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ON A 1-FACTOR OF THE FOURTH POWER OF A CONNECTED GRAPH

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Summary. Let G be a connected graph of even order $p \geqq 4$. Consider a triangle-free subgraph H of G^3 such that the maximum degree of H is less than or equal to two. It is proved that there exists a 1-factor F of G^4 with the property that $E(F) \cap E(H) = \emptyset$.

Keywords: Powers of graphs, 1-factors, hamiltonian paths and cycles.

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By a graph we mean here a graph in the sense of [1] or [2]. Let G be a graph; we denote by $V(G)$, $E(G)$, and $\Delta(G)$ the vertex set, the edge set, and the maximum degree of G, respectively; if u and u' are vertices of G, then we denote by $d_G(u, u')$ the distance between u and u' in G; if $P \subseteq V(G)$ and $P \neq \emptyset$, then the subgraph of G induced by P will be denoted by $\langle P \rangle_G$; by the order of G we mean the number $|V(G)|$. We say that a graph G is triangle-free if no subgraph of G is isomorphic to the complete graph K_3. We say that a graph F is a 1-factor of a graph G if F is a regular graph of degree one, and at the same time a spanning subgraph of G.

Let G be a graph, and let n be a positive integer. We denote by G^n the graph with $V(G^n) = V(G)$ such that two vertices v and v' are adjacent in G^n if and only if $1 \leqq d_G(v, v') \leqq n$. The graph G^n is called the n-th power of G.

In the present paper we shall prove that if G is a connected graph of even order $p \geqq 4$, and H is a triangle-free subgraph of G^3 with $\Delta(H) \leqq 2$, then there exists a 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$.

Remark 1. Let G be the tree in Fig. 1, and let H be the spanning subgraph of G^4 with

$$E(H) = \bigcup_{i=1}^{3} \{u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4}\}.$$

We can see that H is a subgraph of a hamiltonian cycle of G^4. Obviously, H is not a subgraph of G^3. It is easy to show that there exists no 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$.

Remark 2. Let G be the tree in Fig. 2, and let H be the spanning subgraph of G^2

with

$$E(H) = \bigcup_{i=1}^{3} \{v_{i1}v_{i2}, v_{i2}v_{i3}, v_{i3}v_{i1}\} .$$

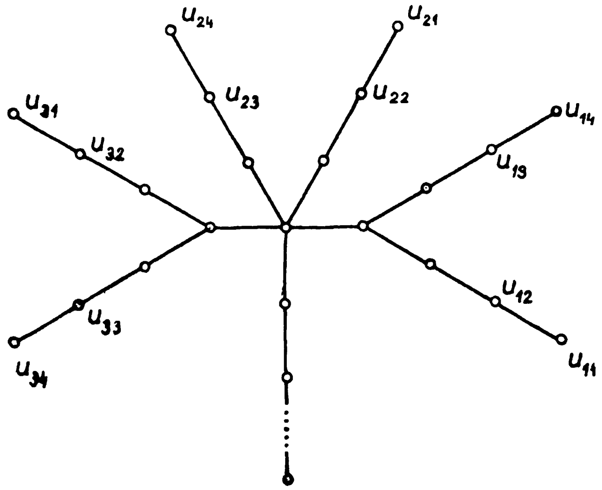

Fig. 1

Obviously, H is not triangle-free. Again, there exists no 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$.

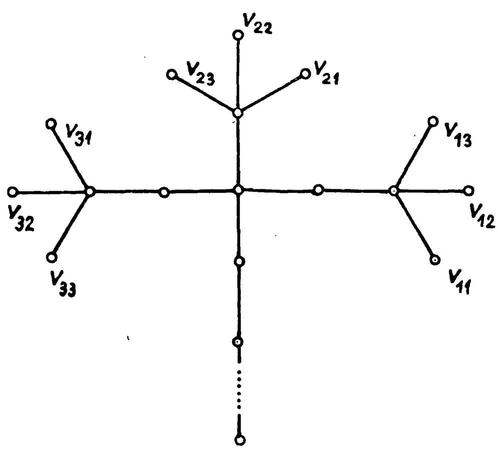

Fig. 2

Let G be a graph. By a D-partition of G we mean a partition $\mathscr{P}$ of $V(G)$ such that for each $P \in \mathscr{P}$, the following condition hold:

(1) $|P| = 4$ or 6;
(2) there exists $u \in V(G)$ such that $\langle P \cup \{u\} \rangle_G$ is connected;
(3) if $|P| = 6$ and $\langle P \rangle_G$ is not connected, then $\langle P \rangle_G$ has two components of order three.

Lemma 1. *Let G be a connected graph of even order $p \geqq 4$. Then there exists a D-paritition of G.*

Proof. Consider an arbitrary spanning tree T of G. The statement of the lemma is obvious for $p = 4$ or 6. Let $p = 8$. Then T is isomorphic to one of 23 trees of order eight presented in the list in [2], p. 233. We can see that T has a D-partition. Therefore, there exists a D-partition of G.

Let $p \geqq 10$. Assume that for any connected graph G' of order $p - 4$ or $p - 6$, we have proved that there exists a D-partition of G'.

Let u_1 and u_2 be distinct vertices of T such that $u_1u_2 \in E(T)$. Then $T - u_1u_2$ has exactly two components, say T_1 and T_2, where T_1 denotes the component which contains u_1. We define $V(u_1, u_2) = V(T_1)$.

As follows from the lemma in [3], there exist distinct vertices u and u_0 of T such that $uu_0 \in E(T)$,
(4) $|V(u, u_0)| \geqq 4$, and
(5) $|V(v, u)| \leqq 3$ for every vertex v of T such that $u \neq v \neq u_0$ and $uv \in E(T)$.
If $|V(u, u_0)| = 4$ or 6, we put $P = V(u, u_0)$. If there exist distinct vertices v_1 and v_2 of $T - u - u_0$ such that $uv_1, uv_2 \in E(T)$ and $|V(v_1, u)| = 3 = |V(v_2, u)|$, then we put $P = V(v_1, u) \cup V(v_2, u)$.

Assume that $|V(u, u_0)| \neq 4, 6$ and that there is at most one vertex v of $T - u - u_0$ such that $uv \in E(T)$ and $|V(v, u)| = 3$. It follows from (4) and (5) that there exist $i \in \{2, 3, 4\}$ and distinct vertices $\bar{v}_1, \ldots, \bar{v}_i$ of $T - u - u_0$ such that $u\bar{v}_1, \ldots, u\bar{v}_i \in \in E(T)$ and

$$|V(\bar{v}_1, u)| + \ldots + |V(\bar{v}_i, u)| = 4\,.$$

We put

$$P = V(\bar{v}_1, u) \cup \ldots \cup V(\bar{v}_i, u)\,.$$

It is clear that $T - P$ is a tree, and therefore, $G - P$ is connected. Since $|P| = 4$ or 6, it follows from the induction hypothesis that there exists a D-partition $\mathscr{P}'$ of $G - P$. Clearly, $\mathscr{P}' \cup \{P\}$ is a D-partition of G, which complete the proof of the lemma.

Lemma 2. *Let G be a graph isomorphic to $K_6 - e$ or K_6, and let H be a triangle-free subgraph of G with $\Delta(H) \leqq 2$. Then there exists a 1-factor F of G such that $E(F) \cap E(H) = \emptyset$.*

Proof. First let G be isomorphic to $K_6 - e$. Then there exist distinct vertices $u_1, \ldots, u_6$ such that

$$V(G) = \{u_1, \ldots, u_6\} \text{ and}$$
$$E(G) = \{u_i u_j;\ i, j \in \{1, \ldots, 6\},\ i \neq j\} - \{u_1 u_4\}\ .$$

There exists a spanning subgraph H^* of G such that $E(H) \subseteq E(H^*)$, H^* is triangle-free, $\Delta(H^*) = 2$, and for any $\bar{e} \in E(G) - E(H^*)$, $\Delta(H + \bar{e}) = 3$. Denote $U = \{u_1, u_4\}$. We distinguish eight cases:

1. H^* is a hamiltonian cycle of G and both components of $H^* - U$ are nontrivial. Without loss of generality we assume that

$$E(H^*) = \{u_1u_2, u_2u_3, \ldots, u_5u_6, u_6u_1\}\ .$$

2. H^* is a hamiltonian cycle of G, and one of the components of $H^* - U$ is trivial. Without loss of generality, let

$$E(H^*) = \{u_1u_2, u_2u_4, u_4u_5, u_5u_3, u_3u_6, u_6u_1\}\ .$$

3. H^* is a hamiltonian $u_1 - u_4$ path of G. Without loss of generality, let

$$E(H^*) = \{u_1u_2, u_2u_6, u_6u_3, u_3u_5, u_5u_4\}\ .$$

4. One of the components of H^* is a cycle C of order five, and $U \subseteq V(C)$. Without loss of generality, let

$$E(H^*) = \{u_1u_2, u_2u_4, u_4u_5, u_5u_6, u_6u_1\}\ .$$

5. One of the components of H^* is a cycle C of order five, and $|U \cap V(C)| = 1$. Without loss of generality, let

$$E(H^*) = \{u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_2\}\ .$$

6. One of the components of H^* is a cycle C of order four, and $U \subseteq V(C)$. Without loss of generality, let

$$E(H^*) = \{u_1u_2, u_2u_4, u_4u_5, u_5u_1, u_3u_6\}\ .$$

7. One of the components of H^* is a cycle C of order four, and $|U \cap V(C)| = 1$. Without loss of generality, let

$$E(H^*) = \{u_1u_2, u_2u_3, u_3u_6, u_6u_1, u_4u_5\}\ .$$

8. One of the components of H^* is a cycle of order four, and $U \cap V(C) = \emptyset$. Without loss of generality, let

$$E(H^*) = \{u_2u_3, u_3u_5, u_5u_6, u_6u_2\}\ .$$

We put $E(F) = \{u_1u_3, u_2u_5, u_4u_6\}$. Clearly, $E(F) \cap E(H^*) = \emptyset$, and thus $E(F) \cap E(H) = \emptyset$.

If G is isomorphic to K_6, then the result of the lemma easily follows.

Now we shall prove the main result of the present paper.

Theorem 1. *Let G be a connected graph of even order $p \geqq 4$, and let H be a triangle-free subgraph of G^3 with $\Delta(H) \leqq 2$. Then there exists a 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$.*

Proof. According to Lemma 1 there exists a D-partition P of G. We shall show that for each $P \in \mathscr{P}$,

(6) there exists a 1-factor F_P of $\langle P \rangle_{G^4}$ such that $E(F_P) \cap E(H) = \emptyset$.

Consider an arbitrary $P \in \mathscr{P}$. According to (1), $|P| = 4$ or 6. Let $|P| = 4$. It follows from (2) that $\langle P \rangle_{G^4}$ is complete, which implies (6).

Let $|P| = 6$. If $\langle P \rangle_{G^4}$ is isomorphic to $K_6 - e$ or K_6, then Lemma 2 implies (6). We shall assume that $\langle P \rangle_{G^4}$ is isomorphic neither to $K_6 - e$ nor to K_6. Then $\langle P \rangle_G$ is disconnected. According to (2) and (3) there exist distinct vertices $u_1, \ldots, u_6$ and v of G such that $P = \{u_1, \ldots, u_6\}$, and

$$\langle\{u_1, \ldots, u_6, v\}\rangle_G, \quad \langle\{u_1, u_2, u_3\}\rangle_G \quad \text{and} \quad \langle\{u_4, u_5, u_6\}\rangle_G$$

are connected. Without loss of generality we assume that $u_2u_3, u_3v, vu_4, u_4u_5 \in E(G)$. Since $\langle P \rangle_{G^4}$ is not complete, at least one of the graphs

$$\langle u_1, u_2, u_3, v\rangle_G \quad \text{and} \quad \langle u_4, u_5, u_6, v\rangle_G$$

is a path. Thus, without loss of generality we assume that $u_1u_2, u_2u_3 \in E(G)$ and $u_1u_3 \notin E(G)$. We can see that for any $i, j \in \{1, \ldots, 6\}$,

$$\text{if} \quad i < j \quad \text{and} \quad (i, j) \notin \{(1, 5), (1, 6), (2, 6)\}, \quad \text{then} \quad u_iu_j \in E(G^4).$$

If $\min(d_G(u_2, u_5), d_G(u_2, u_6)) \leqq 3$, then $\langle P \rangle_{G^4}$ is isomorphic to either $K_6 - e$ or K_6, which is a contradiction. Hence

$$d_G(u_2, u_5) = 4 \quad \text{and} \quad d_G(u_2, u_6) \geqq 4.$$

We distinguish two cases:

1. Let $u_3u_6 \notin E(H)$. If $u_1u_4 \notin E(H)$, then we put $E(F_P) = \{u_1u_4, u_2u_5, u_3u_6\}$. Assume that $u_1u_4 \in E(H)$. Then $d_G(u_1, u_5) \leqq 4$. If $u_1u_5, u_2u_4 \notin E(H)$, we put $E(F_P) = \{u_1u_5, u_2u_4, u_3u_6\}$. If $u_1u_5 \in E(H)$ or $u_2u_4 \in E(H)$, we put $E(F_P) = \{u_1u_2, u_3u_6, u_4u_5\}$. Since H is triangle-free and $\Delta(H) \leqq 2$, we can see that $E(F_P) \cap E(H) = \emptyset$.

2. Let $u_3u_6 \in E(H)$. Then $d_G(u_2, u_6) = 4$. If $d_G(u_1, u_4) \leqq 3$, then $\langle P \rangle_{G^4}$ is isomorphic to $K_6 - e$ or K_6, which is a contradiction. Thus, we assume that $d_G(u_1, u_4) = 4$. If $u_3u_5 \notin E(H)$, we put $E(F_P) = \{u_1u_4, u_2u_6, u_3u_5\}$. If $u_3u_5 \in E(H)$, we put $E(F_P) = \{u_1u_4, u_2u_3, u_5u_6\}$. Clearly, $E(F_P) \cap E(H) = \emptyset$.

Since P was chosen arbitrarily, the proof is complete.

The following result was proved by Sekanina in [4].

Theorem A ([4]). *Let G be a connected graph of order $p \geqq 1$. Then for any distinct vertices u and v of G, there exists a hamiltonian $u - v$ path of G^3. Consequently, if $p \geqq 3$, then there exists a hamiltonian cycle of G^3.*

Combining Theorems 1 and A, we immediately obtain two further results; the latter was proved by Wisztová in [5].

Corollary 1. *Let G be a connected graph of even order $p \geqq 4$, and let u and v be distinct vertices of G. Then there exist a hamiltonian $u - v$ path H of G^3 and a 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$.*

Corollary 2 ([5]). *Let G be a connected graph of even order $p \geqq 4$. Then there exists a hamiltonian cycle H of G^3 and a 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$.*

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Souhrn

O 1-FAKTORU ČTVRTÉ MOCNINY SOUVISLÉHO GRAFU

LADISLAV NEBESKÝ

Nechť G je souvislý graf o sudém počtu uzlů větším než dvě. Uvažujme podgraf H třetí mocniny grafu G a předpokládejme, že H neobsahuje trojúhelník a že jeho maximální stupeň nepřevyšuje dvě. V článku je ukázáno, že existuje 1-faktor F čtvrté mocniny grafu G takový, že F a H jsou hranově disjunktní.

Резюме

ОБ 1-ФАКТОРЕ ЧЕТВЕРТОЙ СТЕПЕНИ СВЯЗНОГО ГРАФА

LADISLAV NEBESKÝ

Пусть G — связный граф с чётным числом узлов большим двух и пусть H — подграф его третей степени, который не содержит треугольник и максимальная степень которого не больше двух. В статье доказано, что существует 1-фактор F четвертой степени графа G такой, что F и H реберно непересекаются.

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