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COMPACT IMBEDDING OF WEIGHTED SOBOLEV SPACE DEFINED ON AN UNBOUNDED DOMAIN I

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Summary. The paper deals with compact imbedding of the weighted Sobolev space $W_0^{k,p}(\Omega, S)$ (S is a collection of weight functions) defined on an unbounded domain in the space of functions $L^p(\Omega, \varrho)$ (ϱ is a weight function). This imbedding is investigated as the limit case of the compact imbeddings of Sobolev spaces defined on bounded domains.

Keywords: Weighted Sobolev space, weighted Lebesgue space, compact imbedding, weight function.

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1. INTRODUCTORY REMARKS

Let Ω be a domain in $\mathbb{R}^N$. By the symbol $\mathscr{W}(\Omega)$ we denote the set of all measurable, a.e. in Ω positive and finite functions $\varrho = \varrho(x)$, $x \in \Omega$. The elements of $\mathscr{W}(\Omega)$ will be called the weight functions.

Let $p \in \langle 1, \infty)$, $\varrho \in \mathscr{W}(\Omega)$. We define the space $L^p(\Omega, \varrho)$ as the set of all measurable functions $u = u(x)$, $x \in \Omega$, such that

$$\|u\|_{p,\Omega,\varrho} = \left(\int_\Omega |u(x)|^p \varrho(x) \, dx\right)^{1/p} < \infty . \tag{1.1}$$

For $\varrho(x) \equiv 1$ we obtain the usual Lebesgue space $L^p(\Omega)$; in this case we write $\|u\|_{p,\Omega}$ instead of $\|u\|_{p,\Omega,\varrho}$. Obviously the space $L^p(\Omega, \varrho)$ with the norm (1.1) is a Banach space.

Let $k \in N$ and let a collection of weight functions

$$S = \{w_\alpha \in \mathscr{W}(\Omega);\ |\alpha| \leqq k\}$$

be given (here α is a multiindex). By the symbol $W^{k,p}(\Omega, S)$ we denote the set of all measurable functions u defined a.e. in Ω which have on Ω distributional derivatives $D^\alpha u$, $|\alpha| \leqq k$, such that

$$\|D^\alpha u\|_{p,\Omega,w_\alpha} < \infty .$$

If

$$w_\alpha^{-1/p} \in L^{p^*}_{\mathrm{loc}}(\Omega)\,, \quad |\alpha| \leqq k\,,^{*})$$

we can easily verify that the space $W^{k,p}(\Omega, S)$ with the norm

$$(1.2) \qquad \|u\|_{k,p,\Omega,S} = \Big(\sum_{|\alpha| \leqq k} \|D^\alpha u\|^p_{p,\Omega,w_\alpha}\Big)^{1/p}$$

is a Banach space.

Now, let us assume that

$$(1.3) \qquad w_\alpha \in L^1_{\mathrm{loc}}(\Omega)\,, \quad |\alpha| \leqq k\,.$$

Then the inclusion

$$C_0^\infty(\Omega) \subset W^{k,p}(\Omega, S)$$

holds so we can introduce the so called "nulled space" $W_0^{k,p}(\Omega, S)$ as the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (1.2). The norm in this space is again given by (1.2).

If M is a subspace of a linear space X, we write $M \subset\subset X$.

Let X and Y be normed linear spaces. The symbol $[X, Y]$ will denote the space of all bounded linear operators mapping X into Y. For $A \in [X, Y]$ we define

$$\|A\| = \sup_{\|x\| \leqq 1} \|Ax\|\,.$$

Further, let Y be a Banach space. The operator $A \in [X, Y]$ is called compact if $A(\{x \in X;\ \|x\| \leqq 1\})$ is totally bounded in Y (i.e. if $\mathrm{cl}\,(A(\{x \in X;\ \|x\| \leqq 1\}))$ is compact in Y).

If $X \subset Y$ and the natural injection of X into Y is compact we write $X \subset\subset Y$. The symbol $X \rightleftarrows Y$ denotes the fact that X and Y are isomorphic.

The aim of this paper is to derive conditions on the collection S of weight functions and on the weight ϱ, which guarantee that the natural injection of $W_0^{k,p}(\Omega, S)$ into $L^p(\Omega, \varrho)$ is compact if the domain Ω is unbounded. The method which was used for a special weight function in [1] is generalized to suit our purpose.

2. PRELIMINARIES

In the subsequent sections we shall use these assertions:

2.1. Lemma. *Let X be a normed linear space and let Y be a Banach space. Let $\{A_n\}_{n=1}^\infty$ be a sequence of compact operators in $[X, Y]$ such that*

$$A_n \to A \quad in \quad [X, Y] \quad (\text{i.e. } \|A - A_n\| \to 0 \ for \quad n \to \infty)\,.$$

Then A is compact.

2.2. Lemma. *Let Z be a normed linear space, $X \subset\subset Z$, $\overline{X} = Z$. Let Y be a Banach*

$^{*})$ p^* denotes the number $p/(p-1)$ with the convention $s/0 = \infty$ for $s \in \mathbb{R} \setminus \{0\}$.

space and $A \in [X, Y]$ *a compact operator. Then there exists a unique operator* $\tilde{A} \in [Z, Y]$ *such that*

a) $\tilde{A}$ *is compact,*

b) $\tilde{A}|_X = A$.*)

2.3. Remark. If the operator A in Lemma 2.2 is the identical map from X into Y, then $\tilde{A}$ is the identical map from Z into Y.

2.4. Lemma. *Let* $a \in \mathbb{R}$, $m \in \mathbb{N}$. *Let us further suppose that* $f \in C^{(m)}((a, \infty))$ *and let* $\operatorname{supp} f$ *be a compact set. Then*

$$f(t) = \frac{(-1)^m}{(m-1)!} \int_t^\infty (s - t)^{m-1} f^{(m)}(s) \, ds \quad \textit{for} \quad t \in (a, \infty) . \tag{2.1}$$

2.5. Remark. Lemma 2.1 is an easy modification of Lemma III.1.5 in [2]. The proofs of the other assertions in this section are left to the reader.

3. COMPACT IMBEDDING OF WEIGHTED SOBOLEV SPACES

3.1. Using the Cartesian coordinates

The points $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ will sometimes be written in the form $x = (x', x_N)$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. If $Q \subset \mathbb{R}^N$, then we denote by $P_N(Q)$ the projection of the set Q into the hyperplane $x_N = 0$.

Let us suppose the following two conditions:

C1. Ω is an unbounded domain in $\mathbb{R}^N$, $\Omega \subset (-a, a)^{N-1} \times (-a, \infty)$ where $a > 0$.

C2. $W^{k,p}(\Omega_n, S) \hookrightarrow\hookrightarrow L^p(\Omega_n, \varrho)$ $\forall n \in \mathbb{N}$,**) where $\Omega_n = \{x \in \Omega; x_N < n\}$ for $n \in \mathbb{N}$.

We shall investigate under what additional assumptions

$$W_0^{k,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho) \tag{3.1.1}$$

holds.

Let us define the operators

$$I_n \colon W_0^{k,p}(\Omega, S) \to L^p(\Omega, \varrho) , \quad n \in \mathbb{N} , \tag{3.1.2}$$

by

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*) $\tilde{A}|_X$ denotes the restriction of the operator $\tilde{A}$ to X.

**) It is sufficient to assume

$$M_n \hookrightarrow\hookrightarrow L^p(\Omega_n, \varrho) \quad \forall n \in \mathbb{N} ,$$

where

$$M_n = \{u;\ u = v|_{\Omega_n},\ v \in W_0^{k,p}(\Omega, S)\} .$$

$$(3.1.3)\qquad (I_n u)(x) = \begin{cases} u(x), & x \in \Omega_n \\ 0, & x \in \Omega \setminus \Omega_n. \end{cases}$$

3.1.1. Lemma. *The operators I_n, $n \in N$, defined by means of (3.1.2) and (3.1.3) are compact.*

Proof. As the space $L^p(\Omega, \varrho)$ is complete, it is sufficient to prove that the set

$$M_n = \{I_n u;\ u \in W_0^{k,p}(\Omega, S),\ \|u\|_{k,p,\Omega,S} \leqq 1\}$$

is totally bounded in $L^p(\Omega, \varrho)$, i.e. that for each $\varepsilon > 0$ the set M_n has a finite ε-net in $L^p(\Omega, \varrho)$.

The condition **C2** implies that the set

$$\tilde{M}_n = \{v;\ v \in W^{k,p}(\Omega_n, S),\ \|v\|_{k,p,\Omega_n,S} \leqq 1\}$$

is totally bounded in $L^p(\Omega_n, \varrho)$ and therefore this set has a finite ε-net in $L^p(\Omega_n, \varrho)$ for each $\varepsilon > 0$.

Let $\varepsilon > 0$ and let

$$\{v_1^n, \ldots, v_i^n\}$$

be a finite ε-net of the set $\tilde{M}_n$. Then the set

$$\{w_1^n, \ldots, w_i^n\},$$

where

$$w_j^n(x) = \begin{cases} v_j^n(x), & x \in \Omega_n, \quad j = 1, \ldots, i, \\ 0, & x \in \Omega \setminus \Omega_n \end{cases}$$

is a finite ε-net of M_n because for $u \in W_0^{k,p}(\Omega, S)$, $\|u\|_{k,p,\Omega,S} \leqq 1$, we have

$$\min_{1 \leqq j \leqq i} \|I_n u - w_j^n\|_{p,\Omega,\varrho} = \min_{1 \leqq j \leqq i} \|u - v_j^n\|_{p,\Omega_n,\varrho} < \varepsilon$$

as $u|_{\Omega_n} \in \tilde{M}_n$.

Further, by the symbol X let us denote the set $C_0^\infty(\Omega)$ with the norm $\|\cdot\|_X = \|\cdot\|_{k,p,\Omega,S}$ and let us consider the operator

$$(3.1.4)\qquad I: X \to L^p(\Omega, \varrho)$$

defined by

$$(3.1.5)\qquad Iu = u, \quad u \in X.$$

In virtue of Lemma 2.2 and Remark 2.3 one can show that (3.1.1) holds if and only if the operator I is compact. To investigate the compactness of I we shall use Lemma 2.1. Therefore we shall try to approximate the operator I by the compact operators $J_n = I_n|_X$ (I_n are the operators defined by means of (3.1.2) and (3.1.3)).

Let us investigate when

$$J_n \to I \quad \text{in} \quad [X, Y] \tag{3.1.6}$$

holds, where we take for the sake of simplicity $Y = L^p(\Omega, \varrho)$.

We easily get

$$\|I - J_n\| = \sup_{\|u\|_X \leqq 1} \|Iu - J_n u\|_Y = \sup_{\|u\|_X \leqq 1} \|u - I_n u\|_Y =$$
$$= \sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega\setminus\Omega_n,\varrho} = \sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho},$$

where $\Omega^n = \{x \in \Omega;\ x_N > n\}$. This yields: (3.1.6) holds if and only if

$$\sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \to 0 \quad \text{for} \quad n \to \infty. \tag{3.1.7}$$

In addition to **C1, C2** we shall suppose that the following condition is fulfilled:

C3. There exist numbers $C > 0$, $m, n_0 \in N$, $1 \leqq m \leqq k$, and nonnegative measurable functions $\mu: (n_0, \infty) \to \mathbb{R}$, $\nu: (n_0, \infty) \to \mathbb{R}$, $\xi: P_N(\Omega^{n_0}) \to \mathbb{R}$ such that

$$\varrho(x) \leqq C\,\mu(x_N)\,\xi(x') \quad \text{for a.e.} \quad x \in \Omega^{n_0}; \tag{3.1.8}$$

$$\nu(x_N)\,\xi(x') \leqq C\,w_{(0,\ldots,0,m)}(x) \quad \text{for a.e.} \quad x \in \Omega^{n_0}; \tag{3.1.9}$$

$$h(n) = h(n; \mu, \nu, p, m) = \tag{3.1.10}$$
$$= \int_n^\infty \mu(t)\, \|(s - t)^{m-1}\, \nu^{-1/p}(s)\|^p_{p^*,(t,\infty)}\, dt \to 0 \quad \text{for} \quad n \to \infty.$$

We shall investigate the validity of (3.1.7). Let $u \in X$ and $n \geqq n_0$, where n_0 is the number from the condition **C3**. We extend the function u outside Ω by zero (then, clearly, $u \in C_0^\infty(\mathbb{R}^N)$) and put $\xi(x') = 1$ for $x' \in \mathbb{R}^{N-1} \setminus P_N(\Omega^{n_0})$. Using the Fubini theorem, in view of (3.1.8), we get for $n \geqq n_0$

$$\|u\|^p_{p,\Omega^n,\varrho} = \int_{\Omega^n} |u(x)|^p\, \varrho(x)\, dx \leqq \tag{3.1.11}$$
$$\leqq C \int_{\mathbb{R}^{N-1}} \left[\int_n^\infty |u(x', x_N)|^p\, \mu(x_N)\, dx_N \right] \xi(x')\, dx'.$$

For a fixed $x' \in \mathbb{R}^{N-1}$ we denote

$$f(t) = u(x', t), \quad t \in \mathbb{R}. \tag{3.1.12}$$

Evidently $f \in C_0^\infty(\mathbb{R})$. Let the number m be from the condition **C3**. Applying Lemma 2.4 we obtain

$$|f(t)| \leqq \frac{1}{(m-1)!} \int_t^\infty (s - t)^{m-1}\, |f^{(m)}(s)|\, ds, \quad t \in \mathbb{R}. \tag{3.1.13}$$

First, let $p \in (1, \infty)$. Then using the Hölder inequality we get for $t \in (n, \infty)$

$$|f(t)| \leqq \frac{1}{(m-1)!}\left(\int_t^\infty |f^{(m)}(s)|^p\, v(s)\, \mathrm{d}s\right)^{1/p} .$$

$$\cdot\left(\int_t^\infty (s-t)^{((m-1)/(p-1))p}\,[v(s)]^{-1/(p-1)}\, \mathrm{d}s\right)^{(p-1)/p} \leqq$$

$$\leqq \frac{1}{(m-1)!}\left(\int_n^\infty |f^{(m)}(s)|^p\, v(s)\, \mathrm{d}s\right)^{1/p} .$$

$$\cdot\left(\int_t^\infty (s-t)^{((m-1)/(p-1))p}\,[v(s)]^{-1/(p-1)}\, \mathrm{d}s\right)^{(p-1)/p} .$$

Raising this inequality to the p-th power, multiplying by the function $\mu(t)$ and integrating by t from n to ∞ we obtain

$$\int_n^\infty |f(t)|^p\, \mu(t)\, \mathrm{d}t \leqq \left[\frac{1}{(m-1)!}\right]^p h(n) \int_n^\infty |f^{(m)}(s)|^p\, v(s)\, \mathrm{d}s\,, \tag{3.1.14}$$

where the function h is defined in (3.1.10). The relations (3.1.11), (3.1.12), (3.1.14) and (3.1.9) imply

$$\|u\|^p_{p,\Omega^n,\varrho} \leqq \tag{3.1.15}$$

$$\leqq C\left[\frac{1}{(m-1)!}\right]^p h(n) \int_{R^{N-1}} \left[\int_n^\infty \left|\frac{\partial^m}{\partial x_N^m} u(x', s)\right|^p v(s)\, \mathrm{d}s\right] \xi(x')\, \mathrm{d}x' \leqq$$

$$\leqq C^2\left[\frac{1}{(m-1)!}\right]^p h(n) \int_{\Omega^n} |D^\alpha u(x)|^p\, w_\alpha(x)\, \mathrm{d}x\,,$$

where $\alpha = (0, \ldots, 0, m)$. From (3.1.15) we conclude

$$\sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \leqq C^{2/p} \frac{1}{(m-1)!}\, h^{1/p}(n)\,. \tag{3.1.16}$$

Now let $p = 1$. Then from (3.1.13) for $t \in (n, \infty)$ we have

$$|f(t)| \leqq \frac{1}{(m-1)!}\,[\operatorname*{ess\,sup}_{s>t}\,(s-t)^{m-1} v^{-1}(s)] \int_n^\infty |f^{(m)}(s)|\, v(s)\, \mathrm{d}s\,.$$

Multiplying this inequality by the function $\mu(t)$ and integrating by t from n to ∞ we obtain

$$\int_n^\infty |f(t)|\, \mu(t)\, \mathrm{d}t \leqq \frac{1}{(m-1)!}\, h(n) \int_n^\infty |f^{(m)}(s)|\, v(s)\, \mathrm{d}s\,, \tag{3.1.17}$$

where

$$h(n) = h(n; \mu, v, 1, m) = \tag{3.1.18}$$

$$= \int_n^\infty \mu(t)\,[\operatorname*{ess\,sup}_{s>t}\,(s-t)^{m-1}\, v^{-1}(s)]\, \mathrm{d}t =$$

$$= \int_n^\infty \mu(t)\, \|(s-t)^{m-1}\, v^{-1/p}(s)\|^p_{p^*,(t,\infty)} \mathrm{d}t\,.$$

From (3.1.17) we again get (3.1.16).

We have just proved:

3.1.2. Theorem. *Let the conditions* **C1 – C3** *be fulfilled. Then*

$$W_0^{k,p}(\Omega, S) \subset\subset L^p(\Omega, \varrho)\,. \tag{3.1.19}$$

3.1.3. Remark. If the condition (3.1.10) in **C3** is replaced by the assumption

$$g(n) = g(n; \mu, \nu, p, m) = \tag{3.1.10*}$$

$$= \int_n^\infty \left\| (s - t)^{m-1} \left[\frac{\mu(s)}{\nu(s)}\right]^{1/p} \right\|^p_{p^*,(t,\infty)} \mathrm{d}t \to 0 \quad \text{for} \quad n \to \infty$$

and if we suppose that, in addition to all the assumptions of Theorem 3.1.2,

(3.1.20) *the function μ is nondecreasing on (n_0, ∞),*

then (3.1.19) holds again.*)

3.1.4. Example. Let Ω satisfy the condition **C1**. Let further $p \in \langle 1, \infty)$, $k = 1$, $\beta > 0$, $\alpha < \beta$. For $x \in \Omega$ we define

$$w_\gamma(x) \equiv 1 \quad \text{for} \quad |\gamma| \leqq 1\,, \quad \gamma \neq (0, \dots, 0, 1)\,, \tag{3.1.21}$$

$$w_{(0,\dots,0,1)}(x) = e^{\beta x_N}\,, \quad \varrho(x) = e^{\alpha x_N}\,. \tag{3.1.22}$$

Let $S = \{w_\gamma; |\gamma| \leqq 1\}$. Because

$$W^{1,p}(\Omega_n, S) \rightleftarrows W^{1,p}(\Omega_n)\,, \quad L^p(\Omega_n, \varrho) \rightleftarrows L^p(\Omega_n)\,, \quad n \in N\,,$$

we obtain from the well-known (unweighted) imbedding theorem

$$W^{1,p}(\Omega_n, S) \subset\subset L^p(\Omega_n, \varrho)\,, \quad n \in N\,,^{**})$$

and so the condition **C2** is satisfied. If we choose $m = 1$, $C = 1$, $n_0 \in N$, $\xi(x') = 1$ for $x' \in P_N(\Omega^{n_0})$,

$$\mu(s) = e^{\alpha s}\,, \quad \nu(s) = e^{\beta s}\,, \quad s \in (n_0, \infty)\,,$$

we can see that (3.1.8) and (3.1.9) from **C3** are satisfied, too.

Let us investigate the validity of (3.1.10). We easily get that

$$h(n) = \left(\frac{p-1}{\beta}\right)^{p-1} \frac{1}{\beta - \alpha} e^{(\alpha-\beta)n}\,, \quad p \in \langle 1, \infty)\,, \quad n \geqq n_0\,,$$

and therefore $h(n) \to 0$ for $n \to \infty$. Then Theorem 3.1.2 implies

$$W_0^{1,p}(\Omega, S) \subset\subset L^p(\Omega, \varrho)\,. \tag{3.1.23}$$

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*) Let us remark that $0 \leqq h(n) \leqq g(n) \to 0$, $n \to \infty$.

**) As we work with the „nulled space" $W_0^{k,p}(\Omega, S)$, we can assume without loss of generality that $\Omega_n \in C^{0,1}$ for each $n \in N$.

3.1.5. Remark. In Example 3.1.4 we do not have to choose the weight functions w_γ for $|\gamma| \leqq 1$, $\gamma \neq (0, \ldots, 0, 1)$ by (3.1.21). It is sufficient that

$$W^{1,p}(\Omega_n, S) \rightleftarrows W^{1,p}(\Omega_n), \quad n \in N .$$

From this relation we see that (3.1.21) can be replaced by

$$\text{(3.1.24)} \qquad w_\gamma(x) = e^{\delta_\gamma x_N}, \quad x \in \Omega, \quad |\gamma| \leqq 1, \quad \gamma \neq (0, \ldots, 0, 1),$$

where δ_γ are some real numbers.

3.1.6. Example. Let Ω satisfy the condition **C1**, $p \in \langle 1, \infty)$, $k, m \in N$, $k \geqq 1$, $1 \leqq m \leqq k$, $\beta > 0$, $\alpha < \beta$. For $x \in \Omega$ we define

$$\text{(3.1.25)} \qquad w_\gamma(x) = e^{\delta_\gamma x_N}, \quad |\gamma| \leqq k, \quad \gamma \neq (0, \ldots, 0, m),$$

where δ_γ are some real numbers,

$$\text{(3.1.26)} \qquad w_{(0, \ldots, 0, m)}(x) = e^{\beta x_N}, \quad \varrho(x) = e^{\alpha x_N} .$$

Let $S = \{w_\gamma; |\gamma| \leqq k\}$. Analogously as in Example 3.1.4 we can verify that the conditions **C2** and (3.1.8), (3.1.9) are satisfied (we choose $\mu(s) = e^{\alpha s}$, $\nu(s) = e^{\beta s}$ and $\xi(x') = 1$).

We shall investigate the validity of (3.1.10) from **C3**. We choose ε in such a way that $0 < \varepsilon < \min(\beta, \beta - \alpha)$. Evidently, there exists a number $n_1 \in N$ such that

$$\text{(3.1.27)} \qquad s^{(m-1)p} \leqq e^{\varepsilon s} \quad \text{for} \quad s > n_1, \ ^{*)}$$

so that for $n \geqq \max(n_0, n_1)$ (the number n_0 is from the condition **C3**) we have

$$h(n) = \int_n^\infty \mu(t) \, \|(s - t)^{m-1} \nu^{-1/p}(s)\|^p_{p^*,(t,\infty)} \, dt \leqq$$

$$\leqq \int_n^\infty e^{\alpha s} \|e^{\varepsilon s/p} e^{-\beta s/p}\|^p_{p^*,(t,\infty)} \, dt =$$

$$= \left(\frac{p-1}{\beta - \varepsilon}\right)^{p-1} \cdot \frac{1}{\beta - \alpha - \varepsilon} \, e^{(\alpha + \varepsilon - \beta)n} \to 0 \quad \text{for} \quad n \to \infty .$$

From Theorem 3.1.2 we obtain (3.1.19).

For $x \in \mathbb{R}^N$ and $\varepsilon \in \mathbb{R}$ let us define

$$z_\varepsilon(x) = \begin{cases} x_N^\varepsilon, & x_N > 1, \\ 1, & x_N \leqq 1 . \end{cases}$$

3.1.7. Example. Let Ω satisfy the condition **C1**, $p \in \langle 1, \infty)$, $k, m \in N$, $k \geqq 1$,

*) Example 3.1.6 generalizes Example 3.1.4. If $m = 1$, then it is possible to choose $\varepsilon = 0$.

$1 \leqq m \leqq k$, $\beta > mp - 1$, $\alpha < \beta - mp$. For $x \in \Omega$ we put

(3.1.28) $$w_\gamma(x) = z_{\delta_\gamma}(x)\,, \quad |\gamma| \leqq k\,, \quad \gamma \neq (0, \dots, 0, m)\,,$$

where $\delta_\gamma \in \mathbb{R}$,

(3.1.29) $$w_{(0,\dots,0,m)}(x) = z_\beta(x)\,, \quad \varrho(x) = z_\alpha(x)\,.$$

Let $S = \{w_\gamma;\ |\gamma| \leqq k\}$. Analogously as in Example 3.1.4 we can verify that the condition **C2** is satisfied. If we choose $C = 1$, $n_0 \in N$, $\xi(x') = 1$ for $x' \in P_N(\Omega^{n_0})$,

$$\mu(s) = s^\alpha\,, \quad \nu(s) = s^\beta\,, \quad s \in (n_0, \infty)\,,$$

we can see that (3.1.8) and (3.1.9) from **C3** are satisfied, too.

Let us investigate the validity of (3.1.10). We easily obtain

$$h(n) = \int_n^\infty \mu(t)\,\|(s - t)^{m-1}\,\nu^{-1/p}(s)\|^p_{p^*,(t,\infty)}\,dt \leqq$$

$$\leqq \int_n^\infty \mu(t)\,\|s^{m-1}\,\nu^{-1/p}(s)\|^p_{p^*,(t,\infty)}\,dt =$$

$$= \int_n^\infty t^\alpha \|s^{m-1} s^{-\beta/p}\|^p_{p^*,(t,\infty)}\,dt =$$

$$= \left(\frac{p-1}{\beta - mp + 1}\right)^{p-1} \cdot \frac{1}{\beta - \alpha - mp}\, n^{\alpha - \beta + mp} \to 0 \quad \text{for} \quad n \to \infty\,.$$

From Theorem 3.1.2 the imbeding (3.1.19) follows.

3.1.8. Remark. (i) If Ω is an unbounded domain, $\Omega \subset (-a, a)^{N-1} \times (-\infty, a)$, where $a > 0$, then it is possible to reduce this case to that investigated in Theorem 3.1.2 by a transformation of variables

$$y' = x'\,, \quad y_N = -x_N\,.$$

(ii) The case when Ω is an unbounded domain,

$\Omega \subset (-a, a)^{N-1} \times \mathbb{R}\,(a > 0)$ and $\inf\{x_N;\ x \in \Omega\} = -\infty$, $\sup\{x_N;\ x \in \Omega\} = +\infty$,

can be investigated analogously as in Theorem 3.1.2 with the only difference of cutting the domain Ω at both ends, i.e. for $n \in N$ we define

$$\Omega_n = \{x \in \Omega;\ |x_N| < n\}\,, \quad \Omega^n = \{x \in \Omega;\ |x_N| > n\}\,.$$

(iii) Theorem 3.1.2 describes the situation when the weight function ϱ or $w_{(0,\dots,0,m)}$ can be bounded from above or from below, respectively, by the product of a positive constant and two nonnegative measurable functions one of which depends on the variable x_N only while the other depends only on x' and the domain is unbounded in the direction of the x_N axis. Let us remark that any of the variables $x_1, x_2, \dots, x_N$

can play the role of the variable x_N. It is even possible to study the case when some curvilinear coordinate takes the role of the variable x_N. In Section 3.2 we shall discuss the case of spherical coordinates.

3.2. USING THE SPHERICAL COORDINATES

We shall consider spherical coordinates (r, Θ) in $\mathbb{R}^N$, where $r = |x|$ is the distance from the point x to the origin and $\Theta = x/|x|$ is a point on the unit sphere $E = \{x \in \mathbb{R}^N;\ |x| = 1\}$. If $Q \subset \mathbb{R}^N$, then $P_E(Q)$ will denote the projection of the set Q into the unit sphere E, i.e.

$$P_E(Q) = \{\Theta \in E;\ \exists r > 0,\ (r, \Theta) \in Q\}\,.$$

Let $W^{k,p}(\Omega, S)$, $L^p(\Omega, \varrho)$ and X be as in Section 3.1. Throughout this section we consider the following two conditions:

C1*. Ω is an unbounded domain in $\mathbb{R}^N$.

C2*. $W^{k,p}(\Omega_n, S) \hookrightarrow\hookrightarrow L^p(\Omega_n, \varrho)$ $\forall n \in N$, where $\Omega_n = \{x \in \Omega;\ |x| < n\}$ for $n \in N$.

Again, we shall look for additional assumptions implying

$$W_0^{k,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho)\,. \tag{3.2.1}$$

Denote $\Omega^n = \{x \in \Omega;\ |x| > n\}$ for $n \in N$. Analogously as in Section 3.1 we can prove:

If

$$\sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \to 0 \quad \text{for} \quad n \to \infty\,, \tag{3.2.2}$$

then (3.2.1) holds.

Moreover, suppose that the following condition is fulfilled:

C3*. There exist numbers $C > 0$, $m, n_0 \in N$, $1 \leqq m \leqq k$ and nonnegative measurable functions $\mu: (n_0, \infty) \to \mathbb{R}$,

$$\nu: (n_0, \infty) \to \mathbb{R}\,, \quad \xi: P_E(\Omega^{n_0}) \to \mathbb{R}$$

such that

$$\varrho(x) \leqq C\, \mu(|x|)\ \ \xi\left(\frac{x}{|x|}\right) \quad \text{for a.e.} \quad x \in \Omega^{n_0}\,; \tag{3.2.3}$$

$$\nu(|x|)\, \xi\left(\frac{x}{|x|}\right) \leqq C \min_{|\alpha| = m} w_\alpha(x) \quad \text{for a.e.} \quad x \in \Omega^{n_0}\,; \tag{3.2.4}$$

$$h(n) = h(n; \mu, \nu, p, m) = \tag{3.2.5}$$

$$= \int_n^\infty \mu(r)\, \|(s - r)^{m-1}\, \nu^{-1/p}(s)\|_{p^*,(r,\infty)}^p\, \mathrm{d}r \to 0 \quad \text{for} \quad n \to \infty\,.$$

Now we shall investigate the validity of (3.2.2). Let $u \in X$ and $n \geqq n_0$, where n_0 is the number from the condition **C3***. We extend the function u outside Ω by zero (then, clearly, $u \in C_0^\infty(\mathbb{R}^N)$) and take $\xi(\Theta) = 1$ for $\Theta \in E \setminus P_E(\Omega^{n_0})$. In view of (3.2.3),

$$(3.2.6) \qquad \|u\|_{p,\Omega^n,\varrho}^p = \int_{\Omega^n} |u(x)|^p \varrho(x) \,\mathrm{d}x \leqq$$

$$\leqq C \int_E \left[\int_n^\infty |u(r, \Theta)|^p \mu(r) r^{N-1} \,\mathrm{d}r \right] \xi(\Theta) \,\mathrm{d}\Theta \quad \text{for} \quad n \geqq n_0 .$$

For a fixed $\Theta \in E$ we denote

$$(3.2.7) \qquad f(r) = u(r, \Theta), \quad r > 0 .$$

Evidently $f \in C^\infty((0, \infty))$, and $\operatorname{supp} f$ is a compact set. Let the number m be from the condition **C3***. Applying Lemma 2.4 we obtain

$$(3.2.8) \qquad f(r) = \frac{(-1)^m}{(m-1)!} \int_r^\infty (s - r)^{m-1} f^{(m)}(s) \,\mathrm{d}s , \quad r > 0 ,$$

which implies

$$(3.2.9) \quad |f(r)| \, \mu^{1/p}(r) \, r^{(N-1)/p} \leqq \frac{1}{(m-1)!} \mu^{1/p}(r) \int_r^\infty (s - r)^{m-1} |f^{(m)}(s)| s^{(N-1)/p} \,\mathrm{d}s .$$

Analogously as in Section 3.1 we obtain from (3.2.9)

$$(3.2.10) \qquad \int_n^\infty |f(r)|^p \mu(r) r^{N-1} \,\mathrm{d}r \leqq$$

$$\leqq \left[\frac{1}{(m-1)!}\right]^p h(n) \int_n^\infty |f^{(m)}(s)|^p v(s) s^{N-1} \,\mathrm{d}s ,$$

where the function h is defined in (3.2.5). The relations (3.2.4), (3.2.6), (3.2.7) and (3.2.10) imply

$$\|u\|_{p,\Omega^n,\varrho}^p \leqq$$

$$\leqq C \left[\frac{1}{(m-1)!}\right]^p h(n) \int_E \left[\int_n^\infty \left| \frac{\partial^m}{\partial r^m} u(s, \Theta) \right|^p v(s) s^{N-1} \,\mathrm{d}s \right] \xi(\Theta) \,\mathrm{d}\Theta \leqq$$

$$\leqq C^2 \left[\frac{1}{(m-1)!}\right]^p h(n) \sum_{|\alpha|=m} \int_{\Omega^n} |D^\alpha u(x)|^p w_\alpha(x) \,\mathrm{d}x \leqq$$

$$\leqq C^2 \left[\frac{1}{(m-1)!}\right]^p h(n) \|u\|_X^p ,$$

hence

$$(3.2.11) \qquad \sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \leqq C^{2/p} \frac{1}{(m-1)!} h^{1/p}(n) , \quad n \geqq n_0 .$$

This and (3.2.5) yield (3.2.2).

From the above consideration we have

3.2.1. Theorem. *Let the conditions* **C1*–C3*** *be fulfilled. Then*

$$W_0^{k,p}(\Omega, S) \subset\subset L^p(\Omega, \varrho). \tag{3.1.12}$$

3.2.2. Remark. (i) The function h from (3.2.5) coincides with the function h from (3.1.10). Therefore we get analogous results here as in Section 3.1. Especially, if $m = 1$, $\mu(r) = r^\alpha$, $\nu(r) = r^\beta$ for $r \in (n_0, \infty)$, then $h(n) \to 0$ for $n \to \infty$ if $\beta > p - 1$, $\alpha < \beta - p$ (cf. Example 3.1.7). Consequently, the number β is always positive (because $p \in \langle 1, \infty)$). If we assume in addition that

$$\text{the function} \quad \mu(r)\, r^{N-1} \quad \text{is nondecreasing on} \quad (n_0, \infty) \tag{3.2.13}$$

we get a larger interval for β – see Example 3.2.3.

(ii) If the condition (3.2.5) in **C3*** is replaced by the assumption

$$g(n) = g(n; \mu, \nu, p, m) = \tag{3.2.5*}$$

$$= \int_n^\infty \left\| (s - r)^{m-1} \left[\frac{\mu(s)}{\nu(s)}\right]^{1/p} \right\|_{p^*,(r,\infty)}^p \mathrm{d}r \to 0 \quad \text{for} \quad n \to \infty$$

and if suppose that, in addition to all the assumptions of Theorem 3.2.1, (3.2.13) is fulfilled, then (3.2.12) holds again.

Really, let us suppose (3.2.13). Then from (3.2.8) we obtain

$$|f(r)|\, \mu^{1/p}(r) r^{(N-1)/p} \leqq \frac{1}{(m-1)!} \int_r^\infty (s - r)^{m-1} |f^{(m)}(s)|\, \mu^{1/p}(s)\, s^{(N-1)/p}\, \mathrm{d}s$$

and further we get

$$\int_n^\infty |f(r)|^p\, \mu(r)\, r^{N-1}\, \mathrm{d}r \leqq \tag{3.2.14}$$

$$\leqq \left[\frac{1}{(m-1)!}\right]^p g(n) \int_n^\infty |f^{(m)}(s)|^p\, \nu(s)\, s^{N-1}\, \mathrm{d}s\,.$$

The relations (3.2.4), (3.2.6), (3.2.7) and (3.2.14) imply

$$\|u\|_{p,\Omega^n,\varrho}^p \leqq C^2 \left[\frac{1}{(m-1)!}\right]^p g(n)\, \|u\|_X^p\,,$$

hence

$$\sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \leqq C^{2/p} \frac{1}{(m-1)!}\, g^{1/p}(n)\,. \tag{3.2.15}$$

This and (3.2.5*) yield the desired assertion.

(iii) Let us remark that while the functions h and g in Section 3.1 (see (3.1.10) and (3.1.10*)) satisfied

$$(3.2.16) \qquad h(n) \leqq g(n) \quad \text{for} \quad n \geqq n_0\,,$$

the functions h and g given by (3.2.5) and (3.2.5*) need not generally satisfy the inequality (3.2.16) (here, the function $\mu(r)\, r^{N-1}$ is nondecreasing in contradiction to Section 3.1, where $\mu(r)$ was nondecreasing).

For $x \in \mathbb{R}^N$ and $\varepsilon \in \mathbb{R}$ let us take

$$(3.2.17) \qquad \omega_\varepsilon(x) = \begin{cases} |x|^\varepsilon\,, & |x| > 1\,, \\ 1\,, & |x| \leqq 1\,. \end{cases}$$

3.2.3. Example. Let Ω be an unbounded domain in $\mathbb{R}^N$, $p \in \langle 1, \infty)$, $k = 1$, $\varepsilon \in \mathbb{R}$

$$(3.2.18) \qquad \beta > 1 - N + p\,, \quad \alpha \in \langle 1 - N, \beta - p)\,.$$

For $x \in \Omega$ we define

$$\varrho(x) = \omega_\alpha(x)\,, \quad w_{(0,\ldots,0)}(x) = \omega_\varepsilon(x)\,,$$
$$w_\gamma(x) = \omega_\beta(x) \quad \text{for} \quad |\gamma| = 1\,.$$

Let $S = \{w_\gamma;\ |\gamma| \leqq 1\}$.

We can easily verify that the conditions **C1***, **C2***, and (3.2.3), (3.2.4) from the condition **C3*** are satisfied (in the condition **C3*** we take $m = 1$, $C = 1$, $n_0 \in N$, $\xi(\Theta) = 1$ for $\Theta \in P_E(\Omega^{n_0})$, $\mu(r) = r^\alpha$, $v(r) = r^\beta$ for $r \in (n_0, \infty)$).

Let us now investigate the validity of (3.2.5*). For $n \in N$, $n \geqq n_0$, we have

$$g(n) = \int_n^\infty \left\| (s - r)^{m-1} \left[\frac{\mu(s)}{v(s)}\right]^{1/p} \right\|^p_{p^*,(r,\infty)} \mathrm{d}r =$$
$$= \left(\frac{p-1}{\beta - \alpha - p + 1}\right)^{p-1} \cdot \frac{1}{\beta - \alpha - p}\, n^{\alpha - \beta + p}\,,$$

hence $g(n) \to 0$ for $n \to \infty$. One can easily verify that (3.2.13) is satisfied as well. Therefore, from Remark 3.2.2 (ii) we get

$$(3.2.19) \qquad W_0^{1,p}(\Omega, S) \subset\subset L^p(\Omega, \varrho)\,.$$

If we use Theorem 3.2.1, we obtain (3.2.19) for

$$(3.2.20) \qquad \beta > p - 1\,, \quad \alpha < \beta - p\,.$$

Let us compare (3.2.18) with (3.2.20). In contrast to (3.2.18) where the interval for β is larger for $N > 2$, the interval for α in (3.2.18) is smaller. The interval for α can, of course, be extended by means of the following remark.

3.2.4. Remark. Let Ω be an unbounded domain in $\mathbb{R}^N$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_2 \geqq \alpha_1$. For $x \in \Omega$ let us take

$$\varrho_i(x) = \omega_{\alpha_i}(x)\,, \quad i = 1, 2\,.$$

Then

$$L^p(\Omega, \varrho_2) \hookrightarrow L^p(\Omega, \varrho_1)\,. \tag{3.2.21}$$

The proof is easy: For $|x| \geqq 1$ we have $|x|^{\alpha_2} \geqq |x|^{\alpha_1}$ and hence $\varrho_2(x) \geqq \varrho_1(x)$ for $x \in \Omega^1$. Consequently, for $u \in L^p(\Omega, \varrho_2)$ we have

$$\|u\|^p_{p,\Omega,\varrho_1} = \|u\|^p_{p,\Omega_1,\varrho_1} + \|u\|^p_{p,\Omega^1,\varrho_1} = \|u\|^p_{p,\Omega_1,\varrho_2} + \|u\|^p_{p,\Omega^1,\varrho_1} \leqq$$
$$\leqq \|u\|^p_{p,\Omega_1,\varrho_2} + \|u\|^p_{p,\Omega^1,\varrho_2} = \|u\|^p_{p,\Omega,\varrho_2}\,.$$

This yields (3.2.21).

3.2.5. Remarks. From Example 3.2.3 and Remark 3.2.4 we get:
(3.2.19) holds if

$$\beta > \min(1 - N + p,\ p - 1)\,, \quad \alpha < \beta - p\,.$$

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Souhrn

KOMPAKTNOST VNOŘENÍ VÁHOVÉHO SOBOLEVOVA PROSTORU DEFINOVANÉHO NA NEOMEZENÉ OBLASTI I

BOHUMÍR OPIC

Článek se zabývá kompaktním vnořením váhového Sobolevova prostoru $W_0^{k,p}(\Omega, S)$ (S je systém váhových funkcí) definovaného na neomezené oblasti do prostoru funkcí $L^p(\Omega, \varrho)$ (ϱ je váhová funkce). Dané vnoření je vyšetřováno jako limitní případ kompaktních vnoření Sobolevových prostorů definovaných na omezených oblastech.

Резюме

КОМПАКТНОЕ ВЛОЖЕНИЕ ВЕСОВОГО ПРОСТРАНСТВА СОБОЛЕВА, ОПРЕДЕЛЕННОГО В НЕОГРАНИЧЕННОЙ ОБЛАСТИ

BOHUMÍR OPIC

В работе исследуется компактность вложения весового пространства Соболева $W_0^{k,p}(\Omega, S)$ (S — система весовых функций), определенного в неограниченной области, в пространство функций $L^p(\Omega, \varrho)$ (ϱ-весовая функция). Это вложение рассматривается как предельный случай компактных вложений пространств Соболева, определенных в ограниченных областях.

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