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ON CONTINUITY AND MONOTONICITY OF DARBOUX FUNCTIONS

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Summary. In this paper we present some conditions under which a weakly monotone Darboux function $f: I^2 \to R^2$ is continuous.

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It is well known that every weakly monotone Darboux function $f: R \to R$ is continuous (see for example [2, Theorem 2, p. 94]). This fact is also true if f is a real function defined on a topological space more general than the real line (see [3], [6]). It is easy to see that if the functions considered assume their values in R^2 then the above theorem is false. In the present paper we investigate conditions under which a weakly monotone Darboux function $f: I^2 \to R^2$ is continuous.

We use the following basic definitions and notation. By the symbol $K(x_0, \delta)$ we will denote the open circle in the plane with the centre at x_0 and the radius δ. The closure of any set A will be denoted by $\bar{A}$ or $\operatorname{cl} A$, the interior of this set by $\operatorname{Int} A$, the interior of A in the subspace K by $\operatorname{Int}_K A$ and the boundary of A by $\operatorname{Fr} A$. The symbol A^d will stand for the set of all accumulation points of the set A. By ϱ we will denote the distance on the plane. We say that a family $\mathscr{F}$ is dense in R^2 if $\operatorname{cl}\left(\bigcup_{A \in \mathscr{F}} A\right) = R^2$. The symbol $[a, b] \parallel L$ denotes that the segment $[a, b]$ is paralel to the line L in the plane, while $K \parallel \mathscr{S}(K \perp \mathscr{S})$, where $\mathscr{S}$ is a family of parallel lines, denotes that K is parallel (vertical) to every line of this family.

To avoid ambiguity and misunderstandings as concerns the notions used in the present paper, we introduce the following definitions.

Definition. A function $f: X \to Y$ (where X, Y are arbitrary topological spaces) is called *monotone relative to the family* $\mathscr{F}$ of subsets of Y, if the set $f^{-1}(F)$ is connected in X for every $F \in \mathscr{F}$.

Definition. Let $C_0(C, S_0)$ denote the class of all open and connected (connected, singleton) subsets of Y. Then a function $f: X \to Y$ is called *almost monotone* (*monotone*, *weakly monotone*) if f is monotone relative to $C_0(C, S_0)$.

In many papers a weakly monotone function is also known as "monotone" ([7] and [1], but in [1] the author additionally assumes that f is a continuous function) or a "semi-monotone" ([8]) function. Our terminology is similar to that in [3] (see also [6]).

Definition [4]. We say that $f\colon X \to Y$ (where X and Y are arbitrary topological spaces) is a *Darboux function* (or *possesses the Darboux property*) if $f(C)$ is a connected set for every connected set $C \subset X$.

In many papers a Darboux function is also known as a connected function ([3], [6]).

We shall consider the functions defined and assuming their values in R^2. Let I denote the interval $[0, 1]$.

Theorem 1. *If a Darboux function $f\colon I^2 \to R^2$ is almost monotone then it is a continuous and weakly monotone function.*

Proof. First, we shall show that f is a continuous function. Let $x_0 \in I^2$, $\alpha = f(x_0)$, and let $\varepsilon > 0$ be an arbitrary number. We shall prove that there exists $\delta > 0$ such that

$$(*) \qquad f(K(x_0, \delta)) \subset K(\alpha, \varepsilon)\,.$$

Consider the set $A = R^2 \setminus \mathrm{cl}\,(K(\alpha, \tfrac{1}{2}\varepsilon))$.

If $f^{-1}(A) = \emptyset$, then $f(x) \in \mathrm{cl}\,(K(\alpha, \tfrac{1}{2}\varepsilon))$ for every $x \in I^2$ and so the condition $(*)$ is fulfilled.

Thus, let $f^{-1}(A) \neq \emptyset$. Since A is an open and connected set, $f^{-1}(A)$ is connected in I^2.

We will prove that $x_0 \notin \mathrm{cl}\,((f^{-1}(A))$. Suppose on the contrary that $x_0 \in \mathrm{cl}\,(f^{-1}(A))$. Thus $f^{-1}(A) \cup \{x_0\}$ is a connected set and so $f(f^{-1}(A) \cup \{x_0\})$ is connected, which is impossible

There exists $\delta > 0$ such that

$$K(x_0, \delta) \cap f^{-1}(A) = \emptyset\,.$$

Therefore

$$f(K(x_0, \delta)) \subset \mathrm{cl}\,(K(\alpha, \tfrac{1}{2}\varepsilon)) \subset K(\alpha, \varepsilon)$$

and $(*)$ is proved.

Now, we shall show that f is weakly monotone. Assume, to the contrary, that there exists $\alpha \in R^2$ such that $f^{-1}(\alpha)$ is not connected. Thus $f^{-1}(\alpha) = A \cup B$, where A and B are disjoint, nonempty and closed sets in $f^{-1}(\alpha)$. Since $f^{-1}(\alpha)$ is a closed set, A and B are closed (in I^2) as well. Therefore there exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Let $\varepsilon_1 = 1$. Consider the open circle $K(\alpha, \varepsilon_1)$ and the inverse image $f^{-1}(K(\alpha, \varepsilon_1))$. We have

$$f^{-1}(K(\alpha, \varepsilon_1)) \cap U \neq \emptyset \neq f^{-1}(K(\alpha, \varepsilon_1)) \cap V$$

and so there exists an element x_1 such that

$$x_1 \in f^{-1}(K(\alpha, \varepsilon_1)) \setminus (U \cup V).$$

Suppose that we have defined the sequence $x_1, \ldots, x_{n-1}$. Put $\varepsilon_n = \frac{1}{2}\varrho(\alpha, f(x_{n-1}))$. Consider the open circle $K(\alpha, \varepsilon_n)$ and its inverse image $f^{-1}(K(\alpha, \varepsilon_n))$. Then

$$f^{-1}(K(\alpha, \varepsilon_n)) \cap U \neq \emptyset \neq f^{-1}(K(\alpha, \varepsilon_n)) \cap V$$

and so there exists an element $x_n \in f^{-1}(K(\alpha, \varepsilon_n)) \setminus (U \cup V)$.

Continuing this procedure we obtain two sequences, $\{K(\alpha, \varepsilon_n)\}$ and $\{x_n\}$. From the sequence $\{x_n\}$ we select a subsequence $\{x_{k_n}\}$ converging to some x. It is easy to see that $\varepsilon_{k_n} \to 0$ and consequently $\lim_{n\to\infty} f(x_{k_n}) = \alpha$, where, by virtue of the continuity of f, $\alpha = f(x)$ and so $x \in f^{-1}(\alpha)$. This is impossible because $\{x_n\} \subset I^2 \setminus (U \cup V)$. The contradiction completes the proof.

It is known ([3]) that a function $f: I^2 \to R$ which is Darboux and weakly monotone is also continuous. On the other hand, it is not difficult to give an example of a function $f: I^2 \to R^2$ which is Darboux and weakly monotone but not continuous. Before presenting the next theorem we formulate a definition and some lemmas.

Definition. We say that a set $A \subset R^2$ is *directionally convex* if there exists a line L such that for every elements $a, b \in A$, the condition $[a, b] \parallel L$ implies $[a, b] \subset A$.

Lemma A (K. M. Garg [3]). *Let X be a topological space, Y a T_1-space and $f: X \to Y$ a Darboux function. Then, if $C \subset Y$ possesses closed components then the inverse image $f^{-1}(C)$ possesses closed components, too.*

Lemma B (R. J. Pawlak [6]). *Let $f: X \to Y$ be a connected function, where X is a connected and locally connected space, Y a T_1-space. If*

1° *a set K cuts Y into sets A and B*

and

2° *$f^{-1}(K)$ is a connected set,*

then the sets $f^{-1}(A \cup K)$ and $f^{-1}(B \cup K)$ are connected. (We say that a nonvoid set K cuts a topological space X if $X \setminus K = A \cup B$, where A and B are nonempty open and disjoint sets.)

Let $L(f, x_0)$ denote the set of all cluster values of f at x_0.

Theorem 2. *Let $f: I^2 \to R^2$ be a Darboux and weakly monotone function. Then f is a continuous function with a directionally convex image if and only if f is monotone relative to a dense (in R^2) set $\mathcal{S}$ of parallel lines such that*

$$(*) \qquad \operatorname{Int}_K L(f, x) = \emptyset$$

for every line $K \parallel \mathcal{S}$ and $x \in f^{-1}(K)$.

Proof. For simplicity we shall write A instead of $A \cap I^2$.

Sufficiency. First, we shall show that $f^{-1}(L)$ is a connected set for every line $L \parallel \mathscr{S}$.

The density of $\mathscr{S}$ implies that there exist two sequences $\{K_n\}$ and $\{M_n\}$ of lines of $\mathscr{S}$ such that $\varrho(K_{n+1}, L) < \varrho(K_n, L)$ and $\varrho(M_{n+1}, L) < \varrho(M_n, L)$ for every n; moreover, $\bigcup_{n=1}^{\infty} K_n$ and $\bigcup_{n=1}^{\infty} M_n$ are contained in the two different open halfplanes determined by L, and the sequences converge to L.

Let P_n denote the closed strip bounded by K_n and M_n, and H_n^1 – the closed halfplane generated by K_n such that $L \not\subset H_n^1$. According to Lemmas A, B we infer that $f^{-1}(H_n^1)$ is closed. It is easy to see that $f^{-1}(P_n)$ is closed. According to Lemma B, $f^{-1}(H_n^1 \cup P_n)$ is a connected set. Moreover, $f^{-1}(H_n^1 \cap P_n) = f^{-1}(K_n)$ is a connected set and so $f^{-1}(P_n)$ is connected, too (see [5]); consequently, $f^{-1}(P_n)$ is a continuum. This means that

$$f^{-1}(L) = \bigcap_{n=1}^{\infty} f^{-1}(P_n)$$

is a continuum.

Now, we shall show that the image $f(I^2)$ is directionally convex. Let L be an arbitrary line of $\mathscr{S}$ and let $a, b \in f(I^2)$ be such that $[a, b] \parallel L$. The elements a, b determine a line $K \parallel L$. According to the first part of this proof $f^{-1}(K)$ is connected and so $f(f^{-1}(K))$ is connected. We infer that $a, b \in f(f^{-1}(K)) \subset K$ and hence

$$[a, b] \subset f(f^{-1}(K)) \subset f(I^2).$$

Now, we shall prove that f is continuous. Let $x \in I^2$, $\alpha = f(x)$, and let ε be an arbitrary positive number. We shall show that there exists $\delta > 0$ such that

$$f(K(x, \delta)) \subset K(\alpha, \varepsilon). \tag{1}$$

Let J denote the interval vertical to the direction of $\mathscr{S}$ with the end-points belonging to the boundary $\mathrm{Fr}(K(\alpha, \varepsilon))$ and such that $\alpha \in J$. Let β_1, β_2 be different points of J such that $\varrho(\beta_1, \alpha) = \frac{1}{2}\varepsilon = \varrho(\beta_2, \alpha)$. Let L_1, L_2 be the parallel lines such that $L_1 \parallel \mathscr{S}$,, $L_2 \parallel \mathscr{S}$ and $\beta_1 \in L_1$, $\beta_2 \in L_2$.

Let K be the line parallel to $\mathscr{S}$ and such that $\alpha \in K$. Let K^+, K^- denote the half-lines of K determined by α. By γ_1 and γ_2 we denote the points of intersection of the half-lines K^- and K^+ with the lines determined by the points of intersection of L_1 with $\mathrm{Fr}(K(\beta_1, \frac{1}{2}\varepsilon))$ and L_2 with $\mathrm{Fr}(K(\beta_2, \frac{1}{2}\varepsilon))$, respectively.

Then, according to $(*)$, there exist $\alpha^- \in [\gamma_1, \alpha) \setminus L(f, x)$ and $\alpha^+ \in (\alpha, \gamma_2] \setminus L(f, x)$. Let K_1, K_2 be the lines vertical to $\mathscr{S}$ such that $\alpha^- \in K_1$ and $\alpha^+ \in K_2$. By α_{ij} we denote the point of intersection of K_i and L_j (for $i, j = 1\ 2$). In this way the points $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ determine a rectangle P such that $\alpha \in P$ and

$$\mathrm{Int}\, P \subset K(\alpha, \varepsilon). \tag{2}$$

It is easy to see that the intervals $[\alpha_{11}, \alpha_{12}]$, $[\alpha_{21}, \alpha_{22}]$ are vertical to $\mathscr{S}$.

We shall show that

$$x \notin (f^{-1}([\alpha_{11}, \alpha_{12}]))^d . \tag{3}$$

Indeed, suppose, on the contrary, that there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to x, $x_n \neq x$ for $n = 1, 2, \ldots$ and $\{x_n\} \subset f^{-1}([\alpha_{11}, \alpha_{12}])$. Since $f^{-1}(z)$ is a connected set for any $z \in [\alpha_{11}, \alpha_{12}]$, $f^{-1}(z)$ is closed by Lemma A. Of course, $x \notin f^{-1}(z)$ and so x is not an accumulation point of any level $f^{-1}(z)$, where $z \in \in [\alpha_{11}, \alpha_{12}]$. It is not difficult to see that there exists a subsequence of $\{x_n\}$ such that $f(x_i) \neq f(x_j)$ for $i \neq j$. Suppose that $\{x_n\}$ is this subsequence. Let $\{f(x_{k_n})\}$ be a subsequence of $\{f(x_n)\}$ such that $\{f(x_{k_n})\}$ converges to some α^*. Obviously $\alpha^* \neq \alpha^-$. Consider the midpoint $m(\alpha^*, \alpha^-)$ of the interval with the end-points α^* and α^-. Let $M \parallel \mathscr{S}$ be the line such that $m(\alpha^*, \alpha^-) \in M$. Let H denote the closed half-plane determined by M and $\alpha^* \in H$. Thus $x \in f^{-1}(H)$. This contradiction proves (3).

Analogously, we can prove that $x \notin (f^{-1}([\alpha_{21}, \alpha_{22}]))^d$.

Thus there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $K(x, \delta_1) \cap f^{-1}([\alpha_{11}, \alpha_{12}]) = \emptyset$ and $K(x, \delta_2) \cap f^{-1}([\alpha_{21}, \alpha_{22}]) = \emptyset$. At the same time $f^{-1}(L_1 \cup L_2)$ is a closed set (see Lemma A) and $x \notin f^{-1}(L_1 \cup L_2)$, hence there exists $\delta_3 > 0$ such that $K(x, \delta_3) \cap \cap f^{-1}(L_1 \cup L_2) = \emptyset$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then

$$f(K(x, \delta)) \cap \operatorname{Fr} P = \emptyset .$$

This together with the connectedness of $f(K(x, \delta))$ yields

$$f(K(x, \delta)) \subset \operatorname{Int} P \subset K(\alpha, \varepsilon)$$

and so the condition (1) is fulfilled. This completes the proof of sufficiency.

Necessity. According to our assumptions there exists a line M such that the image $f(I^2)$ is directionally convex with respect to M. Consider the family $\mathscr{S}$ of all lines $K \parallel M$. We shall show that f is monotone relative to $\mathscr{S}$. It is sufficient to prove that $f^{-1}(K)$ is a connected set for $K \in \mathscr{S}$. Assume, to the contrary, that there exists a line $K \in \mathscr{S}$ such that $f^{-1}(K) = A \cup B$, where A and B are disjoint, nonempty and closed in $f^{-1}(K)$. Since $f^{-1}(K)$ is a closed set then A and B are closed in I^2. Consequently, if $C = K \cap f(I^2)$ then $C = f(A) \cup f(B)$ and there exists $\beta \in f(A) \cap f(B)$. Consider the level $f^{-1}(\beta)$. It is easy to see that $f^{-1}(\beta) \cap A \neq \emptyset$ and $f^{-1}(\beta) \cap B \neq \emptyset$, which contradicts the weak monotonicity of f. The contradiction obtained completes the proof of necessity.

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Souhrn

O SPOJITOSTI A MONOTONNOSTI DARBOUXOVSKÝCH FUNKCÍ

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V článku jsou odvozeny podmínky pro to, aby slabě monotonní darbouxovská funkce $f: I^2 \to R^2$ byla spojitá.

Резюме

О НЕПРЕРЫВНОСТИ И МОНОТОННОСТИ ФУНКЦИЙ ДАРБУ

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В статье найдены условия для того, чтобы слабо монотонная функция Дарбу $f: I^2 \to R^2$ была непрерывной.

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