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ON A CLASS OF GENERALIZATION LAGUERRE'S POLYNOMIALS

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Summary. In the paper the polynomials are defined by the relation (1) $L_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}$, $a_0^{(n)} > 0$, which are orthonormal on the interval $(a, +\infty)$ with regard to the function $L(x) = (x^2)^\alpha (b + x)^\beta e^{-x}$, where $\alpha > 0$, $\beta > 0$, $a \leqq 0$, $b > |a|$.
The relations for the coefficients of these polynomials, the relation (23) and the differential equation (25) are derived.

Keywords: orthonormal polynomials, Laguerre polynomials.

1. INTRODUCTION

In this paper we study orthonormal polynomials $L_n(x)$, $n = 0, 1, 2, \ldots$ on the interval $(a, +\infty)$ with the weight function $L(x) = (x^2)^\alpha (b + x)^\beta e^{-x}$, where $a \leqq 0$, $\alpha > 0$, $\beta > 0$, $b > |a|$.

These polynomials represent a generalization of the classical Laguerre's polynomials which are orthonormal on the interval $(0, +\infty)$ with the weight function $e^{-x}x^\alpha$, $\alpha > -1$.

We will derive some fundamental relations and inequalities and, on their basis, linear differential equations of the second order. The present paper generalizes some results of [1].

Definition. Let $a, b, \alpha, \beta \in (-\infty, \infty)$, $a \leqq 0$, $b > |a|$, $\alpha > 0$, $\beta > 0$. We say that

$$L_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}, \quad a_0^{(n)} > 0 \tag{1}$$

are *generalized Laguerre's polynomials* if they are orthogonal on the interval $I = (a, +\infty)$ with the weight function

$$L(x) = (x^2)^\alpha (b + x)^\beta e^{-x}. \tag{2}$$

Remark 1. The conditions for orthonormality of the system $\{L_n(x)\}$ have the form

$$\int_I x^k L_n(x) L(x) \, dx = 0, \quad k = 0, 1, \ldots, n - 1, \tag{3}$$

$$\int_I L_n^2(x) L(x) \, dx = 1. \tag{4}$$

Remark 2. In the sequel, $\pi_n(x) = \pi_n$ will denote a polynomial of at most n-th degree.

2. FUNDAMENTAL RELATIONS FOR THE POLYNOMIALS $L_n(x)$

Notation. For $n = 0, 1, 2, \ldots$ we define

$$q_n = \frac{a_0^{(n-1)}}{a_0^{(n)}} \quad \text{for} \quad n > 0\,, \quad q_n = 0 \quad \text{for} \quad n \leqq 0\,, \tag{5}$$

$$r_k^{(n)} = \frac{a_k^{(n)}}{a_0^{(n)}} \quad \text{for} \quad k > 0\,, \quad r_0^{(n)} = 1\,, \quad r_k^{(n)} = 0 \quad \text{for} \quad k < 0\,, \tag{6}$$

$$j_n = \int_I x\, L_n^2(x)\, L(x)\, \mathrm{d}x\,, \tag{7}$$

$$h_n = \int_I x^{-1}\, L_n^2(x)\, L(x)\, \mathrm{d}x\,, \tag{8}$$

$$i_n = \int_I (b + x)^{-1}\, L_n^2(x)\, L(x)\, \mathrm{d}x\,. \tag{9}$$

Lemma 1. *Let $P(x)$ be a polynomial of the n-th degree and $P'(x)$ its first derivative. For $k = 0, 1, \ldots,$ let s_k be the sum of the k-th powers of the zero points of $P(x)$. Let r be a non-negative integer and $\pi_n(x) = \pi_n$ a polynomial of at most n-th degree. Then*

$$x^r\, P'(x) = \sum_{\nu=0}^{r-1} s_\nu x^{r-\nu-1}\, P(x) + \pi_{n-1}\,. \tag{10}$$

Proof. Let $x_1, x_2, \ldots, x_n$ be the zeros of

$$P(x) = \sum_{k=0}^{n} a_k x^k\,, \quad a_n \neq 0$$

(some of them may coincide). Then

$$P(x) = a_n \prod_{i=1}^{n} (x - x_i)$$

and

$$\ln |P(x)| = \ln |a_n| + \sum_{i=1}^{n} \ln |x - x_i|\,.$$

Hence for $x > \max_{1 \leqq i \leqq n} |x_i|$ we obtain

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i} = \frac{1}{x} \sum_{i=1}^{n} \frac{1}{1 - \dfrac{x_i}{x}} = \frac{1}{x} \sum_{i=1}^{n} \sum_{k=0}^{\infty} \left(\frac{x_i}{x}\right)^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{x^{k+1}} \sum_{i=1}^{n} x_i^k = \sum_{k=0}^{\infty} \frac{s_k}{x^{k+1}} = \sum_{k=0}^{r-1} \frac{s_k}{x^{k+1}} + R_r(x)$$

and further,

$$x^r P'(x) = \sum_{k=0}^{r-1} s_k x^{r-1-k} P(x) + \pi_m(x),$$

where

$$\pi_m(x) = P(x)\, x^r R_r(x) = x^r P(x) \sum_{k=r}^{\infty} \frac{s_k}{x^{k+1}} = \frac{1}{x} \sum_{k=r}^{\infty} \frac{s_k}{x^{k-r}} P(x) =$$

$$= O(x^{-1})\, O(1)\, O(x^n) = O(x^{n-1}) \quad \text{for} \quad n \to +\infty .$$

Lemma 2. *For $k = 0, 1, \ldots$, let $s_k^{(n)}$ be the sum of the k-th powers of the zeros of $L_n(x)$. Then*

(11) $$s_1^{(n)} = \sum_{\nu=0}^{n-1} j_\nu = -r_1^{(n)} .$$

Proof. Consider the recurrent relation

$$(x - j_n)\, L_n(x) = q_{n+1}\, L_{n+1}(x) + q_n\, L_{n-1}(x)$$

(see [2], p. 77).

If we compare the coefficients at the power x^n, we obtain

$$a_0^{(n)} r_1^{(n)} - j_n a_0^{(n)} = q_{n+1} a_0^{(n+1)} r_1^{(n+1)} .$$

After dividing this equation by $a_0^{(n)}$ and substituting for q_{n+1} we have

(a) $$r_1^{(n+1)} - r_1^{(n)} = -j_n .$$

Put $\delta_k^{(n)} = s_k^{(n)} - s_k^{(n-1)}$ for $k = 0, 1, \ldots$.

Since

(b) $$s_1^{(n)} = -\frac{a_1^{(n)}}{a_0^{(n)}} = -r_1^{(n)} ,$$

we have according to (a)

(c) $$\delta_1^{(\nu)} = s_1^{(\nu)} - s_1^{(\nu-1)} = -r_1^{(\nu)} + r_1^{(\nu-1)} = j_{\nu-1}$$

(because $s_1^{(0)} = 0$).

Further

$$\sum_{\nu=1}^{n} \delta_1^{(\nu)} = \sum_{\nu=1}^{n} [s_1^{(\nu)} - s_1^{(\nu-1)}] = \sum_{\nu=1}^{n} j_{\nu-1}$$

and hence

$$s_1^{(n)} = \sum_{\nu=0}^{n-1} j_\nu .$$

Lemma 3. *Let $n = 1, 2, \ldots$. Then*

(12) $$j_n = 2n + a + 1 + 2\alpha + \beta - 2a\alpha h_n - (a + b)\, \beta i_n ,$$

(13) $$j_n = 2n + 1 + 2\alpha + \beta + a\, L_n^2(a)\, L(a) - b\beta i_n .$$

Proof. On the basis of (7) and (4) we obtain

$$j_n - a = \int_I x\, L_n^2(x)\, L(x)\, dx - a \int_I L_n^2(x)\, L(x)\, dx =$$
$$= \int_I (x - a)\, L_n^2(x)\, L(x)\, dx = -\int_I (x - a)\, L_n^2(x)\, L(x)\, e^x\, de^{-x}\,.$$

Integrating by parts we obtain (12), because

$$\int_I x\, L_n'(x)\, L_n(x)\, L(x)\, dx = n\,.$$

The formula (13) can be derived from the relation (7).

Lemma 4. *Let* $n = 1, 2, \ldots$. *Then*

(14) $$q_n^2 + \delta_n q_n = -r_1^{(n)} - an$$

holds, where

(15) $$\delta_n = \int_I [2\alpha a x^{-1} + (a + b)\, \beta (b + x)^{-1}]\, L_n(x)\, L_{n-1}(x)\, L(x)\, dx\,.$$

Proof. It is easy to see that

(a) $$\int_I L_{n-1}(x)\, L_n'(x)\, L(x)\, dx = \frac{n}{q_n}\,,$$

(b) $$\int_I x\, L_{n-1}(x)\, L_n(x)\, L(x)\, dx = q_n\,,$$

and also

(c) $$q_n = \int_I (a - x)\, L_n(x)\, L_{n-1}(x)\, L(x)\, e^x de^{-x}\,.$$

Integrating the last integral by parts we obtain

(d) $$q_n = \int_I (x - a)\, L_n'(x)\, L_{n-1}(x)\, L(x)\, dx - \delta_n$$

because

$$x\, L_n'(x) = n\, L_n(x) - a_1^{(n)} x^{n-1} + \pi_{n-2}(x)\,,$$

where $\pi_{n-2}(x)$ is a polynomial of the degree $n - 2$. From (d) and (a) the identity

$$q_n = -anq_n^{-1} - r_1^{(n)} q_n^{-1} - \delta_n$$

follows.

Lemma 5. *For* $n \to +\infty$ *the relations*

(16) $$h_n = O(1)\,,$$

and

(17) $$i_n = O(1)$$

hold.

Proof. The relations (16) and (17) follow from (7) and (8).

Lemma 6. *For $n = 1, 2, \ldots$ we have*

$$-r_1^{(n)} = n^2 + (a + 2\alpha + \beta)\, n - \sigma_n^* \,, \tag{18}$$

where

$$\sigma_n^* = \sum_{\nu=0}^{n-1} [2a\alpha h_\nu + (a + b)\, \beta i_\nu] \,. \tag{19}$$

Proof. From (11) for $k = 1$ we get

$$r_1^{(n)} = -\sum_{\nu=0}^{n-1} j_\nu \,,$$

and then using (12) we obtain (18).

Lemma 7. *For $n = 1, 2, \ldots$ we have*

$$-r_1^{(n)} = n^2 + (2\alpha + \beta)\, n + \sigma_n \,, \tag{20}$$

where

$$\sigma_n = \sum_{\nu=0}^{n-1} [a\, L_\nu^2(a)\, L(a) - b\beta i_\nu] \,, \tag{21}$$

and for $|\delta_n| < (b + a)\, |\beta|\, i_n i_{n-1}$ *we have*

$$q_n = n + O(1) \,. \tag{22}$$

Proof. The relation (20) follows from (11) and (13). Using (20) we obtain (22) from (14).

Lemma 8. *For $n = 1, 2, \ldots$ we have*

$$x\, L_n'(x) = n\, L_n(x) + q_n\, L_{n-1}(x) + \sum_{\nu=0}^{n-1} \gamma_\nu\, L_\nu(x) \,, \tag{23}$$

where

$$\gamma_\nu = b\beta \int_I (x - b)^{-1}\, L_n(x)\, L_\nu(x)\, L(x)\, dx - a\, L_n(a)\, L_\nu(a)\, L(a) \,. \tag{24}$$

Proof. We have

$$x\, L_n'(x) = \sum_{\nu=0}^{n} \gamma_\nu'\, L_\nu(x) \,, \tag{a}$$

where

$$\gamma_\nu' = \int_I x\, L_n'(x)\, L_\nu(x)\, L(x)\, dx \,. \tag{b}$$

Integrating (b) by parts we obtain

$$\gamma_\nu' = -a\, L_n(a)\, L_\nu(a)\, L(a) - \int_I L_n(x)\, L_\nu(x)\, L(x)\, dx -$$
$$- 2\alpha \int_I L_n(x)\, L_\nu(x)\, L(x)\, dx + b\beta \int_I (x - b)^{-1}\, L_n(x)\, L_\nu(x)\, L(x)\, dx -$$
$$- \beta \int_I L_n(x)\, L_\nu(x)\, L(x)\, dx + \int_I x\, L_n(x)\, L_\nu(x)\, L(x)\, dx -$$
$$- \int_I x\, L_n(x)\, L_\nu'(x)\, L(x)\, dx \,.$$

By virtue of (3), for $\nu < n - 1$ we conclude

(c) $$\gamma'_\nu = \gamma_\nu .$$

For $\nu = n - 1$, taking into account (3) and (4) we get

(d) $$\gamma'_{n-1} = \gamma_{n-1} + q_n .$$

For $\nu = n$, (13) implies

(e) $$\gamma'_n = \gamma_n - (1 + 2\alpha + \beta) + j_n - n = n .$$

Inserting (b)–(e) into (a) we obtain (23).

3. DIFFERENTIAL EQUATIONS FOR THE POLYNOMIALS $L_n(x)$

Theorem. *Let* $n = 0, 1, 2, \ldots$. *Then*

(25) $$L^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}[x(b + x)L'_n(x)L(x)] + [n(x + b - 1) + \sigma_n]L_n(x) = \sum_{\nu=0}^{n-1}\alpha_\nu L_\nu(x) + q_n L_{n-1}(x),$$

where

(26) $$\alpha_\nu = a(a + b)[L'_n(a)L_\nu(a) + L'_\nu(a)L_n(a)]L(a)$$

and

(27) $$x L''_n(x) + \left[-x + 2\alpha + \beta + 1 - \frac{b\beta}{b + x}\right]L'_n(x) + \left[n + \frac{\sigma_n}{b + x}\right]L_n(x) = (b + x)^{-1}R_n(x),$$

where $R_n(x) = \pi_{n-1}$.

Proof. a) Put

(a) $$A_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}[x(b + x)L'_n(x)L(x)] .$$

Then we have

(b) $$L^{-1}(x)A_n(x) = x(b + x)L''_n(x) + [-x(b + x) + (2\alpha + 1)(b + x) + (\beta + 1)x]L'_n(x) = x(b + x)L''_n(x) + [(b + x)(-x + 2\alpha + 1 + \beta + 1) - b(\beta + 1)]L'_n(x).$$

As

(c) $$x(b + x)L''_n(x) = n(n - 1)L_n(x) + \pi_{n-1},$$

(d) $$-x^2 L'_n(x) = -[s_0^{(n)}x + s_1^{(n)}]L_n(x) + \pi_{n-1} = -[nx + n^2 + (2\alpha + \beta)n + \sigma_n]L_n(x) + \pi_{n-1}$$

(we make use of the relations (10), (11) and (20)), we have

(e) $$x[-b + (2\alpha + 1 + \beta + 1)]\, L_n'(x) = = (-b + 2\alpha + \beta + 2)\, n\, L_n(x) + \pi_{n-1}\,.$$

From (b) after inserting (c)–(e) we obtain

$$L^{-1}(x)\, A_n(x) = [n(n-1) - nx - n^2 - bn - (2\alpha + \beta)\, n + + (2\alpha + \beta + 2)\, n - \sigma_n]\, L_n(x) + \pi_{n-1}\,,$$

i.e.

$$L^{-1}(x)\, A_n(x) + [-n(n-1) + nx + n^2 + bn + (2\alpha + \beta)\, n - - (2\alpha + \beta + 2)\, n + \sigma_n]\, L_n(x) = \pi_{n-1}\,.$$

Rearranging the equation we obtain

(f) $$L^{-1}(x)\, A_n(x) + [n(x + b - 1) + \sigma_n]\, L_n(x) = B_n(x)\,,$$

where

(g) $$B_n(x) = \pi_{n-1} = \sum_{\nu=0}^{n-1} \beta_\nu\, L_\nu(x)\,.$$

From (g) we obtain

(h) $$\beta_\nu = \int_I B_n(x)\, L_\nu(x)\, L(x)\, \mathrm{d}x\,.$$

Denote $\beta_\nu = \alpha_\nu' + \alpha_\nu''$. Then integrating (h) by parts and using (f) and (a) we get

(i) $$\alpha_\nu' = \int_I A_n(x)\, L_\nu(x)\, \mathrm{d}x = \int_I L_\nu(x)\, d[x(b + x)\, L_n'(x)\, L(x)] = = [x(b + x)\, L_n'(x)\, L_\nu(x)\, L(x)]_a^\infty - \int_I L_\nu'(x)\, x(b + x)\, \cdot \cdot L_n'(x)\, L(x)\, \mathrm{d}x = -a(b + a)\, L_n'(a)\, L_\nu(a)\, L(a) - - [L_n(x)\, L_\nu'(x)\, x(b + x)\, L(x)]_a^\infty + + \int_I L_n(x) \frac{\mathrm{d}}{\mathrm{d}x} [x(b + x)\, L_\nu'(x)\, L(x)]\, \mathrm{d}x\,.$$

From (a) and (f) we have

$$\frac{\mathrm{d}}{\mathrm{d}x} [x(b + x)\, L_\nu'(x)\, L(x)] = A_\nu(x) = [-\nu x\, L_\nu(x) + \pi_\nu]\, L(x)\,,$$

thus the relation (i) assumes the form

(j) $$\alpha_\nu' = a(a + b)\, [L_\nu'(a)\, L_n(a) - L_n'(a)\, L_\nu(a)]\, L(a) - - \nu \int_I x\, L_\nu(x)\, L_n(x)\, L(x)\, \mathrm{d}x\,.$$

For $\nu < n - 1$ the last integral equals zero.
For $\nu = n - 1$ the last integral equals q_n.

Further, (f) implies

(k) $$\alpha_\nu'' = \int_I [nx - n + bn + \sigma_n]\, L_n(x)\, L_\nu(x)\, L(x)\, \mathrm{d}x = = n \int_I x\, L_n(x)\, L_\nu(x)\, L(x)\, \mathrm{d}x\,.$$

For $\nu < n - 1$ we get

$$\beta_\nu = \alpha'_\nu = a(a + b)\left[L'_\nu(a)\, L_n(a) - L'_n(a)\, L_\nu(a)\right] L(a)\,.$$

For $\nu = n - 1$ we have

$$\beta_{n-1} = \alpha'_{n-1} + \alpha''_{n-1} =$$
$$= a(a + b)\left[L'_{n-1}(a)\, L_n(a) - L'_n(a)\, L_{n-1}(a)\right] L(a) -$$
$$- (n - 1) \int_I x\, L_{n-1}(x)\, L_n(x)\, L(x)\, dx + n \int_I x\, L_{n-1}(x)\, L_n(x)\, L(x)\, dx =$$
$$= a(a + b)\left[L'_{n-1}(a)\, L_n(a) - L'_n(a)\, L_{n-1}(a)\right] L(a) + q_n\,.$$

Inserting (h)–(k) into (f) we obtain (25).

b) After substituting (23) in (25), rearranging and dividing by the positive term (b + x), we obtain (27).

Literature

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Súhrn

O TRIEDE ZOVŠEOBECNENÝCH LAGUERROVÝCH POLYNÓMOV

FRANTIŠEK PÚCHOVSKÝ

V článku sú definované vzťahom (1) polynómy $L_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}$, $a_0^{(n)} > 0$, ktoré sú ortonormálne na intervale $(a, +\infty)$ vzhľadom na funkciu $L(x) = (x^2)^\alpha (b + x)^\beta e^{-x}$, kde $\alpha > 0$, $\beta > 0$, $a \leqq 0$, $b > |a|$.

Odvodzujú sa vzťahy pre koeficienty týchto polynómov, ďalej vzťah (23) a diferenciálna rovnica (25).

Резюме

О КЛАССЕ ОБОБЩЕННЫХ МНОГОЧЛЕНОВ ЛАГЕРРА

FRANTIŠEK PÚCHOVSKÝ

В статье определяются выражением (1) полиномы $L_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}$, $a_0^{(n)} > 0$, ортонормальные в интеграле (a, ∞) по отношению к функции $L(x)\ (x^2)^\alpha (b + x)^\beta e^{-x}$, где $\alpha > 0$, $\beta > 0$, $a \leqq 0$, $b > |a|$.

Выводятся отношения для коэффициентов этих многочленов, отношение (23) и дифференциальное уравнение (25).

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