Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 112 (1987), No. 3, 261--268

Persistent URL: http://dml.cz/dmlcz/118322

112 (1987) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 3, 261—268

LIFTS OF GENERALIZED SYMMETRIC SPACES TO TANGENT BUNDLES

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Dedicated to Professor Hidekiyo Wakakuwa on the occasion of his 60th birthday

(Received November 8, 1984)

Summary. A simple proof is given of the fact that the complete lift of a simply connected generalized symmetric pseudo-Riemannian space to its tangent bundle is a generalized symmetric pseudo-Riemannian space.

Keywords: complete lift, generalized symmetric pseudo-Riemannian space.

The theory of generalized symmetric spaces and regular s-manifolds was studied many authors (see, for example, [1]–[4], [6]–[10]). A useful tool for this study by the algebraic characterization of regular s-manifolds established by O. Kowalski [6]. M. Toomanian [9] found a construction how to lift the structure of a regular pseudo-Riemannian s-manifold to its tangent bundle. The result is a pseudo-Riemannian regular s-structure on the tangent bundle. His method is analytic, and the calculations involved are rather complicated.

In this paper we give a simple and more algebraic proof of the Toomanian's result in the case when the base manifold is simply connected. We are using only basic facts from the paper [11] by K. Yano and S. Kobayashi and those from the book [6] by O. Kowalski.

Section 1 is a summary of concepts about lifting operations from a base manifold to its tangent bundle. Section 2 deals with the theory of regular s-manifolds. In these first two sections, we restrict ourselves to the facts which are needed in Section 3. Finally, in Section 3 we prove our main theorem.

I would like to express my sincer gratitude to Professor O. Kowalski for suggesting this problem to me, for several useful discussions and for his encouragement. I would like to thank also the Department of Mathematical Analysis, Charles University in Prague, which provided convenience for my research stay in Prague.

1. TANGENT BUNDLES

In this section we give a brief survey on prolongations of tensor fields and connections of manifold to its tangent bundle. We refer to Yano-Kobayashi [11] for more details.

Let M be a smooth manifold of dimension n. Let $\mathfrak{X}(M)$ be the Lie algebra of all smooth vector fields on M and $\mathfrak{T}(M)$ the tensor algebra of all smooth tensor fields on M. For any smooth mapping φ of M into a smooth manifold N, let φ_* denote the differential of φ, φ^* its dual mapping.

Further, let M_x be the tangent space of M at a point x in M and $TM = \bigcup_{x \in M} M_x$ the tangent bundle over M with the natural projection π.

Given a system of local coordinates $(x^1, x^2, \ldots, x^n)$ in M, we denote by $(x^1, x^2, \ldots, x^n, u^1, u^2, \ldots, u^n)$ the system of local coordinates in TM determined as follows: If $x' = \sum b^i(\partial/\partial x^i)_x \in M_x$ and x is a point with the coordinates $(a^1, a^2, \ldots, a^n)$ with respect to $(x^1, x^2, \ldots, x^n)$, then x' has the coordinates $(a^1, a^2, \ldots, a^n, b^1, b^2, \ldots, b^n)$ with respect to $(x^1, x^2, \ldots, x^n, u^1, u^2, \ldots, u^n)$.

For a function f on M, the function $\pi^* f$ on TM induced by the projection π is denoted by f^v and is called the *vertical lift* of the function f from M to TM. Any 1-form ω on M may be regarded, in a natural way, as a function on TM. We denote this function by $\iota\omega$. The value of the function $\iota\omega$ at a point (x, X_x) in TM is $(\iota\omega)(x, X_x) = \omega_x(X_x)$, where X_x is a tangent vector of M at a point x in M. For any vector field Y on M we define a vector field Y^v on TM by $Y^v(\iota\omega) = (\omega(Y))^v$ for all 1-forms ω on M. We call Y^v the *vertical lift* of the vector field Y from M to TM. For any function f on M we denote by $\mathrm{d}f$ the differential of f. $\mathrm{d}f$ is a 1-form on M. We define the *vertical lift* of a 1-form $\mathrm{d}f$ on M by $(\mathrm{d}f)^v = \mathrm{d}(f^v)$ for all functions f on M. We define the *vertical lift* of an arbitrary 1-form ω on M by $\omega^v = \sum(\omega_i)^v (\mathrm{d}x^i)^v$, where $\omega = \sum \omega_i \, \mathrm{d}x^i$. We extend the vertical lifts defined above to a unique linear mapping of the tensor algebra $\mathfrak{T}(M)$ on M to the tensor algebra $\mathfrak{T}(TM)$ on TM under the condition $(T \otimes S)^v = T^v \otimes S^v$ for all tensor fields T and S on M.

For a function f on M we put $f^c = \iota \, \mathrm{d}f$ and call the function f^c on TM the *complete lift* of the function f from M to TM. For a vector field Y on M we define a vector field Y^c on TM by $Y^c f^c = (Yf)^c$ for all functions f on M. We call Y^c the *complete lift* of the vector field Y from M to TM. Given a 1-form ω on M we define a 1-form ω^c on TM by $\omega^c(Y^c) = (\omega(Y))^c$ for all vector fields Y on M. We call ω^c the *complete lift* of the 1-form ω from M to TM. We extend the complete lifts defined above to a unique linear mapping of the tensor algebra $\mathfrak{T}(M)$ on M to the tensor algebra $\mathfrak{T}(TM)$ on TM under the condition $(T \otimes S)^c = T^c \otimes S^v + T^v \otimes S^c$ for all tensor fields T and S on M.

In terms of the system of local coordinates, we easily obtain that

$$Y^c = \sum Y^i \, \partial/\partial x^i + \sum u^r (\partial Y^i/\partial x^r) \, \partial/\partial u^i$$

for all vector fields $Y = \sum Y^i \, \partial/\partial x^i$ on M. From this formula for Y^c we get the following lemma (cf. Yano-Kobayashi [11], Remark in Section 5).

Lemma. *Let x' be a point in TM which is not in the zero-section of TM. Then the set $\{Y^c_{x'} \in (TM)_{x'} \mid Y \in \mathfrak{X}(M)\}$ is the whole tangent space $(TM)_{x'}$.*

Yano and Kobayashi [11] have derived a number of properties of the lifting operations. We sum up here only those which will be used later (see Proposition A to Proposition F below).

Proposition A. *For any tensor field T of type (p, q) on M, we have*

$$T^c(Y_1^c, Y_2^c, \ldots, Y_q^c) = (T(Y_1, Y_2, \ldots, Y_q))^c$$

for all $Y_i \in \mathfrak{X}(M)$ $(i = 1, 2, \ldots, q)$.

Proposition B. *Let g be a pseudo-Riemannian metric on M. Then the complete lift g^c of g is a pseudo-Riemannian metric on TM with n positive and n negative signs.*

Let ∇ be an affine connection on M. Then there exists a unique affine connection ∇^c on TM which satisfies

$$\nabla^c_{X^c} Y^c = (\nabla_X Y)^c$$

for all $X, Y \in \mathfrak{X}(M)$. We call the connection ∇^c the *complete lift* of the connection ∇ from M to TM. Now we have

Proposition C. *If R and T are the curvature tensor field and the torsion tensor field for ∇, then R^c and T^c are the curvature tensor field and the torsion tensor field for ∇^c.*

Proposition D. *If M is complete with respect to an affine connection ∇, then TM is complete with respect to ∇^c, and vice versa.*

Proposition D is an immediate consequence of a result from [11], saying that a Jacobi vector field along a geodesic in (M, ∇) considered as a curve in (TM, ∇^c) is a geodesic, and vice versa.

Proposition E. *If ∇ is the Riemannian connection of M with respect to a pseudo-Riemannian metric g, then ∇^c is the Riemannian connection of TM with respect to the pseudo-Riemannian metric g^c.*

Proposition F. *Let R and T be the curvature tensor field and the torsion tensor field of an affine connection of M. According as $R = 0$, $\nabla R = 0$, $T = 0$ or $\nabla T = 0$, we have $R^c = 0$, $\nabla^c R^c = 0$, $T^c = 0$ or $\nabla^c T^c = 0$.*

2. AFFINE REDUCTIVE SPACES AND REGULAR s-MANIFOLDS

We shall give some preliminaries which can be found in the book [6] by Kowalski.

First of all we shall recall some elementary properties of the reductive homogeneous spaces.

Let K be a connected Lie group and H its closed subgroup. Consider the homogeneous manifold K/H. Let $\mathfrak{k} \supset \mathfrak{h}$ be the Lie algebras of K and H, respectively.

Suppose that there is a subspace $\mathfrak{m} \subset \mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ (direct sum of vector spaces) and $\operatorname{ad}(h)\,\mathfrak{m} = \mathfrak{m}$ for all $h \in H$. Then the homogeneous space K/H is said to be *reductive with respect to the decomposition* $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. Let $\tilde{\nabla}$ be the canonical connection of the reductive homogeneous space K/H. Then the curvature tensor field $\tilde{R}$ and the torsion tensor field $\tilde{T}$ are parallel, that is, $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$ (see, for example, [5] Theorem 2.6, p. 193).

Further, we need the concept of the affine reductive space.

Let $(M, \tilde{\nabla})$ be a connected manifold with an affine connection. The group of all affine transformations of M preserving each holonomy subbundle of the frame bundle $\mathfrak{F}(M)$ is called the *group of transvections* of $(M, \tilde{\nabla})$. It will be denoted by $\operatorname{Tr}(M, \tilde{\nabla})$. Now $(M, \tilde{\nabla})$ is called an *affine reductive space* if the group $\operatorname{Tr}(M, \tilde{\nabla})$ acts transitively on each holonomy bundle. It is known [6, Theorem I.25] that a connected manifold $(M, \tilde{\nabla})$ with an affine connection is an affine reductive space if and only if M can be expressed as a reductive homogeneous space K/H with respect to a decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$, where K is effective on M, and $\tilde{\nabla}$ is the canonical connection of K/H. The following is essentially due to K. Nomizu (cf. [6, Theorem I.40]):

Proposition G. *Let $(M, \tilde{\nabla})$ be a connected and simply connected manifold with a complete affine connection such that $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$. Then $(M, \tilde{\nabla})$ is an affine reductive space.*

Next, we concentrate on the pseudo-Riemannian regular s-manifolds. All definitions and theorems below are slight modifications of those for the Riemannian case given in Kowalski [6]. We also refer to Černý-Kowalski [1].

Let (M, g) be a smooth pseudo-Riemannian manifold. An *s-structure* on (M, g) is a family $\{s_x \mid x \in M\}$ of isometries of (M, g) (called *symmetries*) such that each s_x has the point x as an isolated fixed point. An s-structure $\{s_x\}$ on (M, g) is said to be *regular* if

(i) the mapping $(x, y) \mapsto s_x(y)$ of $M \times M$ into M is smooth,

(ii) for every pair of points $x, y \in M$ we have $s_x \circ s_y = s_z \circ s_x$, where $z = s_x(y)$.

If we define the tangent tensor field S of type (1,1) of $\{s_x\}$ by $S_x = (s_x)_{*x}$ for each $x \in M$, we can see that $\{s_x\}$ is regular if and only if the tensor field S is smooth and invariant with respect to all symmetries s_x.

A *generalized symmetric pseudo-Riemannian space* is a connected pseudo-Riemannian manifold (M, g) admitting at least one regular s-structure. Every generalized symmetric pseudo-Riemannian space is a homogeneous pseudo-Riemannian manifold.

Let (M, g) be a generalized pseudo-Riemannian space and $\{s_x\}$ a fixed regular s-structure on (M, g). Then the triplet $(M, g, \{s_x\})$ will be called a *pseudo-Riemannian regular s-manifold*. Let now ∇ denote the Riemannian connection of (M, g) and let S be the tangent tensor field of $\{s_x\}$. Following [3], we introduce a new linear connection $\tilde{\nabla}$ by the formula

$$\tilde{\nabla}_Y Z = \nabla_Y Z - (\nabla_{(I-S)^{-1}Y} S)(S^{-1}Z)$$

for all $Y, Z \in \mathfrak{X}(M)$. We call this connection the *canonical connection* of $(M, g, \{s_x\})$. The basic properties of the affine manifold $(M, \tilde{\nabla})$ are given in [3], [6]. In particular, $(M, \tilde{\nabla})$ is always an affine reductive space [6, Corollary II.27]:

Proposition H. *The canonical connection of a connected pseudo-Riemannian regular s-manifold $(M, g, \{s_x\})$ is always complete and satisfies $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$, $\tilde{\nabla}g = \tilde{\nabla}S = 0$. Also $(M, \tilde{\nabla})$ is an affine reductive space.*

The next proposition gives sufficient conditions for an affine reductive space to become a pseudo-Riemannian regular s-manifold. It can be easily compiled from Propositions V.3 and V.4 in [6].

Proposition I. *Let $(M, \tilde{\nabla})$ be a simply connected affine reductive space, and $o \in M$ a fixed point. Let g be a pseudo-Riemannian metric on M such that $\tilde{\nabla}g = 0$. Finally, let $S_0: M_0 \to M_0$ be a non-singular linear transformation.*

Suppose that the following conditions hold:

(i) $I_0 - S_0$ *is a non-singular transformation of* M_0,
(ii) $\tilde{R}_0(S_0 Y, S_0 Z) S_0 W = S_0 \tilde{R}_0(Y, Z) W$ *and* $\tilde{T}_0(S_0 Y, S_0 Z) = S_0 \tilde{T}_0(Y, Z)$ *for all* $Y, Z, W \in M_0$,
(iii) $\tilde{R}_0(S_0 Y, S_0 Z) = \tilde{R}_0(Y, Z)$ *for all* $Y, Z \in M_0$,
(iv) $g_0(S_0 Y, S_0 Z) = g_0(Y, Z)$ *for all* $Y, Z \in M_0$.

Then the space (M, g) admits a unique pseudo-Riemannian regular s-structure $\{s_x\}$ such that $(s_0)_{*0} = S_0$.

The converse is also true for the arbitrary choice of the origin o.

3. LIFTED s-STRUCTURES

In this section we show that the structure of a simply connected pseudo-Riemannian regular s-manifold can be lifted to its tangent bundle. We shall start with

Proposition 1. *Let $(M, \tilde{\nabla})$ be a simply connected affine reductive space, and $\tilde{\nabla}^c$ the complete lift of the affine connection $\tilde{\nabla}$ from M to its tangent bundle TM. Then $(TM, \tilde{\nabla}^c)$ is an (simply connected) affine reductive space.*

Proof. Let $\tilde{R}$ and $\tilde{T}$ be the curvature tensor field and the torsion field of the connection $\tilde{\nabla}$ on M. Then $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$ since $(M, \tilde{\nabla})$ is an affine reductive space. Now let $\tilde{R}^c$ and $\tilde{T}^c$ be the complete lifts of $\tilde{R}$ and $\tilde{T}$ from M to TM, respectively. By Proposition C, $\tilde{R}^c$ is the curvature tensor field and $\tilde{T}^c$ the torsion tensor field of $\tilde{\nabla}^c$. Further, $\tilde{\nabla}^c\tilde{R}^c = \tilde{\nabla}^c\tilde{T}^c = 0$ holds in virtue of $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$ and Proposition F.

Since the connection $\tilde{\nabla}$ is complete, the connection $\tilde{\nabla}^c$ is also complete by Proposition D. Hence Proposition 1 follows from Proposition G.

Now we prove the main theorem of this paper.

Theorem. *Let (M, g) be a connected and simply connected pseudo-Riemannian manifold admitting a regular s-structure $\{s_x\}$. Further, let TM be the tangent bundle over M and g^c the complete lift of g from M to TM. Then the pseudo-Riemannian manifold (TM, g^c) admits a regular s-structure $\{s'_{x'}\}$. In other words, the complete lift of a simply connected generalized symmetric pseudo-Riemannian space to its tangent bundle is a generalized symmetric pseudo-Riemannian space.*

Proof. Let $\tilde{\nabla}$ be the canonical connection of the pseudo-Riemannian regular s-manifold $(M, g, \{s_x\})$. Then $(M, \tilde{\nabla})$ is an affine reductive space and $\tilde{\nabla}g = 0$. Hence, by Proposition 1, $(TM, \tilde{\nabla}^c)$ is a simply connected affine reductive space. Moreover, $\tilde{\nabla}^c g^c = 0$. Here $\tilde{\nabla}^c$ is the complete lift of $\tilde{\nabla}$ to TM.

Next, we prove that the space (TM, g^c) has a regular s-structure. Let o' be a fixed point which is in TM but not in the zero-section of TM, and let $o = \pi(o') \in M$. It is sufficient to prove the conditions (i)–(iv) of Proposition I for $S^c_{o'}$, $\tilde{R}^c_{o'}$, $\tilde{T}^c_{o'}$ and $g^c_{o'}$, using the validity of (i)–(iv) for S, $\tilde{R}$, $\tilde{T}$ and g at o, and also at any other point $x \in M$.

Since S_0 is non-singular, the set $\{S_0 Y_0 \mid Y_0 \in M_0\}$ is the whole tangent space M_0. Here Y_0 denotes the value of a vector field Y at o. By Proposition A, $S^c_{o'} Y^c_{o'} = (SY)^c_{o'}$ holds for all $Y \in \mathfrak{X}(M)$. Therefore, by Lemma in Section 1, the set $\{S^c_{o'} Y^c_{o'} \mid Y \in \mathfrak{X}(M)\}$ is the whole tangent space $(TM)_{o'}$ at $o' \in TM$. This implies that $S^c_{o'}$ is non-singular. In a similar way it is proved that $I_{o'} - S^c_{o'}$ is non-singular. Hence the condition (i) of Proposition I is valid. The calculations for (ii)–(iv) are straightforward. For example, we show the proof of the formula (iii) $\tilde{R}^c_{o'}(S^c_{o'} Y', S^c_{o'} Z') = \tilde{R}^c_{o'}(Y', Z')$ for all $Y', Z' \in \in (TM)_{o'}$. By Lemma in Section 1, it is sufficient to show this for vectors $Y' = Y^c_{o'}$ and $Z' = Z^c_{o'}$ which are the values of the complete lifts Y^c and Z^c of any vector fields Y and Z on M. Using Proposition A, we see that

$$\tilde{R}^c_{o'}(S^c_{o'} Y^c_{o'}, S^c_{o'} Z^c_{o'})\, W^c_{o'} = (\tilde{R}(SY, SZ)\, W)^c_{o'} = (\tilde{R}(Y, Z)\, W)^c_{o'} = \tilde{R}^c_{o'}(Y^c_{o'}, Z^c_{o'})\, W^c_{o'}$$

for all $W^c_{o'} \in (TM)_{o'}$, where W^c is the complete lift of some $W \in \mathfrak{X}(M)$. Hence, using Lemma in Section 1 again, we get $\tilde{R}^c_{o'}(S^c_{o'} Y^c_{o'}, S^c_{o'} Z^c_{o'}) = \tilde{R}^c_{o'}(Y^c_{o'}, Z^c_{o'})$.

This completes the proof of the theorem.

Remark 1. This theorem is a generalization, for the simply connected case, of the following result which has been stated without proof in [11]: If M is a pseudo-Riemannian (or affine) symmetric space with a metric g (or a connection ∇), then TM is also a pseudo-Riemannian (affine) symmetric space with a metric g^c (a connection ∇^c, respectively).

Remark 2. As mentioned at the beginning of this paper, Toomanian [9] constructed a pseudo-Riemannian regular s-structure on the tangent bundle over a Riemannian regular s-manifold wihout restriction to the simply connected case. To do this, he first defined transformations $s'_{x'}$, $x' \in TM$, on TM as follows: Let $\{s_x\}$ be the regular s-structure on (M, g). Further, let ψ_x, $x \in M$, be the mapping of the Lie group K of transformations on M to M defined by $\psi_x(a) = ax$ for all $a \in K$, and let T_y, $y \in M$, be the mapping of M to K defined by $T_y(x) = s_y^{-1} \circ s_x$ for all $x \in M$. Now let, for any $x' = (x, X_x)$ and $y' = (y, Y_y)$ in TM,

$$s'_{x'}(y') = (s_x(y), (s_x)_* Y_y + (\psi_{s_x(y)})_* \operatorname{ad}(s_x) \tilde{X}_e),$$

where $a \mapsto \operatorname{ad}(a)$ is the adjoint representation of K on its Lie algebra, and $\tilde{X}_e = (T_x)_{*x} X_x$. Next, he proved that $(TM, g^c, \{s'_{x'}\})$ is a pseudo-Riemannian regular s-manifold [9, Theorem 3.2]. Finally, he showed that the tangent tensor field S' of $\{s'_{x'}\}$ is the complete lift of the tangent tensor field S of $\{s_x\}$ [9, Theorem 3.3]. Therefore, we see that, for each $x' \in TM$ and $x = \pi(x')$, the following diagram is commutative:

$$\begin{array}{ccc} TM & \xrightarrow{\;s'_{x'}\;} & TM \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\;s_x\;} & M \end{array}$$

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Souhrn

LIFTY ZOBECNĚNÝCH SYMETRICKÝCH PROSTORŮ NA TEČNÉ FIBROVANÉ PROSTORY

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Je podán jednoduchý důkaz tvrzení, že úplný lift jednoduše souvislého zobecněného symetrického pseudoriemannovského prostoru na jeho tečný fibrovaný prostor je opět zobecněný symetrický pseudoriemannovský prostor.

Резюме

ПОДЪЕМЫ ОБОБЩЕНЫХ СИММЕТРИЧЕСКИХ ПРОСТРАНСТВ Н КАСАТЕЛЬНЫЕ РАССЛОЕНИЯ

Masami Sekizawa

Приводится простое доказательство того, что полный подъем односвязного обобщенного псевдориманова пространства в касательное расслоение является тоже обобщенным симметрическим псевдоримановым пространством.

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