(M^2-1)^{1/2} & \text{for } M < \sqrt{2}, \\ 2 & \text{for } M > \sqrt{2}, \end{cases}$$
and $\varepsilon_0(X_M) < 1$ if and only if $M < \frac{\sqrt{5}}{2}$;
2) for $1 \le M < \frac{\sqrt{5}}{2}$, the condition $\gamma < [N(X_M)]^{-1/2}$ is weaker than $\gamma < \gamma_0$, where $\gamma_0$ is the unique solution of $x\big(1 - \delta_{X_M}(\frac{1}{x})\big) = 1$;

and

$$N(X_M) = \frac{M}{\sqrt{2}} \ \text{ for } \ 1 \le M \le \sqrt{2}, \ [3].$$

Combining these results we get the following

**Corollary 1.** *Let $A$ be a nonempty bounded closed convex subset of a James space $X_M$, $1 \le M < \sqrt{2}$. If $T : A \to A$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} |||T^n||| < \frac{2^{1/4}}{\sqrt{M}}\,,$$

*then $T$ has a fixed point in $A$.*

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