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Jan Havrda

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A STUDY OF INDEPENDENCE IN A SET WITH ORTHOGONALITY

JAN HAVRDA, Praha

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Summary. We investigate a set with orthogonality $(\Omega, \perp)$ and the corresponding complete lattice with orthogonality $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$. We assume that the lattice $\mathscr{S}$ is orthomodular and that it satisfies some natural assumptions. Let us suppose that $o \notin A \subset \Omega$ and that the set A contains at least two points. We then call the set A j-independent if $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$, k-independent if $B^{\perp\perp} \cap C^{\perp\perp} = \{o\}$ whenever $A = B \cup C$, $B \cap C = \emptyset$, $B \neq \emptyset \neq C$, and l-independent if $x \notin (A - \{x\})^{\perp\perp}$ for all $x \in A$. We call the set A I-independent if each finite subset of A which contains at least two points is i-independent for $(I, i) = (J, j)$, resp. $(I, i) = (K, k)$, resp. $(I, i) = (L, l)$. The article clarifies mutual relations of these concepts.

Keywords: set with orthogonality, orthomodular lattice, i-independent set, I-independent set.

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1. This paper carries on some ideas of [1] and presents three concepts of independent sets in a set with an orthogonality relation $(\Omega, \perp)$. It also pays attention to their interrelations. The motivation comes from the theory of linear spaces.

Let us recall that we call a relation $\perp \subset \Omega \times \Omega$ an orthogonality relation if 1. $\perp$ is symmetric, 2. there is a distinguished element o such that $\{o\} \times \Omega \subset \perp$ and the intersection of $\perp$ with the diagonal is exactly (o, o). The presence of an orthogonality relation on the set Ω gives rise to a complete lattice $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ where S consists of all subsets A of Ω satisfying $A = (A^\perp)^\perp$. Here, Ω plays the role of the unit element and $\{o\}$ plays the role of the nought element.

Throughout the whole paper, we shall assume that the complete lattice $\mathscr{S}$ is orthomodular and satisfies Axiom A and Axiom P:

Axiom A. *For every $x \in \Omega$, $x \neq o$, $\{x\}^{\perp\perp}$ is an atom in $\mathscr{S}$.*

Axiom P. *If $x \in \Omega$, $A \in S$, $x \notin A$, $x \notin A^\perp$, then there exist atoms $A_1 \subset A$ and $A_2 \subset A^\perp$ such that $x \in A_1 \vee A_2$.*

Let us restate here some equivalent conditions on a lattice with an orthogonality relation $\mathscr{P} = (P, \leqq, \perp, 1, 0)$ which we shall use in the sequel.

1.1. $\mathscr{P}$ is orthomodular.

1.2. If $a, b \in P$, $a \leqq b$, then $b = a \vee (a^\perp \wedge b)$.

1.3. If $a, b \in P$, $a \leqq b$, $a^\perp \wedge b = 0$, then $a = b$.

1.4. If $a, b, c \in P$, $a \leqq c$, $b \leqq c^\perp$, then $(a \vee b) \wedge c = a$.

2. Let $(\Omega, \perp)$ be a given set with an orthogonality relation. By the following definition we shall introduce three types of independence of subsets of Ω.

2.1. Definition. Let A be a subset of Ω such that $o \notin A$. Let us assume that the set A contains at least two points. We call the set A *j-independent* if and only if $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$. We call the set A *k-independent* if and only if $B^{\perp\perp} \cap C^{\perp\perp} = \{o\}$ whenever $A = B \cup C, B \neq \emptyset \neq C, B \cap C = \emptyset$. We call the set A *l-independent* if and only if $x \notin (A - \{x\})^{\perp\perp}$ for all $x \in A$.

2.2. Lemma. 1. *Every j-independent set is k-independent.* 2. *Every k-independent set is l-independent.*

Proof. 1. Let A be a j-independent set. We have $\{o\} = \bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \bigcap_{x \in B} (B \cup C - \{x\})^{\perp\perp} \cap \bigcap_{x \in C} (B \cup C - \{x\})^{\perp\perp} \supset C^{\perp\perp} \cap B^{\perp\perp}$ whenever $A = B \cup C$, $B \cap C = \emptyset, B \neq \emptyset \neq C$.

2. Let A be a k-independent set. It is true that $\{o\} = (A - \{x\})^{\perp\perp} \cap \{x\}^{\perp\perp}$ for all $x \in A$, hence $x \notin (A - \{x\})^{\perp\perp}$.

2.3. Lemma. *The following statements are equivalent*: 1. *The set A is j-independent.* 2. *For every element $y \in A^{\perp\perp}$, $y \neq o$, there is an element $a_y \in A$ such that $y \notin (A - \{a_y\})^{\perp\perp}$.*

In addition, we have $A^{\perp\perp} = (A - \{a_y\})^{\perp\perp} \vee \{y\}^{\perp\perp}$ for the elements y and a_y from Statement 2.

Proof. $1 \Rightarrow 2$. If such an element a_y does not exist, then $y \in (A - \{x\})^{\perp\perp}$ for every $x \in A$, hence $y \in \bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$ according to above, contrary to our hypothesis $y \neq o$.

$2 \Rightarrow 1$. If $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} \neq \{o\}$, then there is an element $y \in \bigcap_{x \in A} (A - \{x\})^{\perp\perp}$, $y \neq o$. Hence $o \neq y \in (A - \{x\})^{\perp\perp}$ for all $x \in A$ — a contradiction.

According to Theorem 2.10 of [2], we have $(A - \{a_y\})^{\perp\perp} \prec (A - \{a_y\})^{\perp\perp} \vee \{a_y\}^{\perp\perp} = A^{\perp\perp}$. Since $(A - \{a_y\})^{\perp\perp} \subset (A - \{a_y\})^{\perp\perp} \vee \{y\}^{\perp\perp} \subset A^{\perp\perp}$, we have either $(A - \{a_y\})^{\perp\perp} = (A - \{a_y\})^{\perp\perp} \vee \{y\}^{\perp\perp}$ or $(A - \{a_y\})^{\perp\perp} \vee \{y\}^{\perp\perp} = A^{\perp\perp}$. But the first identity is not valid because, in that case, we should have $y \in (A - \{a_y\})^{\perp\perp}$, contrary to our hypothesis. Lemma is proved.

2.4. Lemma. *The following statements are equivalent*: 1. *The set A is k-independent.* 2. *The identity $B^{\perp\perp} \cap C^{\perp\perp} = \{o\}$ is valid for every subsets $B, C \subset A$, $B \cap C = \emptyset, B \neq \emptyset \neq C$.*

Proof. $1 \Rightarrow 2$. We have $B^{\perp\perp} \cap C^{\perp\perp} \subset B^{\perp\perp} \cap (A - B)^{\perp\perp} = \{o\}$.

$2 \Rightarrow 1$. It suffices to put $C = A - B$.

2.5. Lemma. *The following statements are equivalent*: 1. *The set A is l-independent.* 2. *The inequality $B^{\perp\perp} \neq A^{\perp\perp}$ holds for every subset $B \subset A$, $\emptyset \neq B \neq A$.*

Proof. $1 \Rightarrow 2$. Let us suppose $B \subset A$, $\emptyset \neq B \neq A$ and $B^{\perp\perp} = A^{\perp\perp}$. Then there is an element $x \in A$, $x \notin B$, hence $B \subset A - \{x\} \subset A$. It follows that $B^{\perp\perp} \subset (A - \{x\})^{\perp\perp} \subset A^{\perp\perp}$ which implies $x \in A \subset A^{\perp\perp} = (A - \{x\})^{\perp\perp}$ – a contradiction.

$2 \Rightarrow 1$. Putting $B = A - \{x\}$ we have $(A - \{x\})^{\perp\perp} \subset A^{\perp\perp}$ and $(A - \{x\})^{\perp\perp} \neq A^{\perp\perp}$ for all $x \in A$. It is true that $\{o\} \neq (A - \{x\})^{\perp} \cap A^{\perp\perp} = (A - \{x\})^{\perp} \cap [(A - \{x\})^{\perp\perp} \vee \{x\}^{\perp\perp}]$ in accordance with Statement 1.3. If the set A is not l-indepedent, then there is an element $x \in A$ such that $x \in (A - \{x\})^{\perp\perp}$, hence $\{x\}^{\perp\perp} \subset (A - \{x\})^{\perp\perp}$. We get $\{o\} \neq (A - \{x\})^{\perp} \cap [(A - \{x\})^{\perp\perp} \vee \{x\}^{\perp\perp}] = (A - \{x\})^{\perp} \cap (A - \{x\})^{\perp\perp} = \{o\}$ – a contradiction.

2.6. Lemma. *If a set A is i-independent, then its every subset, which contains at least two points, is also i-independent for $i = j, k, l$.*

Proof. $i = j$. Let $B \subset A$ and let B contain at least two points. Then $\{o\} = \bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \bigcap_{x \in B} [(A - B)^{\perp\perp} \vee (B - \{x\})^{\perp\perp}] \cap \bigcap_{x \in A - B} [B^{\perp\perp} \vee (A - B - \{x\})^{\perp\perp}] \supset \bigcap_{x \in B} (B - \{x\})^{\perp\perp} \cap B^{\perp\perp} = \bigcap_{x \in B} (B - \{x\})^{\perp\perp}$.

$i = k$, $i = l$. Proof is obvious.

2.7. Lemma. *If a set A is i-independent and if $a \in \Omega$, $a \neq o$, $a \perp A$, then $A \cup \{a\}$ is also an i-independent set for $i = j, k, l$.*

Proof. $i = j$. We have $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$, $a \perp A$, hence $\{a\}^{\perp\perp} \perp A^{\perp\perp}$. It is true that $\bigcap_{x \in A \cup \{a\}} (A \cup \{a\} - \{x\})^{\perp\perp} = A^{\perp\perp} \cap \bigcap_{x \in A} [\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}] = \bigcap_{x \in A} A^{\perp\perp} \cap [\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}] = \bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$ where the last but one identity follows from Statement 1.4.

$i = k$. On the one hand we have $A^{\perp\perp} \cap \{a\}^{\perp\perp} = \{o\}$, and on the other, if we have $A = B \cup C$, $B \cap C = \emptyset$, $B \neq \emptyset \neq C$, then $A \cup \{a\} = (B \cup \{a\}) \cup C$ and $(B \cup \{a\}) \cap C = \emptyset$. It is evident that $\{a\}^{\perp\perp} \subset (\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap (\{a\}^{\perp\perp} \vee C^{\perp\perp})$. In accordance with Statements 1.2 and 1.4, we get $(\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap (\{a\}^{\perp\perp} \vee C^{\perp\perp}) = \{a\}^{\perp\perp} \vee [\{a\}^{\perp} \cap (\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap (\{a\}^{\perp\perp} \vee C^{\perp\perp})] = \{a\}^{\perp\perp} \vee [\{a\}^{\perp} \cap (\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap \{a\}^{\perp} \cap (\{a\}^{\perp\perp} \vee C^{\perp\perp})] = \{a\}^{\perp\perp} \vee (B^{\perp\perp} \cap C^{\perp\perp}) = \{a\}^{\perp\perp}$. Thus, $\{a\}^{\perp\perp} = (\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap (\{a\}^{\perp\perp} \vee C^{\perp\perp}) \supset (\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap C^{\perp\perp}$. Since the equality $\{a\}^{\perp\perp} = (\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap C^{\perp\perp}$ cannot hold, we see that $(\{a\}^{\perp\perp} \vee B^{\perp\perp}) \cap C^{\perp\perp} = \{o\}$ (Axiom A). This completes the proof.

$i = l$. We have $A^{\perp\perp} = (A - \{x\})^{\perp\perp} \vee [(A - \{x\})^{\perp} \cap A^{\perp\perp}]$ for every $x \in A$ in view of Statement 1.2. Let B_x be the set $(A - \{x\})^{\perp} \cap A^{\perp\perp}$. Then $B_x \subset (A -$

$-\{x\})^{\perp}$, hence we get $A^{\perp\perp} \cap B_x^{\perp} = [(A - \{x\})^{\perp\perp} \vee B_x] \cap B_x^{\perp} = (A - \{x\})^{\perp\perp}$ in virtue of Statement 1.4. If the set A is l-independent and if $a \neq o$, $a \perp A$, then $\{a\}^{\perp\perp} \subset A^{\perp} = (A - \{x\})^{\perp} \cap B_x^{\perp} \subset B_x^{\perp}$. We have $\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp} = \{a\}^{\perp\perp} \vee \vee (A^{\perp\perp} \cap B_x^{\perp})$. Since $\{a\}^{\perp\perp} \subset (\{a\}^{\perp\perp} \vee A^{\perp\perp}) \cap B_x^{\perp}$ and since $A^{\perp\perp} \subset \{a\}^{\perp}$, according to Statements 1.2 and 1.4 we get $(\{a\}^{\perp\perp} \vee A^{\perp\perp}) \cap B_x^{\perp} = \{a\}^{\perp\perp} \vee [\{a\}^{\perp} \cap (\{a\}^{\perp\perp} \vee \vee A^{\perp\perp}) \cap B_x^{\perp}] = \{a\}^{\perp\perp} \vee (A^{\perp\perp} \cap B_x^{\perp})$. Hence we have $\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp} = = (\{a\}^{\perp\perp} \vee A^{\perp\perp}) \cap B_x^{\perp}$. If the relation $x \in \{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}$ holds for some $x \in A$, it follows that $\{x\}^{\perp\perp} = \{x\}^{\perp\perp} \cap [\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}] = \{x\}^{\perp\perp} \cap (\{a\}^{\perp\perp} \vee \vee A^{\perp\perp}) \cap B_x^{\perp} \subset B_x^{\perp}$. Therefore $\{x\}^{\perp\perp} = \{x\}^{\perp\perp} \cap A^{\perp\perp} \subset B_x^{\perp} \cap A^{\perp\perp} = (A - \{x\})^{\perp\perp}$ – a contradiction.

2.8. Lemma. *An i-independent set $A \subset \Omega$ is a maximal i-independent set with respect to the set inclusion if and only if $A^{\perp\perp} = \Omega$ for $i = j, k, l$.*

Proof. 1. Let A be an i-indepedent set for $i = j, k, l$ and let $A^{\perp\perp} \neq \Omega$. Hence $A^{\perp} \neq \{o\}$ and there exists $a \in A^{\perp}$, $a \neq o$. It is true that $a \perp A$. The set $A \cup \{a\}$ is also an i-independent set according to Lemma 2.7 for $i = j, k, l$.

2. Let A be an i-independent set for $i = j, k, l$ and let $A^{\perp\perp} = \Omega$. If $A \subset B$, $A \neq B$, where the set B is also i-independent for $i = j, k, l$, then the set A as well as the set B are l-indepedent in accordance with Lemma 2.2. It follows that $\Omega = A^{\perp\perp} \subset B^{\perp\perp}$, $A^{\perp\perp} \neq B^{\perp\perp}$ according to Lemma 2.5 However, this contradicts our hypothesis. Hence A is a maximal i-independent set for $i = j, k, l$.

2.9. Theorem. *Every i-independent set $A \subset \Omega$ is a subset of maximal i-independent set for $i = j, k, l$.*

Proof. First, we shall prove the following statement: If $A \in S$, $A \neq \{o\}$, then $A = \bigvee_{i \in I} \{a_i\}^{\perp\perp}$ where $a_i \neq o$, $a_i \perp a_j$ for $i \neq j$, $i, j \in I$. Indeed, let $\{C_k : k \in K\}$ be a chain of orthogonal sets (i.e. when $x, y \in C_k$, $x \neq y$, then $x \perp y$) such that $C_k^{\perp\perp} \subset A$ for all $k \in K$. Hence $\bigcup_{k \in K} C_k = C$ is an orthogonal set. Moreover, $C^{\perp\perp} = (\bigcup_{k \in K} C_k)^{\perp\perp} = \bigvee_{k \in K} C_k^{\perp\perp} \subset A$. It follows that there are maximal orthogonal sets $D \subset A$. Then $D^{\perp\perp} = A$. If not, then $D^{\perp\perp} \subset A$, $D^{\perp\perp} \neq A$. Hence $D^{\perp} \cap A \neq \{o\}$ in virtue of Statement 1.3. Consequently, there is $a \in D^{\perp} \cap A$, $a \neq o$. We have $(D \cup \{a\})^{\perp\perp} \subset A$ and the set $D \cup \{a\}$ is an orthogonal set, therefore the set D is not maximal. Our assertion is proved.

Now, let A be an i-independent set. If $A^{\perp\perp} = \Omega$, then A is a maximal i-independent set in view of Lemma 2.8 for $i = j, k, l$. If $A^{\perp\perp} \neq \Omega$, then in accordance with our assertion above, $A^{\perp} = \bigvee_{h \in I} \{a_h\}^{\perp\perp}$ where $a_h \neq o$, $a_g \perp a_h$ for $g \neq h$, $g, h \in I$. Let B stand for the set $\{a_h : h \in I\}$. The set $A \cup B$ has the property $(A \cup B)^{\perp\perp} = A^{\perp\perp} \vee \vee \bigvee_{h \in I} \{a_h\}^{\perp\perp} = A^{\perp\perp} \vee A^{\perp} = \Omega$. We shall prove that $A \cup B$ is an i-independent set for $i = j, k, l$.

$i = j$. It is true that $\bigcap_{x \in A \cup B} (A \cup B - \{x\})^{\perp\perp} = \bigcap_{x \in A} [A^{\perp} \vee (A - \{x\})^{\perp\perp}] \cap \bigcap_{x \in B} [A^{\perp\perp} \vee (B - \{x\})^{\perp\perp}]$. We have $\bigcap_{x \in A} [A^{\perp} \vee (A - \{x\})^{\perp\perp}] \supset A^{\perp} \vee \bigcap_{x \in A} (A - \{x\})^{\perp\perp}$. According to Statement 1.4, we get $A^{\perp\perp} \cap [\bigcap_{x \in A} (A - \{x\})^{\perp\perp}]^{\perp} \cap \bigcap_{x \in A} [A^{\perp} \vee (A - \{x\})^{\perp\perp}] = [\bigcap_{x \in A} (A - \{x\})^{\perp\perp}]^{\perp} \cap \bigcap_{x \in A} \{A^{\perp\perp} \cap [A^{\perp} \vee (A - \{x\})^{\perp\perp}]\} = [\bigcap_{x \in A} (A - \{x\})^{\perp\perp}]^{\perp} \cap \bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$. Hence, in view of Statement 1.3, we have $\bigcap_{x \in A} [A^{\perp} \vee (A - \{x\})^{\perp\perp}] = A^{\perp} \vee \bigcap_{x \in A} (A - \{x\})^{\perp\perp}$. Furthermore, $\bigcap_{x \in B} [A^{\perp\perp} \vee (B - \{x\})^{\perp\perp}] \supset A^{\perp\perp} \vee \bigcap_{x \in B} (B - \{x\})^{\perp\perp}$. Again in accordance with Statement 1.4, we get $A^{\perp} \cap [\bigcap_{x \in B} (B - \{x\})^{\perp\perp}]^{\perp} \cap \bigcap_{x \in B} [A^{\perp\perp} \vee (B - \{x\})^{\perp\perp}] = [\bigcap_{x \in B} (B - \{x\})^{\perp\perp}]^{\perp} \cap \bigcap_{x \in B} (B - \{x\})^{\perp\perp} = \{o\}$. Hence, in virtue of Statement 1.3, we have $\bigcap_{x \in B} [A^{\perp\perp} \vee (B - \{x\})^{\perp\perp}] = A^{\perp\perp} \vee \bigcap_{x \in B} (B - \{x\})^{\perp\perp}$. Summarizing, we have $\bigcap_{x \in A \cup B} (A \cup B - \{x\})^{\perp\perp} = [A^{\perp} \vee \bigcap_{x \in A} (A - \{x\})^{\perp\perp}] \cap [A^{\perp\perp} \vee \bigcap_{x \in B} (B - \{x\})^{\perp\perp}] = A^{\perp} \cap [A^{\perp\perp} \vee \bigcap_{h \in I} (A^{\perp} \cap \{a_h\}^{\perp})] = A^{\perp} \cap [A^{\perp\perp} \vee (A^{\perp} \cap A^{\perp\perp})] = \{o\}$ according to Statement 1.4.

$i = k$. Let us consider $C \subset A$, $\emptyset \neq C \neq A$ and let $D = (A - C) \cup B$. It is true that $C^{\perp\perp} \cap D^{\perp\perp} = C^{\perp\perp} \cap [(A - C)^{\perp\perp} \vee B^{\perp\perp}] \subset A^{\perp\perp} \cap [(A - C)^{\perp\perp} \vee A^{\perp}] = (A - C)^{\perp\perp}$ as a consequence of Statement 1.4. Since $C^{\perp\perp} \cap D^{\perp\perp} \subset C^{\perp\perp}$ we have $C^{\perp\perp} \cap D^{\perp\perp} \subset C^{\perp\perp} \cap (A - C)^{\perp\perp} = \{o\}$. Now suppose that $D \subset B$ where $\emptyset \neq D \neq B$ and let $C = A \cup (B - D)$. It follows that $C^{\perp\perp} \cap D^{\perp\perp} = [A^{\perp\perp} \vee (B - D)^{\perp\perp}] \cap D^{\perp\perp} = \{o\}$ as a consequence of $[A^{\perp\perp} \vee (B - D)^{\perp\perp}] \perp D^{\perp\perp}$. Finally, let $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, $A_1 \neq \emptyset \neq A_2$, $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, $B_1 \neq \emptyset \neq B_2$. Let us denote $C = A_1 \cup B_1$, $D = A_2 \cup B_2$. We have $C^{\perp\perp} \cap D^{\perp\perp} = (A_1^{\perp\perp} \vee B_1^{\perp\perp}) \cap (A_2^{\perp\perp} \vee B_2^{\perp\perp}) \subset (A^{\perp\perp} \vee B_1^{\perp\perp}) \cap (A_2^{\perp\perp} \vee B_2^{\perp\perp}) = A_2^{\perp\perp}$ where the last identity follows from Statement 1.4 because $A_2^{\perp\perp} \subset A^{\perp\perp} \vee B_1^{\perp\perp}$ and $B_2^{\perp\perp} \subset (A^{\perp\perp} \vee B_1^{\perp\perp})^{\perp} = A^{\perp} \cap B_1^{\perp}$. In a similar way, we can prove that $C^{\perp\perp} \cap D^{\perp\perp} \subset (A_1^{\perp\perp} \vee B_1^{\perp\perp}) \cap (A^{\perp\perp} \vee B_2^{\perp\perp}) = A_1^{\perp\perp}$. Hence $C^{\perp\perp} \cap D^{\perp\perp} \subset A_1^{\perp\perp} \cap A_2^{\perp\perp} = \{o\}$.

$i = l$. Let $x \in A$. Then $x \notin (A - \{x\})^{\perp\perp}$. According to Statement 1.2, we have $A^{\perp\perp} = (A - \{x\})^{\perp\perp} \vee [(A - \{x\})^{\perp} \cap A^{\perp\perp}] = (A - \{x\})^{\perp\perp} \vee A_x$ where $A_x = (A - \{x\})^{\perp} \cap A^{\perp\perp}$. This implies $A_x \subset (A - \{x\})^{\perp}$ and $A_x \subset A^{\perp\perp}$. In accordance with Statement 1.4, it is true that $A^{\perp\perp} \cap A_x^{\perp} = [(A - \{x\})^{\perp\perp} \vee A_x] \cap A_x^{\perp} = (A - \{x\})^{\perp\perp}$. Further, we get $A_x \subset (A - \{x\})^{\perp\perp} \vee A_x = A^{\perp\perp} = B^{\perp}$ or $B^{\perp\perp} \subset A_x^{\perp}$. Hence $(A - \{x\})^{\perp\perp} \vee B^{\perp\perp} \subset A_x^{\perp}$. Suppose $x \in (A - \{x\})^{\perp\perp} \vee B^{\perp\perp}$ which yields $x \in A_x^{\perp}$. But then $x \in \{x\}^{\perp\perp} \subset A^{\perp\perp} \cap A_x^{\perp} = (A - \{x\})^{\perp\perp}$ – a contradiction. Further, let $x \in B = \{a_h : h \in I\}$. It is true that $\{x\}^{\perp\perp} \subset A^{\perp}$ or $A^{\perp\perp} \subset \{x\}^{\perp}$. If $x \in A^{\perp\perp} \vee (B - \{x\})^{\perp\perp} = A^{\perp\perp} \vee (B^{\perp\perp} \cap \{x\}^{\perp}) \subset \{x\}^{\perp}$ then $x \in \{x\}^{\perp}$ or $x = o$. This is an evident contradiction and it completes the proof of our assertion.

The following theorem is a generalization of Lemma 2.7.

2.10. Theorem. *If the set A is i-independent and if $a \notin A^{\perp\perp}$, then $A \cup \{a\}$ is also an i-independent set for $i = j, k, l$.*

Proof. $i = j$. We have $\bigcap_{x \in A \cup \{a\}} (A \cup \{a\} - \{x\})^{\perp\perp} = A^{\perp\perp} \cap \bigcap_{x \in A} [\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}] = \bigcap_{x \in A} A^{\perp\perp} \cap [\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}] = \bigcap_{x \in A} \{[A_x \vee (A - \{x\})^{\perp\perp}] \cap [B_x \vee (A - \{x\})^{\perp\perp}]\}$ where $A_x = (A - \{x\})^{\perp} \cap A^{\perp\perp}$ and $B_x = (A - \{x\})^{\perp} \cap [\{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp}]$. Evidently, $A_x \perp (A - \{x\})^{\perp\perp}$ and $B_x \perp (A - \{x\})^{\perp\perp}$. Applying Statement 1.4 we get $(A - \{x\})^{\perp\perp} \vee [(A - \{x\})^{\perp} \cap A_x^{\perp}] = A_x^{\perp}$ and $(A - \{x\})^{\perp\perp} \vee [(A - \{x\})^{\perp} \cap B_x^{\perp}] = B_x^{\perp}$. Applying again Statement 1.4 we have $(A_x^{\perp} \vee B_x^{\perp}) \cap (A - \{x\})^{\perp} = \{(A - \{x\})^{\perp\perp} \vee [(A - \{x\})^{\perp} \cap A_x^{\perp}] \vee [(A - \{x\})^{\perp} \cap B_x^{\perp}]\} \cap (A - \{x\})^{\perp} = [(A - \{x\})^{\perp} \cap A_x^{\perp}] \vee [(A - x\})^{\perp} \cap B_x^{\perp}]$ or $(A_x \cap B_x) \vee (A - \{x\})^{\perp\perp} = [A_x \vee (A - \{x\})^{\perp\perp}] \cap [B_x \vee (A - \{x\})^{\perp\perp}]$. Therefore, $\bigcap_{x \in A \cup \{a\}} (A \cup \{a\} - \{x\})^{\perp\perp} = \bigcap_{x \in A} [(A_x \cap B_x) \vee (A - \{x\})^{\perp\perp}]$. In accordance with Theorem 2.10 of [2], A_x and B_x are atoms in the lattice $\mathcal{S}$. If $A_x = B_x$, then $a \in \{a\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp} = B_x \vee (A - \{x\})^{\perp\perp} = A_x \vee (A - \{x\})^{\perp\perp} = \{x\}^{\perp\perp} \vee (A - \{x\})^{\perp\perp} = A^{\perp\perp}$ – a contradiction. Thus, $A_x \cap B_x = \{o\}$ and we have $\bigcap_{x \in A \cup \{a\}} (A \cup \{a\} - \{x\})^{\perp\perp} = \bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$.

$i = k$. Let $A = B \cup C$, $B \cap C = \emptyset$, $B \neq \emptyset \neq C$. Using Statement 1.2 we have $(B^{\perp\perp} \vee \{a\}^{\perp\perp}) \cap C^{\perp\perp} \subset (B^{\perp\perp} \vee \{a\}^{\perp\perp}) \cap (B^{\perp\perp} \vee C^{\perp\perp}) = (B^{\perp\perp} \vee B_a) \cap (B^{\perp\perp} \vee C_B)$ where $B_a = B^{\perp} \cap (B^{\perp\perp} \vee \{a\}^{\perp\perp})$ and $C_B = B^{\perp} \cap (B^{\perp\perp} \vee C^{\perp\perp})$. Since $B_a \perp B^{\perp\perp}$ and $C_B \perp B^{\perp\perp}$, it is true that $(B^{\perp\perp} \vee B_a) \cap (B^{\perp\perp} \vee C_B) = B^{\perp\perp} \vee (B_a \cap C_B)$ which can be proved in a similar way as in the first part of this proof. According to Theorem 2.10 of [2], B_a is an atom. If $B_a \cap C_B \neq \{o\}$, then $B_a = B_a \cap C_B = B_a \cap A^{\perp\perp}$. It follows that $a \in B^{\perp\perp} \vee \{a\}^{\perp\perp} = B^{\perp\perp} \vee B_a = B^{\perp\perp} \vee (B_a \cap A^{\perp\perp}) \subset B^{\perp\perp} \vee A^{\perp\perp}$ – a contradiction. Therefore $B_a \cap C_B = \{o\}$, hence $(B^{\perp\perp} \vee \{a\}^{\perp\perp}) \cap C^{\perp\perp} \subset B^{\perp\perp}$. Since $(B^{\perp\perp} \vee \{a\}^{\perp\perp}) \cap C^{\perp\perp} \subset C^{\perp\perp}$ we have $(B^{\perp\perp} \vee \{a\}^{\perp\perp}) \cap C^{\perp\perp} \subset B^{\perp\perp} \cap C^{\perp\perp} = \{o\}$.

$i = l$. Proof of the statement coincides with the proof of Theorem 2.10 of [1]. The theorem is proved.

Let us introduce the following axiom.

Axiom I. *If $A \subset \Omega$ is an l-independent set, $A = B \cup C$, $B \neq \emptyset \neq C$, $B \cap C = \emptyset$, then $\bigcap_{x \in C} (A - \{x\})^{\perp\perp} = B^{\perp\perp}$.*

In accordance with Theorem 2.12 of [1], Axiom I is satisfied when C is a finite set. Let us now suppose that A is an orthogonal set. Then $\bigcap_{x \in C} (B \cup C - \{x\})^{\perp\perp} = \bigcap_{x \in C} [B^{\perp\perp} \vee (C - \{x\})^{\perp\perp}] = B^{\perp\perp} \vee \bigcap_{x \in C} (C - \{x\})^{\perp\perp}$ because $B^{\perp} \cap [B^{\perp\perp} \vee (C - \{x\})^{\perp\perp}] = (C - \{x\})^{\perp\perp}$ for all $x \in C$ according to Statement 1.4. Hence applying again Statement 1.4 we have $B^{\perp} \cap \bigvee_{x \in C} (C - \{x\})^{\perp} = B^{\perp} \cap \bigvee_{x \in C} \{B^{\perp\perp} \vee [B^{\perp} \cap (C - \{x\})^{\perp}]\} = B^{\perp} \cap \{B^{\perp\perp} \vee \bigvee_{x \in C} [B^{\perp} \cap (C - \{x\})^{\perp}]\} = \bigvee_{x \in C} [B^{\perp} \cap (C - \{x\})^{\perp}]$ which gives $B^{\perp\perp} \vee \bigcap_{x \in C} (C - \{x\})^{\perp\perp} = \bigcap_{x \in C} [B^{\perp\perp} \vee (C - \{x\})^{\perp\perp}]$. Further, we have $(C - \{x\})^{\perp\perp} = C^{\perp\perp} \cap \{x\}^{\perp}$ as a consequence of Statement 1.4 which implies $\bigcap_{x \in C} (C - \{x\})^{\perp\perp} = \bigcap_{x \in C} (C^{\perp\perp} \cap \{x\}^{\perp}) = C^{\perp\perp} \cap \bigcap_{x \in C} \{x\}^{\perp} = C^{\perp\perp} \cap C^{\perp} = \{o\}$ Therefore we have $\bigcap_{x \in C} (B \cup C - \{x\})^{\perp\perp} = B^{\perp\perp}$. Thus we see that Axiom I is also satisfied when A is an orthogonal set.

2.11. Lemma. *If A is an l-independent set and if the lattice $\mathscr{S}$ satisfies* Axiom I *then A is also j-independent.*

Proof. For an element $a \in A$ we have $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} = (A - \{a\})^{\perp\perp} \cap \bigcap_{x \in A - \{a\}} (A - \{x\})^{\perp\perp} = (A - \{a\})^{\perp\perp} \cap \{a\}^{\perp\perp} = \{o\}$.

2.12. Definition. Let $\emptyset \neq A \subset \Omega$, $o \notin A$ and let us assume that the set A contains at least two points. We shall say that the set A is *J-independent* if and only if its every finite subset which contains at least two points is j-independent. We shall say that the set A is *K-independent* if and only if its every finite subset which contains at least two points is k-independent. We shall say that the set A is *L-independent* if and only if its every finite subset which contains at least two points is l-independent.

2.13. Theorem. *Let $\emptyset \neq A \subset \Omega$, $o \notin A$ and let us suppose that the set A contains at least two points. Then the following statements are equivalent.* 1. *The set A is J-independent.* 2. *The set A is K-independent.* 3. *The set A is L-independent.*

Proof follows from Lemma 2.2, Theorem 2.12 of [1] and Lemma 2.11.

Let us note that, according to Lemma 2.6, every j-independent set is J-independent, every k-independent set is K-independent and every l-independent set is L-independent.

Let $o \neq a \in \Omega$ and let the set $A = \{a\}^{\perp\perp} - \{o\}$ contain at least two points. Then for all $x \in A$ we have $(A - \{x\})^{\perp\perp} = \{a\}^{\perp\perp}$. If $A = B \cup C$, $B \cap C = \emptyset$, $B \neq \emptyset \neq C$, hence $\bigcap_{x \in C} (B \cup C - \{x\})^{\perp\perp} = \{a\}^{\perp\perp} = B^{\perp\perp}$. This example shows that the assertion of Axiom I may be satisfied even when the set A is not l-independent.

Literature

[1] *J. Havrda:* Independence in a set with orthogonality, Časopis pěst. mat. *107* (1982), 267—272.
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Souhrn

STUDIE NEZÁVISLOSTI V MNOŽINĚ S ORTOGONALITOU

JAN HAVRDA

Uvažuje se množina s ortogonalitou $(\Omega, \perp)$ a jí odpovídající úplný svaz s ortogonalitou $\mathcal{S} = (S, \subset, \perp, \Omega, \{o\})$. Předpokládá se, že svaz $\mathcal{S}$ je ortomodulární a splňuje některé další předpoklady. Nechť $o \notin A \subset \Omega$, A obsahuje alespoň dva prvky. Podmnožina A se nazývá j-nezávislá, když $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$, nazývá se k-nezávislá, když $B^{\perp\perp} \cap C^{\perp\perp} = \{o\}$, kdykoliv $A = B \cup C$, $B \neq \emptyset \neq C$, $B \cap C = \emptyset$, nazývá se l-nezávislá, když $x \notin (A - \{x\})^{\perp\perp}$ pro všechna $x \in A$. Podmnožina A se nazývá I-nezávislá, když každá její konečná podmnožina, která obsahuje alespoň dva prvky, je i-nezávislá, kde $(I, i) = (J, j)$, (K, k), (L, l). Článek se zabývá vlastnostmi těchto pojmů a vztahy mezi nimi.

Резюме

ИЗУЧЕНИЕ НЕЗАВИСИМОСТИ В МНОЖЕСТВЕ С ОРТОГОНАЛЬНОСТЬЮ

JAN HAVRDA

Рассматривается множество с отношением ортогональности $(\Omega, \perp)$ и порожденная им полная решетка с ортогональностю $\mathcal{S} = (S, \subset, \perp, \Omega, \{o\})$. Предполатестся, что решетка $\mathcal{S}$ ортомодулярна и удовлетворяет некоторым дальнейшим предположениям. Пусть $o \notin A \subset \Omega$, где в A по крайней мере два элемента. Множество A называется j-независимым, если $\bigcap_{x \in A} (A - \{x\})^{\perp\perp} = \{o\}$; k-независимым, если $B^{\perp\perp} \cap C^{\perp\perp} = \{o\}$, как только $A = B \cup C$, $B \neq \emptyset \neq C$, $B \cap C = \emptyset$; l-независимым, если $x \notin (A - \{x\})^{\perp\perp}$ для всех $x \in A$. Множество A называется I-независимым, если каждое конечное подмножество, в котором по крайней мере два элемента, i-независимо, $(I, i) = (J, j)$, (K, k), (L, l). Статья занимается взаимными отношениями между этими понятиями.

Author's address: Katedra matematiky FEL ČVUT, Suchbátarova 2, 166 27 Praha 6 - Dejvice.