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ON GENERALIZED OUTERPLANARITY OF LINE GRAPHS

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Summary. A generalized outerplanar graph is a planar graph which can be embedded in the plane in such a way that at least one end-vertex of each edge lies on the boundary of the same face. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be the class of all outerplanar graphs and the class of all generalized outerplanar graphs, respectively. Let $L(G)$ stand for the line graph of a graph G. In this note we show that the following three statements on G are equivalent:

(1) $L(G) \in \mathscr{A}_2$;

(2) G has no subgraph homeomorphic from one of the seven graphs shown in Fig. 2;

(3) the following two conditions hold:

(i) $G \in \mathscr{A}_1$,

(ii) the degree of each vertex is at most four, each vertex c of degree four is a cut-vertex, for every c there are at least two bridges incident with c, and at least one of them is an end-bridge.

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In this paper all graphs are finite and simple. First, let us recall some concepts.

We follow Behzad and Chartrand [1] in saying that a graph H is homeomorphic from G if either $H \cong G$ or H is a subdivision of G. Further, a graph G_1 is homeomorphic with a graph G_2 if there is a graph G_3 such that both G_1 and G_2 are homeomorphic from G_3.

A graph G is said to be outerplanar if it can be embedded in the plane so that all vertices of G lie on the boundary of the same face, say $\Omega_1(G)$. Let $\mathscr{A}_1$ be the class of all outerplanar graphs. Chartrand and Harary [2] showed that G belongs to $\mathscr{A}_1$ if and only if it contains no subgraph homeomorphic from K_4 or $K_{2,3}$.

In [5] the study of the neighborhoods of the second type motivated the following generalization of outerplanar graphs: A generalized outerplanar graph G is a planar graph which can be embedded in the plane in such a way that at least one end-vertex of each edge lies on the boundary of the same face, say $\Omega_2(G)$. Let us always choose $\Omega_2(G)$ in such a way that the boundary of $\Omega_2(G)$ contains the maximum number of vertices of G. In [5] the class of all generalized outerplanar graphs was denoted by $\mathscr{A}_2$. It was also shown that G belongs to $\mathscr{A}_2$ if and only if no subgraph of G is homeomorphic from one of the graphs in Fig. 1. Let these graphs be labeled by *1, 2, ..., 12* as in Fig. 1.

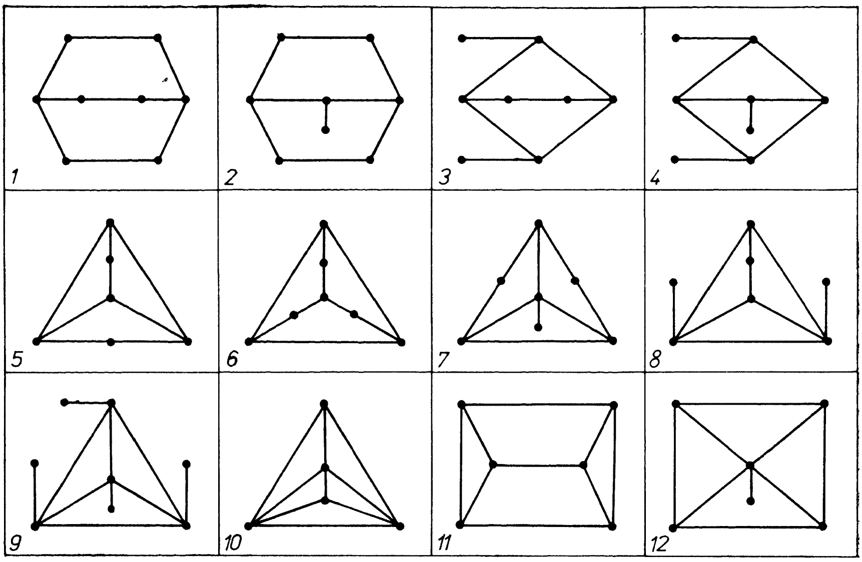

Fig. 1

Let G be a graph with a nonempty edge set. The line graph $L(G)$ of G is the graph having vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G are adjacent. For an edge e of G let $\lambda(e)$ mean the corresponding vertex of $L(G)$. In [6] we characterized the planarity of a line graph $L(G)$ by using the planarity of G and its vertex degrees. Let us recall this result without proof ($\Delta(G)$ means the maximum vertex degree in G).

Theorem 1. *A graph G has a planar line graph $L(G)$ if and only if G is planar, $\Delta(G) \leqq 4$, and every vertex of degree four is a cut-vertex.*

Greenwell and Hemminger [3] gave a characterization of graphs with planar line graphs in terms of forbidden subgraphs. Here we also present their result without proof. For symbols used in Theorem 2 we refer to Harary [4].

Theorem 2. *A graph has a planar line graph if and only if it has no subgraph homeomorphic with $K_{3,3}$, $K_{1,5}$, $P_4 + K_1$ or $K_2 + \bar{K}_3$.*

The aim of this note is to present two necessary and sufficient conditions for a graph G to have a line graph $L(G)$ belonging to $\mathscr{A}_2$. One of them is analogous to Theorem 1, the other to Theorem 2, and both are presented in Theorem 3. First of all let us formulate an auxiliary statement without proof.

Lemma 1. *If $G \in \mathscr{A}_1$ and $\Delta G \leqq 3$ then $L(G) \in \mathscr{A}_2$.*

In Theorem 3 we need the concept of an end-bridge. By an end-bridge of a graph G we mean a bridge uv of G where one of the vertices u and v, say u, has degree 1. The removal of an end-bridge uv is understood to be the removal of both uv and u.

Theorem 3. *The following three statements on a graph G are equivalent:*

(1) $L(G) \in \mathscr{A}_2$;

(2) *G has no subgraph homeomorphic from one of the seven graphs I, II, ..., VII shown in Fig.* 2;

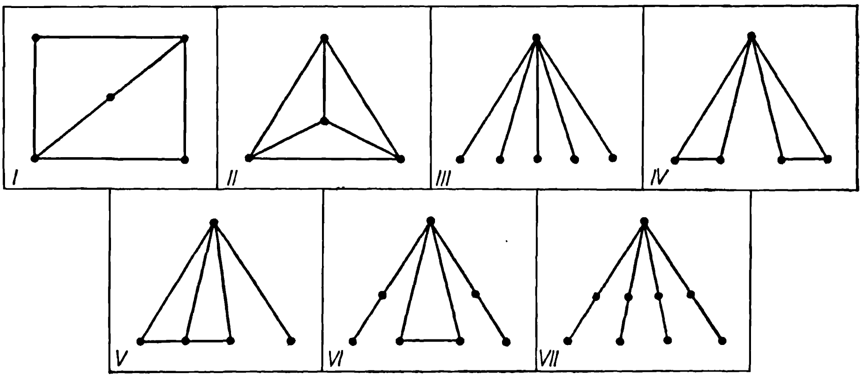

Fig. 2

(3) *the following two conditions hold:*

(i) $G \in \mathscr{A}_1$,

(ii) *the degree of each vertex is at most four, each vertex c of degree four is a cut-vertex, for every c there are at least two bridges incident with c, and at least one of them is an end-bridge.*

Proof. (1) ⇒ (2). It is sufficient to show that the line graph of each graph homeomorphic from a graph in Fig. 2 contains a subgraph homeomorphic from a graph in Fig. 1. Indeed, if we go through the graphs *I, II, III, IV, V, VI, VII* in Fig. 2 then the corresponding graphs in Fig. 1 are successively *11, 11, 10, 5, 10, 8, 9*.

(2) ⇒ (3). Assume that (2) holds and proceed as follows:

(i) If $G \notin \mathscr{A}_1$ then (due to the theorem of Chartrand and Harary [2]) G contains a subgraph homeomorphic from *I* or *II* (Fig. 2), which is a contradiction.

(ii) If $\Delta(G) \geqq 5$ then G contains a subgraph homeomorphic from *III*, a contradiction. Let v be a vertex of degree four in G and let v_i $(1 \leqq i \leqq 4)$ be the vertices adjacent to v.

If v is not a cut-vertex then we denote by C_{ij} a circuit containing the edges vv_i and vv_j in G. If C_{12} and C_{34} had no common vertex but v it would be a contradiction with *IV*. The same contradiction would be obtained if both v_3 and v_4 belonged to C_{12}. If v_3 belongs to C_{12} and v_4 does not we have a contradiction with *V*. Let neither v_3 nor v_4 belong to C_{12} and let w, $w \neq v$, be a vertex belonging to both C_{12} and C_{34}.

The vertices v and w divide C_{34} into two parts. We may assume that no vertex of C_{12} lies inside one of them, say P. Without loss of generality we can assume that v_3 is inside P. If we now add the path P, the edge vv_4, and the vertex v_4 to the circuit C_{12} we get a subgraph contradicting the graph V.

Let v be a cut-vertex of G. If there is at most one bridge incident with v then we have a contradiction with IV or V. If there are at least two bridges incident with v and none of them is an end-bridge then we get a contradiction with VI or VII.

(3) ⇒ (1). If $G \in \mathscr{A}_1$ and, in addition, $\Delta(G) \leqq 3$ then according to Lemma 1 the statement (1) holds. If $\Delta(G) = 4$, let us go through all vertices v of degree four. Let vv_1 be an end-bridge and let vv_i $(2 \leqq i \leqq 4)$ be the other edges incident with v. Let us remove vv_1 for every v. Let G^* be the graph resulting from G. Let us draw the graph $L(G^*) \in \mathscr{A}_2$ in the plane and construct a planar representation of $L(G)$ as follows: For every v place a planar representation of $\lambda(vv_1)$ inside the triangle whose vertices are images of the vertices $\lambda(vv_i)$ $(2 \leqq i \leqq 4)$ of $L(G^*)$ and join v with the vertices of the triangle. This drawing shows that $L(G) \in \mathscr{A}_2$. □

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Souhrn

O ZOBECNĚNÉ VNĚJŠKOVÉ ROVINNOSTI HRANOVÝCH GRAFŮ

JIŘÍ SEDLÁČEK

Zobecněný vnějškově rovinný graf je takový graf G, který je možno vnořit do roviny tak, že existuje oblast vymezená grafem G, na jejíž hranici má každá hrana grafu G obraz aspoň jednoho svého koncového uzlu. Třídu vnějškově rovinných grafů resp. třídu zobecněných vnějškově rovinných grafů označme $\mathscr{A}_1$ resp. $\mathscr{A}_2$. Hranový graf grafu G, označený $L(G)$, je průnikový graf hranové množiny grafu G.

V této poznámce se ukazuje, že tyto tři výroky o grafu G jsou ekvivalentní:

(1) $L(G) \in \mathscr{A}_2$;

(2) G neobsahuje žádný podgraf homeomorfní z některého ze sedmi grafů znázorněných na obr. 2;

(3) současně platí:

(i) $G \in \mathscr{A}_1$,

(ii) maximální uzlový stupeň grafu G je nejvýše 4, každý uzel stupně 4 je artikulace grafu G, incidují s ním aspoň dva mosty a aspoň jeden z nich je koncový.

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